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ON OPTIMALITY CONDITIONS FOR ABSTRACT CONVEX VECTOR OPTIMIZATION PROBLEMS

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Abstract. A sequential optimality condition characterizing the efficient solution without any constraint qualification for an abstract convex vector optimization problem is given in sequential forms using subdifferentials and ϵ-subdifferentials. Another sequential condition involving only the subdifferentials, but at nearby points to the efficient solution for constraints, is also derived. Moreover, we present a proposition with a sufficient condition for an efficient solution to be properly efficient, which are a generalization of the well-known Isermann result for a linear vector optimization problem. An example is given to illustrate the significance of our main results. Also, we give an example showing that the proper efficiency may not imply certain closeness assumption.

1. Introduction

Vector optimization problem consists of vector valued objective function and the constrained set. There are three kinds of solutions for the problem, that is, (properly, weakly) efficient solution. To get an optimality condition for an efficient solution of a vector optimization problem, we often formulate an corresponding scalar problem. However, it is so difficult that such scalar program satisfies a constraint qualification which we need to derive an optimality condition. Hence it is very important to investigate an optimality condition for an efficient solution of a vector optimization problem which holds without any constraint qualification.

Recently Jeyakumar and Zaffaroni ([16]) established necessary and sufficient dual conditions for weakly and properly efficient solutions of an abstract convex vector optimization problems without any constraint qualifications. The optimality conditions are given in asymptotic forms using epigraphs of conjugate functions and subdifferentials. Glover, Jayakumar and Rubinov ([6]) obtained necessary and sufficient dual conditions for efficient solutions of a

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finite-dimensional convex vector optimization problem without any constraint qualification. Very recently, Jeyakumar, Lee and Dinh ([13]) gave sequential optimality conditions characterizing the solution without any constraint qualification for an abstract scalar convex optimization problem.

On the other hand, Isermann ([10]) showed that every efficient solution of a linear vector optimization problem is properly efficient. Many authors ([2], [3], [7], [8], [17]) have tried to get a sufficient condition to extend the Isermann result to several kinds of vector optimization problems.

The aim of this paper is to present a sequential optimality condition characterizing the efficient solution without any constraint qualification for an abstract convex vector optimization problem, which is given in sequential forms using subdifferentials and ϵ-subdifferentials, and to get a sufficient condition to extend the Isermann result to the problem.

This paper is organized as follows. In Section 2, definitions and preliminary results are given for next sections. In Section 3, we obtain a sequential optimality condition characterizing the efficient solution without any constraint qualification for an abstract convex vector optimization problem, and derive another sequential condition involving only the subdifferentials, but at nearby points to the efficient solution for constraints. Moreover, sequential conditions characterizing a properly efficient solution and a weakly efficient solution without any constraint qualification are given. In Section 4, we present a preposition with a sufficient condition for an efficient solution to be properly efficient, which are a generalization of the Isermann result. In Section 5, an example is given to illustrate the significance of our main results. Also, we give an example showing that the proper efficiency may not imply certain closeness assumption.

2. Preliminaries

Now we give notations and preliminary results that will be used later in this chapter. Throughout the chapter, unless otherwise stated, X is a reflexive Banach space, Y and Z are Banach spaces, and the cones $K \subset Y$, $T \subset Z$ are closed and convex. The continuous dual space to X will be denoted by X^* and will be endowed with the weak* topology. When a sequence $\{x_n^*\}$ in X^* converges to $x^* \in X^*$ in the weak* topology, we denote it as $w^*\text{-}\lim_{n\to\infty} x_n^* = x^*$.

The positive polar of the cone $K \subset Y$ is the cone $K^+ = \{\theta \in Y^* \mid \theta(k) \geqq 0 \ \ \forall k \in K\}$, and the strict positive polar of the cone K is $K^{+i} = \{\theta \in Y^* \mid \theta(k) > 0 \ \ \forall k \in K \setminus \{0\}\}$. If $Y = \mathbb{R}^n$ and K is closed and pointed, then $K^{+i} \neq \emptyset$ and $K^{+i} = \operatorname{int} K^+$ ([11]), where $\operatorname{int} K^+$ is the interior of the cone K^+. The core of a subset D of Z is defined by $coreD = \{d \in D : (\forall x \in Z)(\exists \epsilon > 0)(\forall \lambda \in [-\epsilon, \epsilon]) d + \lambda x \in D\}$.

Definition 2.1. Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower-semicontinuous convex function.

(1) The conjugate function of g, $g^* : X^* \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$g^*(v) = \sup\{v(x) - g(x) \mid x \in \operatorname{dom} g\},$$

where the domain of g, $\operatorname{dom} g$, is given by

$$\operatorname{dom} g = \{x \in X \mid g(x) < +\infty\}.$$

(2) The epigraph of g, $\operatorname{epi} g$, is defined by

$$\operatorname{epi} g = \{(x, r) \in X \times \mathbb{R} \mid x \in \operatorname{dom} g,\ g(x) \leq r\}.$$

