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COMPLEX MOMENT MATRICES VIA HALMOS-BRAM AND EMBRY CONDITIONS

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ABSTRACT. By considering a bridge between Bram-Halmos and Embry characterizations for the subnormality of cyclic operators, we extend the Curto-Fialkow and Embry truncated complex moment problem, and solve the problem finding the finitely atomic representing measure μ such that $\gamma_{ij} = \int \bar{z}^i z^j d\mu$, $(0 \le i+j \le 2n, |i-j| \le n+s, 0 \le s \le n)$; the cases of $s = n$ and $s = 0$ are induced by Bram-Halmos and Embry characterizations, respectively. The former is the Curto-Fialkow truncated complex moment problem and the latter is the Embry truncated complex moment problem.

1. Introduction and preliminaries

In [5, Proposition 2.8], it was shown that Bram-Halmos characterization for the subnormality of a cyclic operator on a complex Hilbert space induces a moment matrix $M(n)$ which was considered in [1] and [2]. As a parallel study, in [5] they discussed a matrix $E(n)$ corresponding to the Embry characterization of such an operator. The moment matrices $M(n)$ and $E(n)$ are contained in our new classes of moment matrices $M(n, s)$, $s = 0, 1, \ldots, n$ (which will be defined below). Let

$$\Gamma_{n,s} = \{\gamma_{ij} \in \mathbb{C} : 0 \le i + j \le 2n,\ |i - j| \le n + s,\ 0 \le s \le n\},$$

where $\gamma_{00} > 0$, $\gamma_{ji} = \overline{\gamma_{ij}}$. Notice that the data in $\Gamma_{n,s}$ lie in the gray pentagon in Figure 1.

The *truncated complex moment problem* for $\Gamma_{n,s}$ entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$(1.1) \qquad \gamma_{ij} = \int \bar{z}^i z^j d\mu, \qquad (0 \le i + j \le 2n,\ |i - j| \le n + s).$$

And μ is said to be a *representing measure* for $\Gamma_{n,s}$. In particular, $\Gamma_{n,n}$ induces the Curto-Fialkow moment matrix $M(n)$ ([1], [2]); $\Gamma_{n,0}$ induces the Embry

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moment matrix $E(n)$ ([5]). If $n = 2k$, let

$$\eta(n,s) = \begin{cases} (k+1)^2 + 2mk - m(m-1) & \text{if } s = 2m, \\ (k+1)^2 + (2m-1)k - (m-1)^2 & \text{if } s = 2m-1, \end{cases}$$

and if $n = 2k+1$, let

$$\eta(n,s) = \begin{cases} (k+1)(k+3) + 2mk - m(m-1) & \text{if } s = 2m+1, \\ (k+1)(k+3) + (2m-1)k - (m-1)^2 & \text{if } s = 2m. \end{cases}$$

Let $\mathcal{M}_k(\mathbb{C})$ be the set of all $k \times k$ matrices. For $A \in \mathcal{M}_{\eta(n,s)}(\mathbb{C})$, we introduce the order on the rows and columns of A. For example, if $n = 4$ and $s = 3$, i.e., $\eta(4,3) = 14$, then the order is as follows:

$$\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, Z^3, \bar{Z}Z^2, \bar{Z}^2Z, \bar{Z}^3, Z^4, \bar{Z}Z^3, \bar{Z}^2Z^2, \bar{Z}^3Z.$$

Let

$$\Lambda_{(n,s)} = \{(i,j) : 0 \le i+j \le n, \max\{i-s, 0\} \le j,\ 0 \le s \le n\}$$

and let $\mathcal{P}_{n,s}$ be the set of polynomials $p(z,\bar{z}) = \sum_{(i,j)\in\Lambda_{(n,s)}} a_{ij}\bar{z}^i z^j$, where $a_{ij} \in \mathbb{C}$. Then it is clear that $\mathcal{P}_{n,s}$ is a subspace of $\mathbf{P}_n[z,\bar{z}]$, the vector space of all complex polynomials in $z, \bar{z}$ of total degree $\le n$. Let $\{e_{ij}\}_{(i,j)\in\Lambda_{(n,s)}}$ be a basis for $\mathbb{C}^{\eta(n,s)}$ as follows: $e_{ij} \equiv e_{ij}^{(\eta(n,s))}$ is the vector with 1 in the $\bar{Z}^iZ^j$ entry and 0 in all other positions. For $p(z,\bar{z}) = \sum_{(i,j)\in\Lambda_{(n,s)}} a_{ij}\bar{z}^i z^j$, let $\widehat{p} := \sum_{(i,j)\in\Lambda_{(n,s)}} a_{ij}e_{ij}$. We define a sesquilinear form $\langle\cdot,\cdot\rangle_A$ on $\mathcal{P}_{n,s}$ by $\langle p,q\rangle_A := \langle A\widehat{p}, \widehat{q}\rangle$ $(p, q \in \mathcal{P}_{n,s})$. In particular, $\langle \bar{z}^i z^j, \bar{z}^k z^l\rangle_A = A_{(k,l)(i,j)}$, for $(i,j) \in \Lambda_{(n,s)}$ and $(k,l) \in \Lambda_{(n,s)}$. We define the moment matrix $M(n,s)$ be a $\eta(n,s) \times \eta(n,s)$ matrix that the entry in row $\bar{Z}^kZ^l$ and column $\bar{Z}^iZ^j$ is $M(n,r)_{(k,l)(i,j)} = \gamma_{l+i,j+k}$, where $(k,l),(i,j) \in \Lambda_{(n,s)}$. (Observe that $M(n,n) = M(n)$ and $M(n,0) = E(n)$ whose definitions are in [1] and [5], resp.) For example, if $n = 2, s = 1$, i.e., for

