

## ON TYPES OF NOETHERIAN LOCAL RINGS AND MODULES

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**ABSTRACT.** We investigate some results which concern the types of Noetherian local rings. In particular, we show that if  $r(A_{\mathfrak{p}}) \leq \text{depth } A_{\mathfrak{p}} + 1$  for each prime ideal  $\mathfrak{p}$  of a quasi-unmixed Noetherian local ring  $A$ , then  $A$  is Cohen-Macaulay. It is also shown that the Kawasaki conjecture holds when  $\dim A \leq \text{depth } A + 1$ . At the end, we deal with some analogous results for modules, which are derived from the results studied on rings.

### 1. Introduction

Throughout this paper, we assume that  $(A, \mathfrak{m})$  is a commutative Noetherian local ring of dimension  $d$ , and  $M$  is a finitely generated  $A$ -module. We also assume that all modules are unitary.

For a prime ideal  $\mathfrak{p}$  of  $A$ , the  $i$ -th Bass number of  $M$  at  $\mathfrak{p}$ , denoted  $\mu_i(\mathfrak{p}, M)$ , is defined to be  $\dim_{k(\mathfrak{p})} \text{Ext}_{A_{\mathfrak{p}}}^i(k(\mathfrak{p}), M_{\mathfrak{p}})$ , where  $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ ; we set  $\mu_i(A) = \mu_i(\mathfrak{m}, A)$  for brevity. The *type* of  $A$ , denoted by  $r(A)$ , is defined to be  $\mu_d(A)$ . When a type of  $A$  is known, the conditions which make a ring  $A$  Cohen-Macaulay have been studied by many mathematicians ([1,3,4,8,9], etc).

Bass characterized Gorenstein rings as Cohen-Macaulay rings  $A$  with  $r(A) = 1$  ([2]). Vasconcelos conjectured that the condition  $r(A) = 1$  is sufficient for  $A$  to be Gorenstein, i.e., the condition “ $A$  is Cohen-Macaulay” can be omitted ([12]). In [5], Foxby proved this conjecture for essentially equicharacteristic rings using a version of the Intersection Theorem. The conjecture was proven in general by Roberts ([11]): he showed that local rings of type one are Cohen-Macaulay, (and hence Gorenstein) using a minimal free resolution of a dualizing complex.

By modifying Roberts’ argument, Costa, Huneke and Miller ([4]) showed that when  $A$  is a local ring whose completion is a domain and  $r(A) = 2$ ,  $A$  is Cohen-Macaulay. Expecting the above might be the best possible, they gave two examples: a complete equidimensional local ring of type two that is not Cohen-Macaulay, and a complete reduced local ring of type two that

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is not Cohen-Macaulay. Afterward, they posed a question: *Does there exist a complete, equidimensional, reduced local ring  $A$  with  $r(A) = 2$  that is not Cohen-Macaulay?*

However, Marley answered this question in negative by proving the theorem that if  $A$  is an unmixed local ring of type two, then  $A$  is Cohen-Macaulay ([8]).

Marley also asked that if a complete local ring of type  $n$  satisfies Serre's condition  $(S_{n-1})$ , then it is Cohen-Macaulay. Kawasaki answered this question in the affirmative when rings contain a field and  $n \geq 3$ , and later Aoyama gave a general proof ([1]).

In [6], Kawasaki conjectured the following which is still open:

**Conjecture.** ([6]) *Let  $A$  be a complete unmixed local ring of type  $n$ . If  $A_{\mathfrak{p}}$  is Cohen-Macaulay for all  $\mathfrak{p}$  in  $\text{Spec}(A)$  such that  $ht(\mathfrak{p}) < n$ , then  $A$  is Cohen-Macaulay.*

In [7], the author posed the following question:

**Question.** ([7]) *Let  $(A, \mathfrak{m})$  be a complete unmixed Noetherian local ring. If  $r(A) \leq \text{depth } A + 1$ , where  $r(A)$  is the type of  $A$ , then is a ring  $A$  Cohen-Macaulay?, or equivalently, if  $A$  is not Cohen-Macaulay, then  $r(A) \geq \text{depth } A + 2$ ?*

In [7], the above question was answered in the affirmative under some additional conditions: It was shown that the above question is true provided that  $\hat{A}_{\mathfrak{p}}$  is Cohen-Macaulay for every non-maximal prime ideal  $\mathfrak{p}$  in  $\hat{A}$ . The question is also true if we add the conditions,  $\dim A \leq \text{depth } A + 1$ , and  $\ell(H_{\mathfrak{m}}^{d-1}(A)) < \infty$ .