(3) The subdifferential of g at $a \in \operatorname{dom} g$ is defined as the non-empty weak* compact convex set

$$\partial g(a) = \{v \in X^* \mid g(x) - g(a) \geq v(x - a),\ \forall x \in \operatorname{dom} g\}$$

and for $\epsilon \geq 0$, the ϵ-subdifferential of g at $a \in \operatorname{dom} g$ is defined as the non-empty weak* closed convex set

$$\partial_\epsilon g(a) = \{v \in X^* \mid g(x) - g(a) \geq v(x - a) - \epsilon\ \forall x \in \operatorname{dom} g\}.$$

See Hiriart-Urruty and Lamarechal [9] for a detailed discussion on the ϵ-subdifferential. Note that

$$\bigcap_{\epsilon > 0} \partial_\epsilon g(a) = \partial g(a).$$

If g is sublinear (i.e., convex and positively homogeneous of degree one), then $\partial_\epsilon g(0) = \partial g(0)$ for all $\epsilon \geq 0$. If $\tilde{g}(x) = g(x) - k,\ x \in X,\ k \in \mathbb{R}$, then $\operatorname{epi} \tilde{g}^* = \operatorname{epi} g^* + (0, k)$. It is worth nothing that if g is sublinear, then $\operatorname{epi} g^* = \partial g(0) \times \mathbb{R}^+$.

Moreover, if g is sublinear and if $\widetilde{g}(x) = g(x) - k,\ x \in X,\ k \in \mathbb{R}$, then

$$\operatorname{epi} \widetilde{g}^* = \partial g(0) \times [k, \infty).$$

The following proposition describes the relationship between the epigraph of a conjugate function and the ϵ-subdifferential and plays a key role in proving the main results.

Proposition 2.1. ([12]) *If $g : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous convex function and if $a \in \operatorname{dom} g$, then*

$$\operatorname{epi} g^* = \bigcup_{\epsilon \geq 0} \left\{ \begin{pmatrix} v \\ v(a) + \epsilon - g(a) \end{pmatrix}^T \in X^* \times \mathbb{R} \ :\ v \in \partial_\epsilon g(a) \right\},$$

where superscript T denotes the transpose.

The mapping $g : X \to Z$ is T-convex if for every $u, v \in X$ and every $t \in [0, 1]$ it holds:

$$g(tu + (1 - t)v) - tg(u) - (1 - t)g(v) \in -T.$$

For a continuous T-convex mapping g it is easy to show that the set $\bigcup_{\lambda \in T^+} \operatorname{epi}(\lambda g)^*$ is a convex cone ([15]).

Throughout this paper, we will consider the following abstract convex vector optimization problem:

$$\textbf{(ACVP)} \qquad \text{Minimize} \quad f(x) \qquad \text{subject to} \quad g(x) \in -T,$$

where $f : X \to Y$ and $g : X \to Z$ are continuous K-convex and T-convex functions respectively.

Let $S = \{x \in X \mid g(x) \in -T\}$. Then $a \in S$ is said to be an efficient solution of **(ACVP)** if

$$[f(S) - f(a)] \cap (-K) = \{0\}.$$

We denote the set of all the efficient solutions of **(ACVP)** by Eff**(ACVP)**. The point $a \in S$ is called a properly efficient solution of **(ACVP)** if there exists a convex cone K' such that

$$K \setminus \{0\} \subset \operatorname{int} K' \quad \text{and} \quad [f(S) - f(a)] \cap (-K') = \{0\}.$$

We denote the set of all the properly efficient solutions of **(ACVP)** by

$$PrEff(\textbf{ACVP}).$$

The point $a \in S$ is said to be a weakly efficient solution of **(ACVP)** if

$$\operatorname{int} K \neq \emptyset \quad \text{and} \quad [f(S) - f(a)] \cap (-\operatorname{int} K) = \emptyset.$$

We denote the set of all the weakly efficient solutions of **(ACVP)** by

$$WEff(\textbf{ACVP}).$$

It is clear that $PrEff(\textbf{ACVP}) \subset Eff(\textbf{ACVP}) \subset WEff(\textbf{ACVP})$.

By separation theorem ([11]), we can obtain the following proposition:

Proposition 2.2. ([16]) *If in* **(ACVP)** *the mapping $f : X \to Y$ and $g : X \to Z$ are K-convex and T-convex, respectively, then*

(i) $a \in PrEff(\textbf{ACVP})$ *if and only if there exists $\theta \in K^{+i}$ such that*

$$(\theta f)(a) = \min\{\theta f(x) \mid x \in S\}.$$

(ii) $a \in WEff(\textbf{ACVP})$ *if and only if there exists $\theta \in K^{+} \setminus \{0\}$ such that*

$$(\theta f)(a) = \min\{\theta f(x) \mid x \in S\}.$$

Extending the result of Corley ([4]), we can obtain the following proposition:

Proposition 2.3. *Let $\theta \in K^{+i}$. If in* **(ACVP)** *the mapping $f : X \to Y$ and $g : X \to Z$ are K-convex and T-convex respectively, then $a \in Eff$***(ACVP)** *if and only if a is an optimal solution of the following scalar optimization problem:*

$$\textbf{(P)}_\theta \qquad \textit{Minimize} \quad (\theta f)(x) \qquad \textit{subject to} \quad g(x) \in -T, \quad f(x) - f(a) \in -K.$$

Proof. Suppose that $a \notin Eff(\mathbf{ACVP})$. Then there exists $x_1 \in S$ such that

$$f(x_1) - f(a) \in -K \setminus \{0\}.$$

Since $\theta \in K^{+i}$,

$$(\theta f)(x_1) - (\theta f)(a) < 0,$$

which implies that a is not an optimal solution of $(\mathbf{P})_\theta$.