$$\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \gamma_{03}, \gamma_{12}, \gamma_{21}, \gamma_{30}, \gamma_{13}, \gamma_{22}, \gamma_{31},$$

the associated moment matrix is

$$M(2,1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

In particular, $M(2,s)$ is referred to the quartic moment matrix here.

The paper consists of five sections. In Section 2 we consider a bridge between Bram-Halmos' and Embry characterizations for cyclic subnormal operators, which is related to complex moment matrices $M(n,s)$. In Section 3, we prove that if $\Gamma_{n,s}$ is double flat (i.e., rank $M(n,s) =$ rank $M(n-2,s)$) and $M(n,s) \ge 0$, then $M(n,s)$ admits a unique flat extension of the form $M(n+k,s)$ for all $k \in \mathbb{N}$. And also we consider several useful examples. Let $M(1,0)$ be any positive quadratic moment matrix. Then it always has a flat extension $M(1,1)$.

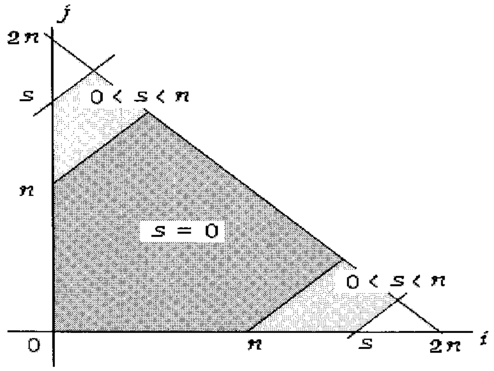

FIGURE 1

But, in general the case of $M(2,0)$ is independent to the case $M(1,s)$. For example, in Section 4 we show that there exists a moment matrix $M(2,0)$ with a representing measure μ such that the number of atoms is different from rank $M(2,0)$. In addition, we discuss singular quartic moment matrix $M(2,s)$ and related examples. Finally, in Section 5 we obtain an algorithm finding a moment measure from the nonsingular quartic moment matrix $M(2,0)$ which will be applied to $M(2,s)$ by a similar method.

Some of the calculations in this article were obtained through computer experiments using the software tool *Mathematica* [8].

2. A bridge between Bram-Halmos and Embry characterizations

Lemma 2.1. *Let* $A := \{\gamma_{ij}\}_{i,j=0}^{\infty}$ *be an infinite matrix of complex numbers. Suppose* $n \in \mathbb{N}$ *and* $0 \le s \le n$. *Then the following assertions are equivalent :*

(i) *there exists a linear functional* $\Lambda : \mathbf{P}[z,\bar{z}] \to \mathbb{C}$ *defined by* $\Lambda(\bar{z}^i z^j) = \gamma_{ij}$ *such that*

$$\Lambda(|\sum_{(i,j)\in\Lambda_{(n,s)}} a_{ij}\bar{z}^i z^j|^2) \ge 0$$

for any $p(z,\bar{z}) = \sum_{(i,j)\in\Lambda_{(n,s)}} a_{ij}\bar{z}^i z^j \in \mathcal{P}_{n,s}$;

(ii) $M(n,s) \ge 0$.

Proof. (i) $\Rightarrow$ (ii): Let $p(z,\bar{z}) = \sum_{(i,j)\in\Lambda_{(n,s)}} a_{ij}\bar{z}^i z^j$. Then

$$\begin{aligned}\Lambda(|p(z,\bar{z})|^2) &= \Lambda(\sum_{(k,l)\in\Lambda_{(n,s)}} \overline{a_{kl}\bar{z}^k z^l}\cdot \sum_{(i,j)\in\Lambda_{(n,s)}} a_{ij}\bar{z}^i z^j) \\ &= \sum_{(k,l),(i,j)\in\Lambda_{(n,s)}} \bar{a}_{kl}a_{ij}\Lambda(\bar{z}^{l+i}z^{j+k})\end{aligned}$$

$$= \sum_{(k,l),(i,j)\in\Lambda_{(n,s)}} \bar{a}_{kl} a_{ij} \gamma_{l+i,j+k}$$
$$= \sum_{(k,l),(i,j)\in\Lambda_{(n,s)}} M(n,s)_{(k,l),(i,j)} \bar{a}_{kl} a_{ij}$$
$$= \langle M(n,s)\widehat{p}, \widehat{p}\rangle \geq 0.$$