In this article, we can't give a complete answer to the question, but we give a positive answer under the assumption,  $r(A_{\mathfrak{p}}) \leq \text{depth } A_{\mathfrak{p}} + 1$  for each prime ideal  $\mathfrak{p}$  of  $A$ . We also prove that the Kawasaki's conjecture holds for rings  $A$  with  $\dim A \leq \text{depth } A + 1$ .

In the last section, we show that all results studied on rings give the analogous results for modules after some modifications.

## 2. Two main theorems

We recall that  $A$  is *equidimensional* if  $\dim A/\mathfrak{p} = \dim A$  for every minimal prime  $\mathfrak{p}$  of  $A$ .  $A$  is said to be *quasi-unmixed* (or formally equidimensional) if its completion  $\hat{A}$  is equidimensional, and to be *unmixed* if  $\dim \hat{A}/\mathfrak{p} = \dim \hat{A}$  for each associated prime  $\mathfrak{p} \in \text{Ass}(\hat{A})$ .

In this section, we concern the following question described in the introduction:

**Question 2.1.** *Let  $(A, \mathfrak{m})$  be a complete unmixed Noetherian local ring. If  $r(A) \leq \text{depth } A + 1$ , where  $r(A)$  is the type of  $A$ , then is a ring  $A$  Cohen-Macaulay?*

We note that if  $\dim A \leq 2$ , then the above question is positive: if  $\dim A = 1$ , then  $r(A) \leq 2$ , and so  $A$  is Cohen-Macaulay. For a ring  $A$  of dimension 2, if  $\text{depth } A \leq 1$  then  $r(A) \leq 2$ , which implies that  $A$  should be Cohen-Macaulay.

In [7], the following theorem was proved:

**Theorem 2.2.** ([7]) *Let  $(A, \mathfrak{m})$  be a quasi-unmixed Noetherian local ring of dimension  $d$  such that  $r(A) \leq \text{depth } A + 1$ . Suppose that  $\widehat{A}_{\mathfrak{p}}$  is Cohen-Macaulay for every non-maximal prime  $\mathfrak{p}$  in  $\widehat{A}$ . Then  $A$  is Cohen-Macaulay.*

We may restate Proposition 2.4 in [7] in the form of our question as follows:

**Proposition 2.3.** ([7]) *Let  $(A, \mathfrak{m})$  be a Noetherian local ring of dimension  $d$  such that  $r(A) \leq \text{depth } A + 1$ . Suppose  $\dim A \leq \text{depth } A + 1$ , and  $\ell(H_{\mathfrak{m}}^{d-1}(A)) < \infty$ . Then  $A$  is Cohen-Macaulay.*

We now recall that if  $A$  is quasi-unmixed, then (i)  $A_{\mathfrak{p}}$  is quasi-unmixed for every prime  $\mathfrak{p}$  of  $A$ , and (ii)  $A/I$  is equidimensional if and only if  $A/I$  is quasi-unmixed for an ideal  $I$  of  $A$  ([9, Theorem 31.6]).

Now, we prove one of the main theorems in this article.

**Theorem 2.4.** *Let  $(A, \mathfrak{m})$  be a complete unmixed Noetherian local ring, and  $\mathfrak{q}$  a prime ideal of  $A$ . Suppose that  $r(A_{\mathfrak{p}}) \leq \text{depth } A_{\mathfrak{p}} + 1$  for each prime ideal  $\mathfrak{p} \subseteq \mathfrak{q}$ . Then  $A_{\mathfrak{q}}$  is Cohen-Macaulay. In particular, if  $r(A_{\mathfrak{p}}) \leq \text{depth } A_{\mathfrak{p}} + 1$  for each prime ideal  $\mathfrak{p}$  of  $A$ , then a ring  $A$  is Cohen-Macaulay.*