Conversely, suppose that $a \in Eff(\mathbf{ACVP})$ and let x be any feasible solution of $(\mathbf{P})_\theta$. Then we have,

$$f(x) - f(a) \in -K.$$

Since $a \in Eff(\mathbf{ACVP})$, $f(x) = f(a)$. Hence a is an optimal solution of $(\mathbf{P})_\theta$. □

The following lemmas are needed to prove the main results.

Lemma 2.1. ([13], [14]) *Let $T \subseteq Z$ be a closed convex cone, let $u : X \to \mathbb{R}$ be a continuous linear mapping, and let $g : X \to Z$ be a continuous T-convex mapping. Suppose that the system $g(x) \in -T$ is consistent. Let $\alpha \in \mathbb{R}$. Then the following statements are equivalent:*

(i) $\{x \in X : g(x) \in -T\} \subseteq \{x \in X : u(x) \leq \alpha\}$

(ii) $\begin{pmatrix} u \\ \alpha \end{pmatrix} \in cl\left(\bigcup_{\lambda \in T^+} \mathrm{epi}(\lambda g)^*\right).$

We can obtain the following Lemma 2.2.

Lemma 2.2. *Let $f : X \to \mathbb{R}$ be a continuous convex function and $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then*

$$\mathrm{epi}(f+g)^* = \mathrm{epi}\, f^* + \mathrm{epi}\, g^*.$$

Lemma 2.3. ([1], [18]) *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then for any real number $\epsilon > 0$ and any $x^* \in \partial_\epsilon f(\bar{x})$ there exist $x_\epsilon \in X$, $x_\epsilon^* \in \partial f(x_\epsilon)$ such that*

$$\|x_\epsilon - \bar{x}\| \leqq \sqrt{\epsilon},$$
$$\|x_\epsilon^* - x^*\| \leqq \sqrt{\epsilon} \quad \text{and}$$
$$|f(x_\epsilon) - x_\epsilon^*(x_\epsilon - \bar{x}) - f(\bar{x})| \leqq 2\epsilon.$$

3. Optimality conditions

Now we give sequential optimality conditions for (properly, weakly) efficient solutions of the abstract convex vector optimization problem $(\mathbf{ACVP})$.

Theorem 3.1. *Let $\theta \in K^{+i}$ and $a \in S$. Then the following are equivalent:*

(i) $a \in Eff(\mathbf{ACVP})$.

(ii) *there exists $u \in \partial(\theta f)(a)$ such that*

$$-\begin{pmatrix} u \\ u(a) \end{pmatrix}^T \in cl\left(\bigcup_{\lambda \in T^+} \mathrm{epi}(\lambda g)^* + \bigcup_{\mu \in K^+}\left[\mathrm{epi}(\mu f)^* + (0, (\mu f)(a))\right]\right).$$

(iii) *there exist* $u \in \partial(\theta f)(a)$, $\lambda_n \in T^+$, $\delta_n \geqq 0$, $v_n \in \partial_{\delta_n}(\lambda_n g)(a)$, $\mu_n \in K^+$, $\epsilon_n \geqq 0$, $w_n \in \partial_{\epsilon_n}(\mu_n f)(a)$ *such that*

$$u + \omega^*\text{-}\lim_{n\to\infty}(v_n + w_n) = 0, \ \lim_{n\to\infty}\delta_n = \lim_{n\to\infty}\epsilon_n = 0 \quad \textit{and}$$
$$\lim_{n\to\infty}(\lambda_n g)(a) = 0.$$

Proof. $a \in Eff(\mathbf{ACVP})$

$\Longleftrightarrow$ (by Proposition 2.2) a is an optimal solution of the problem $(\mathbf{P})$:

$$(\mathbf{P}) \qquad \text{Minimize} \quad (\theta f)(x)$$
$$\text{subject to} \quad g(x) \in -T$$
$$f(x) - f(a) \in -K.$$

$\Longleftrightarrow$ there exists $u \in \partial(\theta f)(a)$ such that $\forall x \in Z := \{x \in X : g(x) \in -T, f(x) - f(a) \in -K\}$, $u(x) \geqq u(a)$.

$\Longleftrightarrow$ there exists $u \in \partial(\theta f)(a)$ such that $\forall x$ with $\begin{pmatrix} g(x) \\ f(x) - f(a) \end{pmatrix}^T \in -(T\times K)$, $-u(x) \leqq -u(a)$.