(ii) $\Rightarrow$ (i): Define a linear functional $\Lambda(\bar{z}^i z^j) = \gamma_{ij}$ on $\mathbf{P}[z,\bar{z}]$. Let

$$p(z,\bar{z}) = \sum_{(k,l),(i,j)\in\Lambda_{(n,s)}} a_{ij} \bar{z}^i z^j.$$

Since $M(n,s) \geq 0$ for all $n \in \mathbb{N}$, the above computation shows that

$$\Lambda(|p(z,\bar{z})|^2) \geq 0.$$

□

Theorem 2.2. *Let T be an operator with a cyclic vector x_0 in $\mathcal{H}$ and let $\gamma_{ij} := (T^{*i}T^j x_0, x_0)$ for any $i,j \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$. Suppose $n \in \mathbb{N}$ and $0 \leq s \leq n$. Then the following assertions are equivalent:*

(i) *for any $p_i(z) \in \mathbf{P}[z]$ with $\deg p_i(z) \leq n - i - \max\{i-s, 0\}$ $(i = 1,\dots,n)$,*

$$\sum_{0\leq i,j\leq n} (T^{*i+\max\{j-s,0\}} T^{j+\max\{i-s,0\}} p_i(T)x_0, p_j(T)x_0) \geq 0;$$

(ii) $M(n,s) \geq 0$.

Proof. Let $\Lambda : \mathbf{P}[z,\bar{z}] \to \mathbb{C}$ be a linear functional satisfying $\Lambda(\bar{z}^i z^j)=(T^{*i}T^j x_0, x_0)$ for any $i,j \in \mathbb{N}_0$. By Lemma 2.1, $M(n,s) \geq 0$ is equivalent to

(2.1) $\Lambda(|p(z,\bar{z})|^2) \geq 0$ for any polynomial $p(z,\bar{z}) \in \mathcal{P}_{n,s}$.

Since $0 \leq i+j \leq n$ and $\max\{i-s,0\} \leq j$ if and only if $0 \leq i \leq n$ and $\max\{i-s,0\} \leq j \leq n-i$, (2.1) is equivalent to

$$\Lambda(|p_0(z) + \bar{z}p_1(z) + \cdots + \bar{z}^n p_n(z)|^2) \geq 0$$

for any polynomial $p_i(z)$ with $\max\{i-s,0\} \leq \deg p_i(z) \leq n-i$ $(i = 0,\dots,n)$, which is equivalent to

$$\sum_{0\leq i,j\leq n} \langle T^{*i}T^j p_i(T)x_0, p_j(T)x_0\rangle \geq 0$$

for any polynomial $p_i(z)$ with $\max\{i-s,0\} \leq \deg p_i(z) \leq n-i$ $(i = 0,\dots,n)$. That is,

$$\sum_{0\leq i,j\leq n} \langle T^{*i+\max\{j-s,0\}} T^{j+\max\{i-s,0\}} p_i(T)x_0, p_j(T)x_0\rangle \geq 0$$

for any polynomial $p_i(z)$ with $\deg p_i(z) \leq n-i-\max\{i-s,0\}$ $(i = 0,\dots,n)$. □

Given an infinite matrix $A := \{\gamma_{ij}\}_{i,j=0}^{\infty}$ of complex numbers, the full complex moment problem (write: CMP) entails finding a positive Borel measure μ on the closed unit disk $\mathbf{D}$ in $\mathbb{C}$ such that $\gamma_{ij} = \int_{\mathbf{D}} \bar{z}^i z^j \, d\mu(z)$. Let $\{\gamma_{ij}\}_{i,j=0}^{\infty}$ and $\{\delta_{ij}\}_{i,j=0}^{\infty}$ be two infinite matrices of complex numbers. For brevity, we write $\{\gamma_{ij}\}_{i,j=0}^{\infty} \geq \{\delta_{ij}\}_{i,j=0}^{\infty}$ if $\sum_{0\leq i,j\leq n}(\gamma_{ij} - \delta_{ij})\bar{a}_i a_j \geq 0$ for any $a_i \in \mathbb{C}$ and all $n \in \mathbb{N}$. In [5], ones obtained that if $\{\gamma_{ij}\}_{i,j=0}^{\infty}$ is an infinite matrix of complex numbers, then $\{\gamma_{ij}\}_{i,j=0}^{\infty}$ solves CMP if and only if $\{\gamma_{ij}\}_{i,j=0}^{\infty} \geq \{\gamma_{i+1,j+1}\}_{i,j=0}^{\infty}$ and $M(n,n) \geq 0$ for all $n \in \mathbb{N}$ if and only if $\{\gamma_{ij}\}_{i,j=0}^{\infty} \geq \{\gamma_{i+1,j+1}\}_{i,j=0}^{\infty}$ and $M(n,0) \geq 0$, for all $n \in \mathbb{N}$.