*Proof.* Since  $A$  is unmixed (and so quasi-unmixed),  $A_{\mathfrak{q}}$  is quasi-unmixed. Thus it is enough to show that  $(\widehat{A_{\mathfrak{q}}})_{\tilde{\mathfrak{p}}}$  is Cohen-Macaulay for every prime ideal  $\tilde{\mathfrak{p}}$  (of  $\widehat{A_{\mathfrak{q}}}$ ), which is properly contained in  $\widehat{\mathfrak{q}A_{\mathfrak{q}}}$ : if then, Theorem 2.2 implies that  $A_{\mathfrak{q}}$  is Cohen-Macaulay since  $r(A_{\mathfrak{q}}) \leq \text{depth } A_{\mathfrak{q}} + 1$ . Since  $A$  is complete,  $A$  is a homomorphic image of a Cohen-Macaulay ring, and so is  $A_{\mathfrak{q}}$ . Thus by Lemma 2.8 in [7] (or see Lemma 3.1 in Section 3), showing that  $(\widehat{A_{\mathfrak{q}}})_{\tilde{\mathfrak{p}}}$  is Cohen-Macaulay for every prime ideal  $\tilde{\mathfrak{p}}$  (of  $\widehat{A_{\mathfrak{q}}}$ ), which is properly contained in  $\widehat{\mathfrak{q}A_{\mathfrak{q}}}$  is equivalent to showing that  $A_{\mathfrak{p}} (= (A_{\mathfrak{q}})_{\mathfrak{p}})$  is Cohen-Macaulay for every prime ideal  $\mathfrak{p}$ , which is properly contained in  $\mathfrak{q}$ . Thus it suffices to show the latter part.

If  $ht(\mathfrak{q}) = 1$ , then  $A_{\mathfrak{q}}$  is Cohen-Macaulay since  $A_{\mathfrak{q}_0}$  is Cohen-Macaulay for every minimal prime  $\mathfrak{q}_0 \subseteq \mathfrak{q}$ . Suppose  $ht(\mathfrak{q}) = t > 1$  and let  $\mathfrak{q}_1 (\subseteq \mathfrak{q})$  be any prime ideal of height 1. Then we can show that  $A_{\mathfrak{q}_1}$  is Cohen-Macaulay by the same reasoning as above. By induction,  $A_{\mathfrak{q}_s}$  is Cohen-Macaulay for any prime ideal  $\mathfrak{q}_s (\subseteq \mathfrak{q})$  of height  $s < t$ . Hence,  $A_{\mathfrak{q}}$  is Cohen-Macaulay by Theorem 2.2. In particular,  $A$  is Cohen-Macaulay if  $\mathfrak{q}$  is a maximal ideal  $\mathfrak{m}$ . This completes the proof.  $\square$

We now turn to Kawasaki's conjecture, and answer it affirmatively when the inequality  $\dim A \leq \text{depth } A + 1$  holds.

**Theorem 2.5.** *Let  $A$  be a complete unmixed Noetherian local ring of type  $n$  with  $\dim A \leq \text{depth } A + 1$ . Suppose that  $A_{\mathfrak{p}}$  is Cohen-Macaulay for all  $\mathfrak{p}$  in  $\text{Spec}(A)$  with  $ht(\mathfrak{p}) < n$ . Then  $A$  is Cohen-Macaulay.*

*Proof.* We may assume that  $r(A) \leq \dim A$  since if  $r(A) > \dim A$ , then  $A$  is Cohen-Macaulay by assumption. Then  $r(A) \leq \dim A \leq \text{depth } A + 1$ . Thus by Theorem 2.2, it is enough to show that  $A_{\mathfrak{p}}$  is Cohen-Macaulay for every prime ideal  $\mathfrak{p} \neq \mathfrak{m}$  of  $A$ . We first claim that

$$\dim A_{\mathfrak{p}} \leq \text{depth } A_{\mathfrak{p}} + 1 \text{ for every prime ideals } \mathfrak{p} \text{ of } A.$$

Indeed, since  $\mu_i(\mathfrak{p}, A) \leq \mu_{i+t}(\mathfrak{m}, A)$  for each  $i$  and  $t = ht(\mathfrak{m}/\mathfrak{p})$  ([13]), if  $\text{depth } A_{\mathfrak{p}} = i$  and  $ht(\mathfrak{m}/\mathfrak{p}) = t$ , then

$$\text{depth } A = \min\{s : \mu_s(\mathfrak{m}, A) \neq 0\} \leq i + t = \text{depth } A_{\mathfrak{p}} + ht(\mathfrak{m}/\mathfrak{p}).$$

Thus

$$\begin{aligned} \dim A_{\mathfrak{p}} - \text{depth } A_{\mathfrak{p}} &\leq (\dim A - ht(\mathfrak{m}/\mathfrak{p})) - (\text{depth } A - ht(\mathfrak{m}/\mathfrak{p})) \\ &= \dim A - \text{depth } A. \end{aligned}$$

Since  $\dim A - \text{depth } A \leq 1$  by assumption, the claim is true.