$\Longleftrightarrow$ (by Lemma 2.1) there exists $u \in \partial(\theta f)(a)$ such that

$$-\begin{pmatrix} u \\ u(a) \end{pmatrix}^T \in cl \bigcup_{(\lambda,\mu)\in T^+\times K^+} \text{epi}(\lambda g + \mu f - \mu f(a))^*.$$

$\Longleftrightarrow$ (by Lemma 2.2) there exists $u \in \partial(\theta f)(a)$ such that $-\begin{pmatrix} u \\ u(a) \end{pmatrix}^T$
$$\in cl\left(\bigcup_{\lambda\in T^+}\text{epi}(\lambda g)^* + \bigcup_{\mu\in K^+}\left[\text{epi}(\mu f)^* + \begin{pmatrix} 0 \\ (\mu f)(a) \end{pmatrix}^T\right]\right).$$

$\Longleftrightarrow$ (by Proposition 2.1) there exists $u \in \partial(\theta f)(a)$ such that $-\begin{pmatrix} u \\ u(a) \end{pmatrix}^T$
$$\in cl\left(\bigcup_{\lambda\in T^+}\bigcup_{\delta\geqq 0}\left\{\begin{pmatrix} v \\ v(a) + \delta - (\lambda g)(a) \end{pmatrix}^T \mid v \in \partial_\delta(\lambda g)(a)\right\}\right.$$
$$+\bigcup_{\mu\in K^+}\bigcup_{\epsilon\geqq 0}\left[\left\{\begin{pmatrix} w \\ w(a) + \epsilon - (\mu f)(a) \end{pmatrix}^T : w \in \partial_\epsilon(\mu f)(a)\right\}\right.$$
$$\left.\left.+\begin{pmatrix} 0 \\ (\mu f)(a) \end{pmatrix}^T\right]\right).$$

$\Longleftrightarrow$ there exist $u \in \partial(\theta f)(a)$, $\lambda_n \in T^+$, $\delta_n \geqq 0$, $v_n \in \partial_{\delta_n}(\lambda_n g)(a)$, $\mu_n \in K^+$, $\epsilon_n \geqq 0$, $w_n \in \partial_{\epsilon_n}(\mu_n f)(a)$ such that $-\begin{pmatrix} u \\ u(a) \end{pmatrix}^T =$

$$\lim_{n\to\infty}\left(\begin{pmatrix} v_n \\ v_n(a)+\delta_n-(\lambda_n g)(a)\end{pmatrix}^T + \begin{pmatrix} w_n \\ w_n(a)+\epsilon_n-(\mu_n f)(a)\end{pmatrix}^T + \begin{pmatrix} 0 \\ (\mu_n f)(a)\end{pmatrix}^T\right)$$

$\Longleftrightarrow$ (iii).

□

Theorem 3.2. *Let $\theta \in K^{+i}$ and $a \in A$. Then $a \in Eff$ **(ACVP)** if and only if there exist $u \in \partial(\theta f)(a)$, $\lambda_n \in T^+$, $\mu_n \in K^+$, $x_n \in X$, $s_n \in \partial(\lambda_n g+\mu_n f)(x_n)$ such that*

$$u + w^* - \lim_{n\to\infty} s_n = 0, \ \lim_{n\to\infty}[(\lambda_n g+\mu_n f)(x_n) - (\mu_n f)(a)] = 0 \quad \textit{and}$$
$$\lim_{n\to\infty} \|x_n - a\| = 0.$$

Proof. Let $a \in Eff$**(ACVP)**. Then by Theorem 3.1, there exist $u \in \partial(\theta f)(a)$, $\lambda_n \in T^+$, $\delta_n \geq 0$, $\epsilon_n \geq 0$, $\mu_n \in K^+$, $v_n \in \partial_{\delta_n}(\lambda_n g)(a)$, $w_n \in \partial_{\epsilon_n}(\mu_n f)(a)$ such that

$$u + w^*\text{-}\lim_{n\to\infty}(v_n + w_n) = 0$$
$$\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}\epsilon_n = 0$$
$$\lim_{n\to\infty}(\lambda_n g)(a) = 0.$$

Notice that it follows the definition of $\partial_\epsilon f$ that

$$\partial_{\delta_n}(\lambda_n g)(a) + \partial_{\epsilon_n}(\mu_n f)(a) \subset \partial_{\alpha_n}(\lambda_n g + \mu_n f)(a),$$

where $\alpha_n = \delta_n + \epsilon_n$. Without loss of generality, we may assume that $\alpha_n > 0$. Letting $t_n = v_n + w_n$, $t_n \in \partial_{\alpha_n}(\lambda_n g+\mu_n f)(a)$ and hence by Lemma 2.3, there exists $x_n \in X$, $s_n \in \partial(\lambda_n g + \mu_n f)(x_n)$ such that

$$\|x_n - a\| \leqq \sqrt{\alpha_n}$$
$$\|s_n - t_n\| \leqq \sqrt{\alpha_n} \quad \text{and}$$
$$|(\lambda_n g + \mu_n f)(x_n) - s_n(x_n - a) - (\lambda_n g + \mu_n f)(a)| \leqq 2\alpha_n.$$

Since $\alpha_n \to 0$ and $s_n(x_n - a) \to 0$, we have

$$\lim_{n\to\infty}\|x_n - a\| = 0$$
$$u + w^*\text{-}\lim_{n\to\infty} s_n = 0 \quad \text{and}$$
$$\lim_{n\to\infty}\Big[(\lambda_n g + \mu_n f)(x_n) - (\mu_n f)(a)\Big] = 0.$$