Corollary 2.3. *Let T be an operator with a cyclic vector x_0 in $\mathcal{H}$ and let $\gamma_{ij} := (T^{*i}T^j x_0, x_0)$, for any $i, j \in \mathbb{N}_0$. Then the following assertions are equivalent:*

(i) *for any $p_i(z) \in \mathbf{P}[z]$ $(i = 1, \ldots, n)$ and any $n \in \mathbb{N}$,*

$$\sum_{0\leq i,j\leq n} \langle T^{*i+\max\{j-s,0\}}T^{j+\max\{i-s,0\}}p_i(T)x_0, p_j(T)x_0\rangle \geq 0;$$

(ii) *$M(n,s) \geq 0$ for all $n \in \mathbb{N}$ and any $s = 0, \ldots, n$,*

(iii) *$M(n,s) \geq 0$ for all $n \in \mathbb{N}$ and some $s = 0, \ldots, n$.*

Moreover, the following two assertions are equivalent:

(iv) *$\{\gamma_{ij}\}_{i,j=0}^{\infty}$ solves CMP,*

(v) *$\{\gamma_{ij}\}_{i,j=0}^{\infty} \geq \{\gamma_{i+1,j+1}\}_{i,j=0}^{\infty}$ and $M(n,s) \geq 0$ for all $n \in \mathbb{N}$ and some $s = 0, \ldots, n$.*

3. Double flat extension theorem

We review some useful properties which can be obtained by the similar proofs in [3] and [5]. We omit the detail proof here.

(P_1) If μ is a representing measure, then

$$\langle M(n,s)\widehat{p}, \widehat{p}\rangle = \int |p(z,\bar{z})|^2 \, d\mu, p(z,\bar{z}) \in \mathcal{P}_{n,s}.$$

(P_2) If μ is a representing measure, then $M(n,s) \geq 0$.

(P_3) If μ is a representing measure, then supp $\mu \subseteq \mathcal{Z}(p) \Longleftrightarrow p(Z,\bar{Z}) = 0$ for $p \in \mathcal{P}_{n,s}$, where $\mathcal{Z}(p) = \{z \in \mathbb{C} : p(z,\bar{z}) = 0\}$.

(P_4) Let $M(n,s)$ be a moment matrix. If $p(z,\bar{z}) \in \mathcal{P}_{n,s}$ with $\bar{p}(z,\bar{z}) \in \mathcal{P}_{n,s}$, then $p(Z,\bar{Z}) = 0$ if and only if $\bar{p}(Z,\bar{Z}) = 0$.

(P_5) Let $M(n,s) \geq 0$. If f, g, $fg \in \mathcal{P}_{n-1,s}$ and $f(Z,\bar{Z}) = 0$, then $(fg)(Z,\bar{Z}) = 0$ in the column space $\mathcal{C}_{M(n,s)}$.

(P_6) If μ is a representing measure for γ, then card supp $\mu \geq$ rank $M(n,s)$.

(P_7) Let $M(\infty,s)$ be an infinite moment matrix with representing measure μ. Then card supp $\mu =$ rank $M(\infty,s)$.

(P_8) Let $M(\infty,s)$ be a finite-rank positive infinite moment matrix. Then $M(\infty,s)$ has a unique representing measure, which is rank $M(\infty,s)$-atomic. In this case, let $r :=$ rank $M(\infty,s)$; there exist unique scalars $\alpha_0, \ldots, \alpha_{r-1}$such that $Z^r = \alpha_0 1 + \cdots + \alpha_{r-1}Z^{r-1}$. The unique representing measure for $M(\infty, s)$ has support equal to the r distinct roots $z_0, \ldots, z_{r-1}$ of the polynomial $z^r - (\alpha_0 + \cdots + \alpha_{r-1}z^{r-1})$, and densities $\rho_0, \ldots, \rho_{r-1}$ determined by the Vandermonde equation

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_0 & z_1 & \cdots & z_{r-1} \\ \vdots & \vdots & & \vdots \\ z_0^{r-1} & z_1^{r-1} & \cdots & z_{r-1}^{r-1} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_{r-1} \end{pmatrix} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \vdots \\ \gamma_{0,r-1} \end{pmatrix}.$$

The followings is a parallel theorem corresponding to [1].