Now let  $\mathfrak{p}$  be any non maximal prime ideal of  $A$ . If  $ht(\mathfrak{p}) < n$ , then  $A_{\mathfrak{p}}$  is Cohen-Macaulay by assumption. Suppose  $ht(\mathfrak{p}) \geq n$ . Since  $A$  is unmixed and catenary, we know  $r(A_{\mathfrak{p}}) \leq r(A)$  ([13]) and thus

$$r(A_{\mathfrak{p}}) \leq r(A) \leq ht(\mathfrak{p}) \leq \text{depth } A_{\mathfrak{p}} + 1.$$

As in the proof of Theorem 2.4, since  $A_{\mathfrak{p}_0}$  is Cohen-Macaulay for every minimal prime  $\mathfrak{p}_0$ , we may assume that  $A_{\mathfrak{q}} (= (A_{\mathfrak{p}})_{\mathfrak{q}})$  is Cohen-Macaulay for every prime  $\mathfrak{q}$ , which is properly contained in  $\mathfrak{p}$  using induction. Since  $A$  is complete,  $A$  is a homomorphic image of a Cohen-Macaulay ring, and so is  $A_{\mathfrak{p}}$ . Thus  $(\widehat{A_{\mathfrak{p}}})_{\mathfrak{q}}$  is Cohen-Macaulay for every prime  $\mathfrak{q}$ , which is properly contained in  $\mathfrak{p}A_{\mathfrak{p}}$  by Lemma 2.8 in [7]. Since  $A_{\mathfrak{p}}$  is quasi-unmixed and  $r(A_{\mathfrak{p}}) \leq \text{depth } A_{\mathfrak{p}} + 1$ , by Theorem 2.2  $A_{\mathfrak{p}}$  is Cohen-Macaulay. Again using Theorem 2.4, we can conclude that  $A$  is Cohen-Macaulay. □

We close this section with a remark that Question 2.1 is obviously true by Theorem 2.5 if the condition ‘ $r(A) \leq \text{depth } A + 1$ ’ implies the condition ‘ $r(A_{\mathfrak{p}}) \leq \text{depth } A_{\mathfrak{p}} + 1$  for each prime ideal  $\mathfrak{p}$  of  $A$ ’. So, it is natural to inquire the following question:

*For a complete unmixed Noetherian local ring  $A$ , does the condition ‘ $r(A) \leq \text{depth } A + 1$ ’ always imply the condition ‘ $r(A_{\mathfrak{p}}) \leq \text{depth } A_{\mathfrak{p}} + 1$  for each prime ideal  $\mathfrak{p}$  of  $A$ ’?*

### 3. Analogous results for modules

We have mostly focused on the case of rings so far, but in this section we construct a similar theory for modules. Although the proofs used in the case of rings give the proofs of the results for modules after some modifications, we include detailed proofs for completeness since there is no complete proofs of them (in particular, for Corollary 3.3) in the literature.

We start this section with recalling some definitions and fact used in the sequel: An  $A$ -module  $M$  is said to be *equidimensional* if  $\dim A/\mathfrak{p} = \dim M$  for every minimal prime  $\mathfrak{p}$  in  $\text{Supp}(M)$ .  $M$  is said to be *quasi-unmixed* if its completion  $\hat{M}$  is equidimensional, and to be *unmixed* if  $\dim \hat{A}/\mathfrak{p} = \dim \hat{M}$  for all  $\mathfrak{p} \in \text{Ass}(\hat{M})$ . It is known ([6]) that if  $A$  is a homomorphic image of a Cohen-Macaulay ring, then  $M$  is a module with finite local cohomologies if and only if  $M$  is equidimensional and  $M_{\mathfrak{p}}$  is a Cohen-Macaulay  $A_{\mathfrak{p}}$ -module for all  $\mathfrak{p}(\neq \mathfrak{m})$  in  $\text{Supp}(M)$ .

The following lemma enables us to localize  $M$  at a prime ideal when we need to localize its completion  $\hat{M}$  at a prime ideal.