Conversely, suppose that there exist $u \in \partial(\theta f)(a)$, $\lambda_n \in T^+$, $\mu_n \in K^+$, $x_n \in X$, $s_n \in \partial(\lambda_n g + \mu_n f)(x_n)$ such that

$$\begin{aligned} & u + w^*\text{-}\lim_{n\to\infty} s_n = 0, \\ & \lim_{n\to\infty}\Big[(\lambda_n g + \mu_n f)(x_n) - (\mu_n f)(a)\Big] = 0 \quad \text{and} \\ & \lim_{n\to\infty} \|x_n - a\| = 0. \end{aligned}$$

Since $s_n \in \partial(\lambda_n g + \mu_n f)(x_n)$ we have

$$(\lambda_n g + \mu_n f)^*(s_n) = s_n(x_n) - (\lambda_n g + \mu_n f)(x_n).$$

So,

$$\begin{aligned} & (s_n, s_n(x_n) - (\lambda_n g + \mu_n f)(x_n) + (\mu_n f)(a)) \\ & \in \operatorname{epi}(\lambda_n g + \mu_n f)^* + (0, (\mu_n f)(a)). \end{aligned}$$

Moreover, $w^*\text{-}\lim_{n\to\infty} s_n = -u$, and

$$\begin{aligned} & \lim_{n\to\infty}\Big[s_n(x_n) - (\lambda_n g + \mu_n f)(x_n) + (\mu_n f)(a)\Big] \\ = & \lim_{n\to\infty} s_n(x_n) = -u(a). \end{aligned}$$

Thus

$$-\begin{pmatrix} u \\ u(a) \end{pmatrix}^T \in cl \bigcup_{\lambda\in T^+,\mu\in K^+} \left(\operatorname{epi}(\lambda g + \mu f)^* + \begin{pmatrix} 0 \\ (\mu f)(a) \end{pmatrix}^T\right).$$

Thus by Lemma 2.2,

$$-\begin{pmatrix} u \\ u(a) \end{pmatrix}^T \in cl\left(\bigcup_{\lambda\in T^+} \operatorname{epi}(\lambda g)^* + \bigcup_{\mu\in K^+}\left[\operatorname{epi}(\mu f)^* + \begin{pmatrix} 0 \\ (\mu f)(a) \end{pmatrix}^T\right]\right).$$

So, by Theorem 3.1, $a \in Eff$**(ACVP)**. □

By using Propositions 2.1 and 2.2, and following methods of proofs in Theorems 3.1 and 3.2, we can obtain the following theorems.

Theorem 3.3. *The following are equivalent:*

(i) $a \in PrEff$**(ACVP)**.

(ii) *there exist* $\theta \in K^{+i}$, $u \in \partial(\theta f)(a)$, $\epsilon_n \geqq 0$, $\lambda_n \in T^+$, $v_n \in \partial_{\epsilon_n}(\lambda_n g)(a)$ *such that*

$$\begin{aligned} & u + w^*\text{-}\lim_{n\to\infty} v_n = 0, \\ & \lim_{n\to\infty} (\lambda_n g)(a) = 0 \quad \textit{and} \\ & \lim_{n\to\infty} \epsilon_n = 0. \end{aligned}$$

(iii) *there exist* $\theta \in K^{+i}$, $u \in \partial(\theta f)(a)$, $\epsilon_n \geqq 0$, $\lambda_n \in T^+$, $x_n \in X$, $v_n \in \partial(\lambda_n g)(x_n)$ *such that*

$$u + w^*\text{-}\lim_{n\to\infty} v_n = 0,$$
$$\lim_{n\to\infty} \lambda_n g(x_n) = 0 \quad \textit{and}$$
$$\lim_{n\to\infty} x_n = a.$$

Theorem 3.4. *The following are equivalent:*

(i) $a \in WEff$**(ACVP)**.

(ii) *there exist* $\theta \in K^+ \setminus \{0\}$, $u \in \partial(\theta f)(a)$, $\epsilon_n \geqq 0$, $\lambda_n \in T^+$, $v_n \in \partial_{\epsilon_n}(\lambda_n g)(a)$ *such that*

$$u + w^*\text{-}\lim_{n\to\infty} v_n = 0,$$
$$\lim_{n\to\infty} (\lambda_n g)(a) = 0 \quad \textit{and}$$
$$\lim_{n\to\infty} \epsilon_n = 0.$$

(iii) *there exist* $\theta \in K^+ \setminus \{0\}$, $u \in \partial(\theta f)(a)$, $\lambda_n \in T^+$, $x_n \in X$, $v_n \in \partial(\lambda_n g)(x_n)$ *such that*

$$u + w^*\text{-}\lim_{n\to\infty} v_n = 0,$$
$$\lim_{n\to\infty} \lambda_n g(x_n) = 0 \quad \textit{and}$$
$$\lim_{n\to\infty} x_n = a.$$

Now we consider the closed cone constraint qualification which requires that the convex cone $\bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda g)^*$ is weak* closed. This constraint qualification holds under the Robinson Regularity Condition, that is, $0 \in core(g(X)+S)$ (see [16]) or the generalized Slater condition that $\operatorname{int} S$ is nonempty and $\exists x_0 \in X$ such that $-g(x_0) \in \operatorname{int} S$ (see [15]).

Following the proof in [13], we can obtain the following Kuhn-Tucker theorems (the Lagrange multiplier theorems) for **(ACVP)** under the closed cone constraint qualification.