Theorem 3.1. *If $A \in M_{\eta(n,s)}(\mathbb{C})$, then there exists a truncated moment sequence $\Gamma_{n,s}$ with $\gamma_{00} > 0$ and $\gamma_{ji} = \overline{\gamma_{ij}}$ such that $A = M(n,s)(\Gamma_{n,s})$ if and only if*

(0) $\langle 1, 1\rangle_A > 0$;
(1) $A = A^*$;
(2) $\langle p, q\rangle_A = \langle \bar{q}, \bar{p}\rangle_A \quad (p, \bar{p}, q, \bar{q} \in \mathcal{P}_{n,s})$;
(3) $\langle zp, q\rangle_A = \langle p, \bar{z}q\rangle_A \quad (p, q \in \mathcal{P}_{n-1,s}, \bar{z}q \in \mathcal{P}_{n,s})$;
(4) $\langle zp, zq\rangle_A = \langle \bar{z}p, \bar{z}q\rangle_A \quad (p, q \in \mathcal{P}_{n-1,s}, \bar{z}p, \bar{z}q \in \mathcal{P}_{n,s})$.

Proof. The proof is very similar to [1] and [5], and so we will omit the proof. □

Proposition 3.2. *Suppose n is even number. If $\Gamma_{n,s}$ is flat and $M(n,s) \geq 0$, $0 \leq s \leq n$, then $M(n,s)$ admits a unique flat extension of the form $M(n+k,s)$ for all $k \in \mathbb{N}$.*

Proof. We want construct a moment matrix $M(n+1,s)$ of the form

$$\widetilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

where $A = M(n,s), B = AW$, and $C = W^*AW$.

If $n = 2w$, we denote the columns of B by the flatness of

$$Z^{2w+1}, \bar{Z}Z^{2w}, \ldots, \bar{Z}^w Z^{w+1}, \bar{Z}^{w+1}Z^w, \ldots, \bar{Z}^{w+[\frac{s}{2}]}Z^{w-[\frac{s}{2}]+1}.$$

And so $M(2w,s)$ is flat, i.e.,

$$\bar{Z}^i Z^{2w-i} = p_i(Z,\bar{Z}), \quad i = 0, 1, \ldots, w, w+1, \ldots, w+[\frac{s}{2}], \quad p_i(z,\bar{z}) \in \mathcal{P}_{2w-1,s}.$$

Let

$$\begin{aligned} Z^{2w+1} &= (zp_0)(Z,\bar{Z}), \\ \bar{Z}Z^{2w} &= (zp_1)(Z,\bar{Z}), \\ \bar{Z}^2 Z^{2w-1} &= (zp_2)(Z,\bar{Z}), \\ &\vdots \\ \bar{Z}^{w+[\frac{s}{2}]}Z^{w-[\frac{s}{2}]+1} &= (zp_{w+[\frac{s}{2}]})(Z,\bar{Z}). \end{aligned}$$

We first show that $\widetilde{A}$ is an extension of $M(2w,s)$. We need to check $\langle \bar{z}^k z^l, \bar{z}^i z^j\rangle_{\widetilde{A}} = \gamma_{j+k,i+l}$, for $k+l = 2w+1$, $\max\{k-s,0\} \le l$, $0 \le i+j \le 2w-1$, $\max\{i-s,0\} \le j$, $|(j+k)-(i+l)| \le 2w$. In fact,

$$\begin{aligned}\langle \bar{z}^k z^l, \bar{z}^i z^j\rangle_{\widetilde{A}} &= \langle z p_k(z,z), \bar{z}^i z^j\rangle_{\widetilde{A}} = \langle p_k(z,\bar{z}), \bar{z}^{i+1} z^j\rangle_{\widetilde{A}} \\ &= \langle \bar{z}^k z^{2w-k}, \bar{z}^{i+1} z^j\rangle_A = A_{(i+1,j)(k,2w-k)} \\ &= \gamma_{j+k,i+1+2w-k} = \gamma_{j+k,i+l}.\end{aligned}$$

We now show that $\widetilde{A}$ is a moment matrix. By Theorem 3.1, we must show that

(a) $\widetilde{A}$ is self-adjoint;

(b) $\langle p,q\rangle_{\widetilde{A}} = \langle \bar{q},\bar{p}\rangle_{\widetilde{A}} \quad (p,\bar{p},q,\bar{q} \in \mathcal{P}_{n,s})$;

(c) $\langle zp,q\rangle_{\widetilde{A}} = \langle p,\bar{z}q\rangle_{\widetilde{A}} \quad (p,q \in \mathcal{P}_{n-1,s}, \bar{z}q \in \mathcal{P}_{n,s})$;

(d) $\langle zp,zq\rangle_{\widetilde{A}} = \langle \bar{z}p,\bar{z}q\rangle_{\widetilde{A}} \quad (p,q \in \mathcal{P}_{n-1,s}, \bar{z}p,\bar{z}q \in \mathcal{P}_{n,s})$.

Indeed, (a) clear.

(b) Since $n = 2w+1$ and $p,\bar{p},q,\bar{q} \in \mathcal{P}_{n,s}$, the polynomials p,q must be the form

$$p(z,\bar{z}) = \sum_{(i,j)\in\Lambda'_{(n,s)}} a_{ij}\bar{z}^i z^j, \quad q(z,\bar{z}) = \sum_{(i,j)\in\Lambda'_{(n,s)}} b_{ij}\bar{z}^i z^j,$$

where

$$\Lambda'_{(n,s)} = \{(i,j) : 0 \le i+j \le n, \max\{i-s,0\} \le j \le i+s,\ 0 \le s \le n\}$$

so (b) is clear.