**Lemma 3.1.** *Let  $(A, \mathfrak{m})$  be a Noetherian local ring, and  $M$  a finitely generated  $A$ -module. Suppose that  $A$  is a homomorphic image of a Cohen-Macaulay ring, and  $M$  is quasi-unmixed. Then  $M_{\mathfrak{p}}$  is Cohen-Macaulay for every prime  $\mathfrak{p} \neq \mathfrak{m}$  in  $\text{Supp}(M)$  if and only if  $\hat{M}_{\mathfrak{p}}$  is Cohen-Macaulay for every  $\mathfrak{p} \neq \hat{\mathfrak{m}}$  in  $\text{Supp}(\hat{M})$ .*

*Proof.* If  $M$  is quasi-unmixed, then  $M$  is equidimensional ([9, Theorem 31.6]), and so together with the assumptions, we know that  $M$  is a ring with finite local cohomologies by the above note. Thus  $\hat{M}$  is also a ring with finite local cohomologies since  $H_{\mathfrak{m}}^i(M) \cong H_{\hat{\mathfrak{m}}}^i(\hat{M})$  for each  $i$ . Hence  $\hat{M}_{\mathfrak{p}}$  is Cohen-Macaulay for every  $\mathfrak{p} \neq \hat{\mathfrak{m}}$  in  $\text{Supp}(\hat{M})$ .

For the converse, since  $\hat{M}$  is equidimensional and  $\hat{A}$  is also a homomorphic image of a Cohen-Macaulay ring, if  $\hat{M}_{\mathfrak{p}}$  is Cohen-Macaulay for every  $\mathfrak{p} \neq \hat{\mathfrak{m}}$  in  $\text{Supp}(\hat{M})$ , then  $\hat{M}$  is also a ring with finite local cohomologies, and thus  $M$  is a ring with finite local cohomologies. Hence  $M_{\mathfrak{p}}$  is Cohen-Macaulay for every prime  $\mathfrak{p} \neq \mathfrak{m}$  in  $\text{Supp}(M)$  again by the above note.  $\square$

**Theorem 3.2.** *Let  $(A, \mathfrak{m})$  be a Noetherian local ring, and  $M$  a finitely generated quasi-unmixed  $A$ -module of dimension  $d$  such that  $r(M) \leq \text{depth } M + 1$ . Suppose that  $\hat{M}_{\mathfrak{p}}$  is Cohen-Macaulay for every prime  $\mathfrak{p} \neq \hat{\mathfrak{m}}$  in  $\text{Supp}(\hat{M})$ . Then  $M$  is Cohen-Macaulay.*

*Proof.* We may assume that  $A$  is complete since  $r(M) = \mu_d(\mathfrak{m}, M) = \mu_d(\hat{\mathfrak{m}}, \hat{M}) = r(\hat{M})$ , and  $\text{depth } M = \text{depth } \hat{M}$ . Suppose that  $M$  is not Cohen-Macaulay and  $\text{depth } M = t < d$ . Let

$$(I^{\bullet}, \phi^{\bullet}) : 0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \dots \rightarrow I^t \rightarrow \dots$$

be a minimal injective resolution of  $M$ , where

$$I^i = \bigoplus_{\mathfrak{p} \in \text{Supp}(M)} E(A/\mathfrak{p})^{\mu_i(\mathfrak{p}, M)}.$$

By applying  $\text{Hom}_A(-, E(A/\mathfrak{m}))$  to  $H_{\mathfrak{m}}^0(I^\bullet) (= \varinjlim \text{Hom}_A(A/\mathfrak{m}^r, I^\bullet))$ , we have

$$(F_\bullet, f_\bullet) : \dots \rightarrow A^{\mu_i(\mathfrak{m})} \xrightarrow{f_{i-1}} A^{\mu_{i-1}(\mathfrak{m})} \rightarrow \dots \rightarrow A^{\mu_{t+1}(\mathfrak{m})} \rightarrow A^{\mu_t(\mathfrak{m})} \rightarrow 0.$$

We note that the  $i$ -th homology of  $F_\bullet$  is  $H_{\mathfrak{m}}^i(M)^\vee$ , where  $(-)^\vee$  denotes the Matlis dual. Since  $A$  is complete,  $A/\text{ann}(M) \cong S/J$  such that  $(S, \mathfrak{m}_S)$  is a Gorenstein local ring,  $J$  is an ideal of  $S$  and  $\dim S = \dim A/\text{ann}(M) = \dim M$ . By local duality, we have

$$H_{\mathfrak{m}}^i(M)^\vee \cong H_{\mathfrak{m}_S}^i({}_sM)^\vee \cong \text{Ext}_S^{d-i}({}_sM, S).$$