Theorem 3.5. *Let* $a \in S$ *and assume that the closed cone constraint qualification holds. Then the following are equivalent:*

(i) $a \in PrEff$**(ACVP)**.

(ii) *there exist* $\theta \in K^{+i}$, $\lambda \in T^+$ *such that*

$$0 \in \partial(\theta f)(a) + \partial(\lambda g)(a) \quad \textit{and} \quad (\lambda g)(a) = 0.$$

Theorem 3.6. *Let* $a \in S$ *and assume that the closed cone constraint qualification holds. Then the following are equivalent:*

(i) $a \in WEff$**(ACVP)**.

(ii) *there exist* $\theta \in K^+$, $\lambda \in T^+$ *such that*

$$0 \in \partial(\theta f)(a) + \partial(\lambda g)(a) \quad \textit{and}$$
$$(\lambda g)(a) = 0.$$

4. Generalization of Isermann's result

The following proposition, which is a generalization of the Isermann's result ([10], [5], [19]), gives a sufficient condition that an efficient solution can be properly efficient.

Proposition 4.1. *Let* $a \in S$ *and assume that*

$$\bigcup_{\lambda \in T^+} \operatorname{epi}(\lambda g)^* + \bigcup_{\mu \in K^+} \left[\operatorname{epi}(\mu f)^* + (0, (\mu f)(a))\right]$$

is closed in $X^* \times \mathbb{R}$. *Then if* $a \in Eff(\mathbf{ACVP})$, $a \in PrEff(\mathbf{ACVP})$.

Proof. Let $a \in Eff(\mathbf{ACVP})$ and $\theta \in K^{+i}$. Then by Lemma 2.2 and Theorem 3.1, there exists $u \in \partial(\theta f)(a)$ such that

$$-\begin{pmatrix} u \\ u(a) \end{pmatrix}^T \in \bigcup_{\lambda \in T^+,\ \mu \in K^+} \left(\operatorname{epi}(\lambda g + \mu f)^* + \begin{pmatrix} 0 \\ (\mu f)(a) \end{pmatrix}^T\right).$$

Thus there exist $\lambda \in T^+$ and $\mu \in K^+$ such that

$$\begin{pmatrix} -u \\ -u(a) - \mu f(a) \end{pmatrix}^T \in \operatorname{epi}(\lambda g + \mu f)^*$$

and hence $(\lambda g + \mu f)^*(-u) \leqq -u(a) - \mu f(a)$. Thus for any $x \in X$, $-u(x) - (\lambda g + \mu f)(x) \leqq -u(a) - \mu f(a)$. So, for any $x \in S$, $0 \leqq u(x) - u(a) + (\mu f)(x) - (\mu f)(a)$. Since $u \in \partial(\theta f)(a), (\theta f)(x) - (\theta f)(a) \geqq u(x-a)$ and we have, for any $x \in S$,

$$\begin{aligned} 0 &\leqq (\theta f)(x) - (\theta f)(a) + (\mu f)(x) - (\mu f)(a) \\ &= (\theta + \mu) f(x) - (\theta + \mu) f(a). \end{aligned}$$

Hence for any $x \in S$, $(\theta + \mu) f(a) \leqq (\theta + \mu) f(x)$.
Since $\theta \in K^{+i}$ and $\mu \in K^+$, $\theta + \mu \in K^{+i}$. So, by Proposition 2.2, $a \in PrEff(\mathbf{ACVP})$. □

The following proposition is the well-known Isermann theorem ([10]) for a linear vector optimization problem.

Proposition 4.2. *Let* $X = \mathbb{R}^n$, $Y = \mathbb{R}^p$, $Z = \mathbb{R}^m$, $K = \mathbb{R}^p_+$, $T = \mathbb{R}^m_+$ $f(x) = (c_1 x, \ldots, c_p x)$ *and* $g(x) = (a_1 x - b_1, \ldots, a_m x - b_m)$, *where* $c_i \in \mathbb{R}^n$, $i = 1, \ldots, p$, $a_j \in \mathbb{R}^m$, $j = 1, \ldots, m$ *and* $b_j \in \mathbb{R}$, $j = 1, \ldots, m$. *Then*

$$PrEff(\mathbf{ACVP}) = Eff(\mathbf{ACVP}).$$

Proof. Let $\bar{a} \in Eff(\mathbf{ACVP})$. Then we have,

$$\begin{aligned}&\bigcup_{\lambda \in T^+} \mathrm{epi}(\lambda g)^* + \bigcup_{\mu \in K^+} \left[\mathrm{epi}(\mu f)^* + (0, (\mu f)(\bar{a}))\right] \\ = \quad & \mathrm{cone}\ co\left(\left\{\begin{pmatrix} c_i \\ c_i\bar{a}\end{pmatrix}^T : i = 1, \ldots, p\right\} \cup \left\{\begin{pmatrix} a_j \\ b_j\end{pmatrix}^T : j = 1, \ldots, m\right\} \right. \\ & \left. \cup \left\{\begin{pmatrix} 0 \\ 1\end{pmatrix}^T\right\}\right).\end{aligned}$$

Thus the above set is closed, and hence it follows from Theorem 4.1, $\bar{a} \in PrEff(\mathbf{ACVP})$. Since $PrEff(\mathbf{ACVP}) \subset Eff(\mathbf{ACVP})$, $PrEff(\mathbf{ACVP}) = Eff(\mathbf{ACVP})$. □

5. Examples

Now we give an example to illustrate Theorems 3.1 and 3.2, and Theorem 4.1.