(c) We take $\bar{z}^k z^l, \bar{z}^i z^j$ instead of p,q. Since $p,q \in \mathcal{P}_{2w,s}, \bar{z}q \in \mathcal{P}_{2w+1,s}$, we must have $0 \le k+l \le 2w, \max\{k-s,0\} \le l, 0 \le i+j \le 2w, \max\{i-s,0\} < j$.

For $0 \le k+l \le 2w-1, \max\{k-s,0\} \le l, 0 \le i+j \le 2w, \max\{i-s,0\} < j$, we have

$$\begin{aligned}\langle z\cdot\bar{z}^k z^l, \bar{z}^i z^j\rangle_{\widetilde{A}} &= \langle \bar{z}^k z^{l+1}, \bar{z}^i z^j\rangle_{\widetilde{A}} = \langle \bar{z}^k z^{l+1}, \bar{z}^i z^j\rangle_A \\ &= \langle \bar{z}^k z^l, \bar{z}^{i+1} z^j\rangle_A = \langle \bar{z}^k z^l, \bar{z}\cdot\bar{z}^i z^j\rangle_{\widetilde{A}}.\end{aligned}$$

For $k+l = 2w, \max\{k-s,0\} \le l, 0 \le i+j \le 2w-1, \max\{i-s,0\} < j$, we have

$$\begin{aligned}\langle z\cdot\bar{z}^k z^l, \bar{z}^i z^j\rangle_{\widetilde{A}} &= \langle \bar{z}^k z^{l+1}, \bar{z}^i z^j\rangle_{\widetilde{A}} \\ &= \langle \bar{z}^k z^{2m-k+1}, \bar{z}^i z^j\rangle_{\widetilde{A}} \\ &= \langle z p_k(z,\bar{z}), \bar{z}^i z^j\rangle_{\widetilde{A}} \quad (p_k \in \mathcal{P}_{2w-1,s}) \\ &= \langle z p_k(z,\bar{z}), \bar{z}^i z^j\rangle_A \\ &= \langle p_k(z,\bar{z}), \bar{z}^{i+1} z^j\rangle_A \\ &= \langle p_k(z,\bar{z}), \bar{z}\cdot\bar{z}^i z^j\rangle_A \\ &= \langle p_k(z,\bar{z}), \bar{z}\cdot\bar{z}^i z^j\rangle_{\widetilde{A}} \\ &= \langle \bar{z}^k z^{2w-k}, \bar{z}\cdot\bar{z}^i z^j\rangle_{\widetilde{A}} \\ &= \langle \bar{z}^k z^l, \bar{z}\cdot\bar{z}^i z^j\rangle_{\widetilde{A}}.\end{aligned}$$

For $k+l=2w, \max\{k-s,0\} \le l, 0 \le i+j=2w, \max\{i-s,0\} < j$, we have

$$\begin{aligned}
\langle z \cdot \bar{z}^k z^l, \bar{z}^i z^j \rangle_{\widetilde{A}} &= \langle \bar{z}^k z^{l+1}, \bar{z}^i z^j \rangle_{\widetilde{A}} \\
&= \langle \bar{z}^k z^{2w-k+1}, \bar{z}^i z^{2w-i} \rangle_{\widetilde{A}} \\
&= \langle z p_k(z,\bar{z}), p_i(z,\bar{z}) \rangle_{\widetilde{A}} \quad (p_k, p_i \in \mathcal{P}_{2w-1,s}) \\
&= \langle z p_k(z,\bar{z}), p_i(z,\bar{z}) \rangle_{A} \\
&= \langle p_k(z,\bar{z}), \bar{z} p_i(z,\bar{z}) \rangle_{A} \\
&= \langle p_k(z,\bar{z}), \bar{z} p_i(z,\bar{z}) \rangle_{\widetilde{A}} \\
&= \langle \bar{z}^k z^l, \bar{z} \cdot \bar{z}^i z^j \rangle_{\widetilde{A}}.
\end{aligned}$$

(d) For $0 \le k+l \le 2w-1, \max\{k-s,0\} < l, 0 \le i+j \le 2w-1, \max\{i-s,0\} < j$, we have

$$\begin{aligned}
\langle z \cdot \bar{z}^k z^l, z \cdot \bar{z}^i z^j \rangle_{\widetilde{A}} &= \langle \bar{z}^k z^{l+1}, \bar{z}^i z^{j+1} \rangle_{\widetilde{A}} \\
&= \langle z \cdot \bar{z}^k z^l, z \cdot \bar{z}^i z^j \rangle_{A} \\
&= \langle \bar{z} \cdot \bar{z}^k z^l, \bar{z} \cdot \bar{z}^i z^j \rangle_{A} \\
&= \langle \bar{z} \cdot \bar{z}^k z^l, \bar{z} \cdot \bar{z}^i z^j \rangle_{\widetilde{A}}.
\end{aligned}$$