For a prime ideal  $\mathfrak{p} (\neq \mathfrak{m})$  in  $\text{Supp}(M)$ , since  $M_{\mathfrak{p}}$  is Cohen-Macaulay by Lemma 3.1 and  $S_{\mathfrak{p}_s}$  is Gorenstein, it follows that

$$\text{Ext}_S^j(M, S) \otimes S_{\mathfrak{p}_s} \cong \text{Ext}_{S_{\mathfrak{p}_s}}^j(M_{\mathfrak{p}}, S_{\mathfrak{p}_s}) = 0 \text{ if } j \neq \dim S_{\mathfrak{p}_s} - \dim M_{\mathfrak{p}}.$$

We note that  $\dim S_{\mathfrak{p}_s} = \dim M_{\mathfrak{p}}$  since  $M$  is quasi-unmixed and  $\dim S = \dim M$ . Thus it is obtained that  $H_i(F_\bullet \otimes A_{\mathfrak{p}}) = 0$  if  $i \neq d$  since

$$\begin{aligned} H_i(F_\bullet \otimes A_{\mathfrak{p}}) &\cong H_i(F_\bullet) \otimes A_{\mathfrak{p}} \cong H_{\mathfrak{m}}^i(M)^\vee \otimes A_{\mathfrak{p}} \\ &\cong \text{Ext}_S^{d-i}(M, S) \otimes S_{\mathfrak{p}_s} \cong \text{Ext}_{S_{\mathfrak{p}_s}}^{d-i}(M_{\mathfrak{p}}, S_{\mathfrak{p}_s}). \end{aligned}$$

Thus  $(F_d \rightarrow F_{d-1} \rightarrow \dots \rightarrow F_t \rightarrow 0) \otimes A_{\mathfrak{p}}$  is exact for each prime  $\mathfrak{p} (\neq \mathfrak{m})$ , and so split. Now we consider two cases, (i)  $\text{depth } A = t < d - 1$  and (ii)  $t = d - 1$ .

Case 1: Suppose  $\text{depth } M = t < d - 1$ , and let

$$G_\bullet : 0 \rightarrow G_{d-t} \xrightarrow{g_{d-t}} G_{d-t-1} \rightarrow \dots \rightarrow G_1 \xrightarrow{g_1} G_0,$$

where  $G_i = \text{Hom}_A(F_{d-i}, A)$ , and  $g_i = \text{Hom}_A(f_{d-i}, A)$ . For  $j = 1, \dots, d - t$ , let  $r_j = \sum_{i=j}^{d-t} \text{rank}(G_i)$  and  $I_j$  the ideal generated by the  $r_j$ -minors of  $g_j$ . Then it can be shown  $\text{rank}(F_d) - r_1 > 1$ , and  $r_1 \geq t$ . Thus we have

$$r(M) = \mu_d(\mathfrak{m}) = \text{rank}(F_d) > r_1 + 1 \geq t + 1 = \text{depth } M + 1 \geq r(M),$$

which is a contradiction.

Case 2: Suppose  $\text{depth } M = t = d - 1$ . In this case, we show  $r(M) \geq \text{depth } M + 2$ , which contradicts the assumption. Suppose to the contrary that  $r(M) < \text{depth } M + 2$ . We first note  $\ell(H_{\mathfrak{m}}^{d-1}(M)) < \infty$  since  $\text{Supp}(H_{d-1}(F_\bullet)) = \{\mathfrak{m}\}$ , and  $H_{d-1}(F_\bullet)$  is finitely generated. If  $I_1$  is an ideal generated by the maximal minors of  $f_{d-1}$ , then  $I_1$  is  $\mathfrak{m}$ -primary since  $(F_d \rightarrow F_{d-1} \rightarrow 0) \otimes A_{\mathfrak{p}}$  is exact for all primes  $\mathfrak{p} \neq \mathfrak{m}$ . By Theorem 13.10 in [10] and the assumption, we know that

$$\dim A = \text{ht } I_1 \leq \text{rank}(F_d) - \text{rank}(F_{d-1}) + 1 \leq r(M) \leq \text{depth } M + 1 = d.$$