Example 5.1. Consider the following convex vector optimization problem:

$$\begin{array}{lll}(\mathbf{ACVP}) & \text{Minimize} & f(x) := (x, x^2) \\ & \text{subject to} & g(x) := x \leqq 0.\end{array}$$

Let $T = \mathbb{R}^+$ and $K = \mathbb{R}^+_2$. Then $0 \in Eff(\mathbf{ACVP})$, but $0 \notin PrEff(\mathbf{ACVP})$.

(1) We will show that the condition (ii) in Theorem 3.1 holds for $(\mathbf{ACVP})$ at $a = 0$. We can check that

$$\begin{aligned}&\bigcup_{\lambda \in T^+} \mathrm{epi}(\lambda g)^* + \bigcup_{\mu \in K^+} \left[\mathrm{epi}(\mu f)^* + (0, (\mu f)(0))\right] \\ = \quad & \{(x, y) \in \mathbb{R}^2 : x < 0,\ y > 0\} \cup \{(x, y) \in \mathbb{R}^2 : x \geqq 0,\ y \geqq 0\}.\end{aligned}$$

Take $\theta = (1, 1)$. Then $u := \nabla(\theta f)(0) = 1$ and

$$-\begin{pmatrix} 1 \\ 0\end{pmatrix} \notin \bigcup_{\lambda \in T^+} \mathrm{epi}(\lambda g)^* + \bigcup_{\mu \in K^+} \left[\mathrm{epi}(\mu f)^* + (0, (\mu f)(0))\right].$$

But

$$-\begin{pmatrix} 1 \\ 0\end{pmatrix} \in cl\left\{\bigcup_{\lambda \in T^+} \mathrm{epi}(\lambda g)^* + \bigcup_{\mu \in K^+} \left[\mathrm{epi}(\mu f)^* + (0, (\mu f)(0))\right]\right\}.$$

Thus the condition (ii) in Theorem 3.1 holds at $a = 0$.

(2) We will show that the condition (iii) in Theorem 3.1. holds for $(\mathbf{ACVP})$ at $a = 0$. Take $\theta = (1, 1)$, $\lambda_n = \frac{1}{n}$, $\delta_n = \frac{1}{n}$, $\mu_n = (0, \frac{1}{2}n)$, $\epsilon_n = \frac{1}{2n}$. Then

$\partial_{\delta_n}(\lambda_n g)(0) = \{\frac{1}{n}\}$, and $\partial_{\epsilon_n}(\mu_n f)(0) = [-1,1]$. Let $v_n = \frac{1}{n}$ and $w_n = -1$. Then $v_n \in \partial_{\delta_n}(\lambda_n g)(0)$ and $w_n \in \partial_{\epsilon_n}(\mu_n f)(0)$. Moreover, we have

$$\nabla(\theta f)(0) + \lim_{n\to\infty}(v_n + w_n) = 0,$$
$$\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}\epsilon_n = 0 \quad \text{and}$$
$$\lim_{n\to\infty}(\lambda_n g)(0) = 0.$$

Thus the condition (iii) holds at $a = 0$.

(3) Now we will show that the conclusion of Theorem 3.2 holds for **(ACVP)** at $a = 0$. Take $\theta = (1,1)$, $x_n = -\frac{1}{n}$, $\mu_n = (0, \frac{1}{2}n)$, $\lambda_n = 0$. Then $\lim_{n\to\infty} x_n = 0$,

$$\begin{aligned}\nabla(\theta f)(0) + \lim_{n\to\infty}\nabla(\mu_n f + \lambda_n g)(x_n) &= 1 + \lim_{n\to\infty}\Big[2\times\frac{1}{2}n\times(-\frac{1}{n})\Big]\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\Big[(\mu_n f + \lambda_n g)(x_n) - (\mu_n f)(0)\Big] &= \lim_{n\to\infty}\Big[\frac{1}{2}n\times(-\frac{1}{n})^2\Big]\\ &= 0.\end{aligned}$$

Hence the conclusion of Theorem 3.2 holds.

(4) We will show that if $a > 0$, $-a \in PrEff$**(ACVP)**. First we will prove that if $a > 0$, then there exists $u \in \partial(\theta f)(-a)$ such that

$$-\begin{pmatrix} u \\ u(-a)\end{pmatrix}^T \in \left(\bigcup_{\lambda\in T^+}\operatorname{epi}(\lambda g)^* + \bigcup_{\mu\in K^+}\Big[\operatorname{epi}(\mu f)^* + (0,(\mu f)(-a))\Big]\right)$$

and $\bigcup_{\lambda\in T^+}\operatorname{epi}(\lambda g)^* + \bigcup_{\mu\in K^+}\Big[\operatorname{epi}(\mu f)^* + (0,(\mu f)(-a))\Big]$ is closed.
Indeed, let $a > 0$,

$$\bigcup_{\mu\in K^+}\Big[\operatorname{epi}(\mu f)^* + (0,(\mu f)(-a))\Big] = \bigcup_{\mu\in K^+}\Big[\operatorname{epi}(\mu f)^* + (0, -\mu_1 a + \mu_2 a^2)\Big].$$