For $k+l=2w, \max\{k-s,0\} < l, 0 \le i+j \le 2w-1, \max\{i-s,0\} < j$, we have

$$\begin{aligned}
\langle z \cdot \bar{z}^k z^l, z \cdot \bar{z}^i z^j \rangle_{\widetilde{A}} &= \langle \bar{z}^k z^{l+1}, \bar{z}^i z^{j+1} \rangle_{\widetilde{A}} \\
&= \langle z p_k(z,\bar{z}), \bar{z}^i z^{j+1} \rangle_{\widetilde{A}} \\
&= \langle z p_k(z,\bar{z}), \bar{z}^i z^{j+1} \rangle_{A} \\
&= \langle \bar{z} p_k(z,\bar{z}), \bar{z}^{i+1} z^j \rangle_{A} \\
&= \langle \bar{z} p_k(z,\bar{z}), \bar{z}^{i+1} z^j \rangle_{\widetilde{A}} \\
&= \langle \bar{z} \cdot \bar{z}^k z^l, \bar{z} \cdot \bar{z}^i z^j \rangle_{\widetilde{A}}.
\end{aligned}$$

Similarly, we can prove the case of $0 \le k+l = 2w-1, \max\{k-s,0\} < l, i+j = 2w, \max\{i-s,0\} < j$.

For $k+l=2w, \max\{k-s,0\} < l, i+j=2w, \max\{i-s,0\} < j$, we have

$$\begin{aligned}
\langle z \cdot \bar{z}^k z^l, z \cdot \bar{z}^i z^j \rangle_{\widetilde{A}} &= \langle z p_k(z,\bar{z}), z p_i(z,\bar{z}) \rangle_{\widetilde{A}} \\
&= \langle z p_k(z,\bar{z}), z p_i(z,\bar{z}) \rangle_{A} \\
&= \langle \bar{z} p_k(z,\bar{z}), \bar{z} p_i(z,\bar{z}) \rangle_{A} \\
&= \langle \bar{z} p_k(z,\bar{z}), \bar{z} p_i(z,\bar{z}) \rangle_{\widetilde{A}} \\
&= \langle \bar{z} \cdot \bar{z}^k z^l, \bar{z} \cdot \bar{z}^i z^j \rangle_{\widetilde{A}}.
\end{aligned}$$

Moreover, the matrix $M(n, s)$ also admits a flat extension of the form $M(n+2, s)$. In fact, we have

$$\begin{aligned} Z^{2m+2} &= (z^2 p_0)(Z, \bar{Z}), \\ \bar{Z}Z^{2m+1} &= (z^2 p_1)(Z, \bar{Z}), \\ \bar{Z}^2 Z^{2m} &= (z^2 p_2)(Z, \bar{Z}), \\ &\vdots \\ \bar{Z}^{m+1} Z^{m+1} &= (\bar{z} z p_m)(Z, \bar{Z}). \end{aligned}$$

Hence $M(n, s)$ admits a flat extension of the form $M(n+k, s)$ for all $k \in \mathbb{N}$. □

Now we have the following

Theorem 3.3. *Let $n > 1$. If $\Gamma_{n,s}$ is double flat (i.e., $\operatorname{rank} M(n, s) = \operatorname{rank} M(n-2, s)$) and $M(n, s) \geq 0$, then $M(n, s)$ admits a unique flat extension of the form*

$$M(n + k, s), k \in \mathbb{N}.$$

Proof. If n is even number, the result follows from Proposition 3.2. If $n \geq 3$ is odd number, then $n - 1$ is even and $M(n - 1, s)$ is flat and positive, thus by Proposition 3.2, $M(n - 1, s)$ admits a unique flat extension of the form $M(n + k, s)$ for all $k \in \mathbb{N}$. □

Theorem 3.4. *The truncated complex moment sequence $\Gamma_{n,s}$ has a $\operatorname{rank} M(n, s)$-atomic representing measure if and only if $M(n, s) \geq 0$ and $M(n, s)$ admits a double flat extension $M(n + 2, s)$, i.e.,*

$$\operatorname{rank} M(n, s) = \operatorname{rank} M(n + 2, s).$$

Proof. Suppose $M(n, s) \geq 0$ and $M(n, s)$ admits a double flat extension $M(n+2, s)$, i.e., rank $M(n, s) =$ rank $M(n+2, s)$. By Theorem 3.3, $M(n+2, s)$ admits a unique flat extension of the form $M(n+3, s)$. Thus, the unique flat extension of the form $M(\infty, s)$ may be constructed by successive application of Theorem 3.3, and rank $M(\infty, s) =$ rank $M(n, s)$. (P_7) implies that $M(\infty, s)$ has a rank $M(\infty, s)$-atomic representing measure μ, and μ is clearly also a rank $M(n, s)$-atomic representing measure for $\Gamma_{n,s}$.