This implies that  $\dim A = \dim M$ ,  $d = r(M) = \text{rank}(F_d)$ , and  $\text{rank}(F_{d-1}) = 1$ . Thus  $H_{d-1}(F_\bullet) = M/(x_1, \dots, x_d)M$ , where  $x_1, \dots, x_d \in \mathfrak{m}$ . Since  $\ell(H_{d-1}(F_\bullet)) < \infty$ ,  $x_1, \dots, x_d$  is a system of parameters of  $M$ . Note that  $(x_1, \dots, x_d) \subseteq \text{Ann}(H_{\mathfrak{m}}^{d-1}(M))$  since  $H_{\mathfrak{m}}^{d-1}(M) = (H_{d-1}(F_\bullet))^\vee = (M/(x_1, \dots, x_d)M)^\vee$ . Using Lemma 2.(c) in [11], we can show  $H_{\mathfrak{m}}^{d-1}(M) \cong H_1(K_\bullet(\underline{x}))$

since  $\ell(H_{\mathfrak{m}}^{d-1}(M)) < \infty$ . Since  $\text{depth } M = d - 1$ , we have  $H_i(K_{\bullet}(\underline{x})) = 0$  for all  $i > 1$ . Hence

$$e(\underline{x}; M) = \ell(M/\underline{x}M) - \ell(H_1(K_{\bullet}(\underline{x}))) = \ell(M/\underline{x}M) - \ell((M/\underline{x}M)^{\vee}) = 0,$$

which is a contradiction to the fact  $e(\underline{x}; M) \neq 0$  if  $\dim A = \dim M$ , and so  $r(M) \geq \text{depth } M + 2$ . This concludes that  $\text{depth } M = t = d = \dim M$ , i.e.,  $M$  is Cohen-Macaulay.  $\square$

The following corollary is stated in [1] (in fact, he assumes that  $\hat{M}$  is  $(S_n)$ ) without proof. Here we give its proof.

**Corollary 3.3.** ([1]) *Let  $M$  be a finitely generated  $A$ -module, and  $n \geq 3$  be an integer. If  $r(M) \leq n$  and  $\hat{M}$  is  $(S_{n-1})$  and equidimensional, then  $M$  is Cohen-Macaulay.*

*Proof.* We may assume that  $A$  is complete. Suppose that  $M$  is not Cohen-Macaulay. If  $M_{\mathfrak{p}}$  is Cohen-Macaulay for each prime  $\mathfrak{p} (\neq \mathfrak{m})$  in  $\text{Supp}(M)$ , then by Theorem 3.2,  $\text{depth } M < r(M) - 1$ , which implies  $\text{depth } M < n - 1$ . This contradicts that  $M$  is  $(S_{n-1})$ , and hence  $M$  is Cohen-Macaulay.

Suppose that  $M_{\mathfrak{p}}$  is not Cohen-Macaulay for some prime  $\mathfrak{p} (\neq \mathfrak{m})$  in  $\text{Supp}(M)$ , but  $M_{\mathfrak{q}}$  is Cohen-Macaulay for every prime  $\mathfrak{q} \subsetneq \mathfrak{p}$ . Since  $M$  is quasi-unmixed,  $M_{\mathfrak{p}}$  is also quasi-unmixed by Theorem 31.6 in [10]. We note that since  $A$  is complete,  $A$  is a homomorphic image of a Cohen-Macaulay local ring, and so is  $A_{\mathfrak{p}}$  (since  $R \rightarrow A \rightarrow 0$  implies  $R_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}} \rightarrow 0$ ). Thus  $(\widehat{M}_{\mathfrak{p}})_{\mathfrak{q}'}$  is Cohen-Macaulay for every prime  $\mathfrak{q}'$  which is not maximal in  $\text{Supp}(\widehat{M}_{\mathfrak{p}})$  by Lemma 3.1. Since  $M$  is  $(S_{n-1})$ , unmixed, and  $A/\text{ann}(M)$  is catenary, we have

$$r(M_{\mathfrak{p}}) \leq r(M) \leq n \leq \text{depth } M_{\mathfrak{p}} + 1.$$

Thus  $M_{\mathfrak{p}}$  is Cohen-Macaulay by Theorem 3.2, which is a contradiction. This completes the proof.  $\square$

We note that a finitely generated  $A$ -module  $M$  is quasi-unmixed if and only if  $A/\text{ann}(M)$  is quasi-unmixed. Also, it is easy to obtain that if  $M$  is quasi-unmixed, then  $M_{\mathfrak{p}}$  is also quasi-unmixed for any  $\mathfrak{p} \in \text{Supp}(M)$  using Theorem 31.6 [9]. Hence we have the following result for modules, which is analogous to Theorem 2.4. With the above facts, the proof can be completed by the same way as the proof of Theorem 2.4.