(i) The case that $\mu_1 \geqq 0,\ \mu_2 = 0$:
$\operatorname{epi}(\mu f)^* + (0, -\mu_1 a + \mu_2 a^2) = \{\mu_1\}\times[0,\infty) + (0, -\mu_1 a) = \{\mu_1\}\times[-\mu_1 a, \infty)$
(ii) The case that $\mu_1 \geqq 0,\ \mu_2 > 0, -\mu_1 a + \mu_2 a^2 = 0$:
$\operatorname{epi}(\mu f)^* + (0, -\mu_1 a + \mu_2 a^2) = \operatorname{epi}(\mu f)^* + (0,0)$ and $(\mu f)^*(v) = \frac{(\mu_1 - v)^2}{4\cdot\frac{\mu_1}{a}}$. Notice

that $\mu_2 = \frac{\mu_1}{a}$ and $\frac{(v-\mu_1)^2}{4\cdot\frac{\mu_1}{a}} \geqq -av\ \forall v \in \mathbb{R}$. Hence

$$\bigcup_{\mu_1 \geqq 0, \mu_2 = 0} \left[\operatorname{epi}(\mu f)^* + (0, -\mu_1 a + \mu_2 a^2)\right]$$
$$\cup \bigcup_{\mu_2 > 0, -\mu_1 a + \mu_2 a^2 = 0} \left[\operatorname{epi}(\mu f)^* + (0,0)\right]$$
$$= \{(v, \alpha) \in \mathbb{R}^2 : \alpha \geqq -av\}.$$

(iii) The case that $\mu_1 \geqq 0,\ \mu_2 > 0, -\mu_1 a + \mu_2 a^2 \neq 0$:

$(\mu f^*)(v) = \frac{(\mu_1 - v)^2}{4\mu_2}$ and $\frac{(v-\mu_1)^2}{4\mu_2} - \mu_1 a + \mu_2 a^2 + av = (v - \mu_1 + 2\mu_2 a)^2 \geqq 0\ \forall v \in \mathbb{R}$. Hence $\bigcup_{\mu \in K^+} \left[\operatorname{epi}(\mu f)^* + (0, (\mu f)(-a))\right] = \{(v, \alpha) \in \mathbb{R}^2 : \alpha \geqq -av\}$.

So,

$$\bigcup_{\lambda \in T^+} \operatorname{epi}(\lambda g)^* + \bigcup_{\mu \in K^+} \left[\operatorname{epi}(\mu f)^* + (0, (\mu f)(-a))\right]$$
$$= \{(x,y) \in \mathbb{R}^2 : x \geqq 0,\ y \geqq 0\} + \{(x,y) \in \mathbb{R}^2 : y \geq -ax\}$$
$$= \{(x,y) \in \mathbb{R}^2 : y \geqq -ax\}$$

and hence this set is closed. Since $(\theta f)(x) = x + x^2$, $\partial(\theta f)(-a) = \{1 - 2a\}$. Let $u = 1 - 2a$. Then

$$-\begin{pmatrix} u \\ u(-a) \end{pmatrix}^T \in \left(\bigcup_{\lambda \in T^+} \operatorname{epi}(\lambda g)^* + \bigcup_{\mu \in K^+} \left[\operatorname{epi}(\mu f)^* + (0, (\mu f)(-a))\right] \right).$$

Thus by Theorem 3.1, $-a \in Eff$**(ACVP)**. Moreover, by Proposition 4.1, $-a \in PrEff$**(ACVP)**.

We give an example to show that $a \in PrEff$**(ACVP)** may not imply the closeness assumption of Proposition 4.1.

Example 5.2. Let $f(x) = (-x, -x)$ and $g(x) = \begin{cases} 0 & \text{if } x \leqq 0 \\ x^2 & \text{if } x > 0. \end{cases}$ Consider the following vector convex optimization problem:

(ACVP)	Minimize	$f(x)$
	subject to	$g(x) \leqq 0.$

Then $0 \in PrEff$**(ACVP)**.

$$\bigcup_{\mu \in \mathbb{R}^2_+} [\operatorname{epi}(\mu f)^* + (0, (\mu f)(0))] = \{(x,y) \in \mathbb{R}^2 : x \leqq 0,\ y \geqq 0\}$$

and

$$\bigcup_{\lambda \geqq 0} \operatorname{epi}(\lambda g)^* + \bigcup_{\mu \in \mathbb{R}^2_+} \Big[\operatorname{epi}(\mu f)^* + (0, (\mu f)(0))\Big]$$
$$= \{(x, y) : x \leqq 0,\ y \geqq 0\} \bigcup \{(x, y) : x > 0,\ y > 0\}.$$

Therefore $\bigcup_{\lambda \geqq 0} \operatorname{epi}(\lambda g)^* + \bigcup_{\mu \in \mathbb{R}^2_+} \Big[\operatorname{epi}(\mu f)^* + (0, (\mu f)(0))\Big]$ is not closed.

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