**Theorem 3.4.** *Let  $(A, \mathfrak{m})$  be a complete Noetherian local ring, and  $M$  a finitely generated unmixed  $A$ -module. Suppose  $r(M_{\mathfrak{p}}) \leq \text{depth } M_{\mathfrak{p}} + 1$  for each prime ideal  $\mathfrak{p}$  in  $\text{Supp}(M)$ . If  $\hat{M}_{\mathfrak{p}}$  is Cohen-Macaulay for every prime  $\mathfrak{p} \neq \hat{\mathfrak{m}}$  in  $\text{Supp}(\hat{M})$ , then  $M$  is Cohen-Macaulay.*

The following theorem shows that the conjecture 4.6 in [6] holds when  $\dim M \leq \text{depth } M + 1$ , which is analogous to Theorem 2.5. The proof is almost same as the proof of Theorem 2.5. But for a complete proof, we still need to check two facts: (1)  $\dim M_{\mathfrak{p}} \leq \text{depth } M_{\mathfrak{p}} + 1$  for all  $\mathfrak{p}$  in  $\text{Supp}(M)$  when

$\dim M \leq \text{depth } M + 1$ , and (2)  $r(M) \leq r(M_{\mathfrak{p}})$  for all  $\mathfrak{p}$  in  $\text{Supp}(M)$ . We can show the first part using the fact that  $\mu_i(\mathfrak{p}, M) \leq \mu_{i+t}(\mathfrak{m}, M)$  for each  $i$  and  $t = ht(\mathfrak{m}/\mathfrak{p})$ . It is easy to obtain the second part since  $\dim M_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim M$  for all  $\mathfrak{p}$  in  $\text{Supp}(M)$ , provided that  $M$  is unmixed, and  $A$  is complete.

Before we state the theorem, we remark that the condition ‘ $\dim M_{\mathfrak{p}} < n$ ’ (without equality) is sufficient: ‘ $\dim M_{\mathfrak{p}} \leq n$ ’ is used in the conjecture.

**Theorem 3.5.** *Let  $A$  be a complete Noetherian local ring, and  $M$  be a finitely generated unmixed  $A$ -module of type  $n$  with  $\dim M \leq \text{depth } M + 1$ . If that  $M_{\mathfrak{p}}$  is Cohen-Macaulay for all  $\mathfrak{p}$  in  $\text{Supp}(M)$  such that  $\dim M_{\mathfrak{p}} < n$ , then  $M$  is Cohen-Macaulay.*

We close this section by restating some of problems together, which are still open, in the form of conjectures.

**Conjecture 1.** ([6]) Let  $A$  be a complete unmixed Noetherian local ring of type  $n$ . If  $A_{\mathfrak{p}}$  is Cohen-Macaulay for all  $\mathfrak{p}$  in  $\text{Spec}(A)$  such that  $ht(\mathfrak{p}) < n$ , then  $A$  is Cohen-Macaulay.

**Conjecture 2.** ([6]) Let  $A$  be a complete Noetherian local ring, and  $M$  be a finitely generated unmixed  $A$ -module of type  $n$ . If that  $M_{\mathfrak{p}}$  is Cohen-Macaulay for all  $\mathfrak{p}$  in  $\text{Supp}(M)$  such that  $\dim M_{\mathfrak{p}} \leq n$ , then  $M$  is Cohen-Macaulay.

**Conjecture 3.** ([7]) Let  $(A, \mathfrak{m})$  be a complete unmixed Noetherian local ring. If  $r(A) \leq \text{depth } A + 1$ , where  $r(A)$  is the type of  $A$ , then  $A$  is Cohen-Macaulay.

**Conjecture 4.** Let  $(A, \mathfrak{m})$  be a complete Noetherian local ring, and  $M$  a finitely generated unmixed  $A$ -module. If  $r(M) \leq \text{depth } M + 1$ , where  $r(M)$  is the type of  $M$ , then  $M$  is Cohen-Macaulay.

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