Conversely, suppose that μ is a rank $M(n, s)$-atomic representing measure for $\Gamma_{n,s}$. Consider $M(n + 2, s)[\mu]$;then rank $M(n, s) =$ card supp $\mu \geq$ rank $M(n + 2, s)[\mu]$(by (P_6), since μ is a representing measure for $M(n + 2, s)[\mu]$) $\geq$ rank $M(n, s)$, and thus $M(n+2, s)[\mu]$ is a double flat extension of $M(n, s)$. □

We discuss a simple example.

Example 3.5. Let

$$M(3,1)=\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & i & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & i & 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 1 & 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 & 1 & i & 0 & 0 \\ -i & 0 & 0 & 0 & -i & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & i & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that rank $M(3,1)$ = rank $M(1,1) = 3$. By Theorem 3.4, $\Gamma_{3,1}$ admits a 3-atomic representing measure. Since $Z^2 = i\bar{Z}$ and $\bar{Z}Z = \mathit{1}$, we obtain $Z^3 = i\mathit{1}$, the three atoms are the roots of $z^3 - i = 0$, i.e., $z_0 = -i$, $z_1 = \frac{\sqrt{3}}{2} + \frac{i}{2}$, $z_2 = -\frac{\sqrt{3}}{2} + \frac{i}{2}$. From

$$\begin{pmatrix} 1 & 1 & 1 \\ z_0 & z_1 & z_2 \\ z_0^2 & z_1^2 & z_2^2 \end{pmatrix}\begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \gamma_{02} \end{pmatrix},$$

we have $\rho_0 = \rho_1 = \rho_2 = \frac{1}{3}$. Thus we obtain the representing measure $\mu = \frac{1}{3}(\delta_{z_0} + \delta_{z_1} + \delta_{z_2})$. We can check that the measure does satisfy

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \qquad (0 \le i+j \le 6,\ |i-j| \le 4).$$

4. Singular quartic moment matrices

Let $M(1,0)$ be any positive quadratic moment matrix. Then it has a flat extension $M(1,1)$. Indeed, to obtain such $M(1,1)$, take γ_{02} among roots of the equation $\left|z - \gamma_{10}^2/\gamma_{00}\right| = (\gamma_{00}\gamma_{11} - |\gamma_{10}|^2)/\gamma_{00}$. But, in general this flat extension can not be always constructed (see Proposition 4.1). So the study of $M(n,s)$ is worthwhile in the truncated complex moment problem.

Proposition 4.1. *There exists a moment matrix $M(2,0)$ with a representing measure μ such that the number of atoms is different from* rank $M(2,0)$. *This matrix $M(2,0)$ has an extension $M(2,2)$ which has a* rank $M(2,2)$*-representing measure.*

Proof. Let us consider a positive matrix

$$M(2,0)=\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1-i \\ 1 & 0 & 1+i & 2 \end{pmatrix}$$

with rank $M(2,0) = 3$. Since $\bar{Z}Z = 1 + (\frac{1}{2} - \frac{1}{2}i)Z^2$ and $Z^3 = \gamma_{03}1 + (1+i)Z + \frac{1}{2}\gamma_{23}Z^2$, by [6, Algorithms 2.7 and 2.10] we may define

$$\begin{aligned}
\gamma_{14} &= 2\gamma_{03},\\
\gamma_{23} &= \gamma_{03} - i\gamma_{03},\\
\gamma_{24} &= 2 + 2i - i\gamma_{03}^2,\\
\gamma_{33} &= 2\gamma_{03}\gamma_{30} + 2,\\
\gamma_{34} &= 2\gamma_{03} - 2i\gamma_{03} + 2i\gamma_{30} + \gamma_{03}^2\gamma_{30} - i\gamma_{03}^2\gamma_{30},\\
\gamma_{35} &= 4\gamma_{03}\gamma_{30} + 4i\gamma_{03}\gamma_{30} + 2 + 2i - 2i\gamma_{03}^2 - i\gamma_{03}^3\gamma_{30}.
\end{aligned}$$

By [6, Theorem 2.11], we need to consider the following equations

$$3i\gamma_{30}^2 - \gamma_{30}^2 - i\gamma_{03}\gamma_{30} + i\gamma_{30}^3\gamma_{03} = 3\gamma_{03}\gamma_{30} - i\gamma_{03}^2 - \frac{1}{2}i\gamma_{03}^3\gamma_{30} - \gamma_{03}^2 - \frac{1}{2}\gamma_{03}^3\gamma_{30},$$

$$2\gamma_{03}\gamma_{30} = 2 - \frac{1}{2}(1+i)\gamma_{03}^2.$$

From the second equality, we have $\gamma_{03}^2 = 2k(1-i)$ for some real k, and so $|\gamma_{03}|^2 = 1 - k$. Substituting them in the first equation, we obtain $7k - 1 = 2k^2$ and $k = -3$, which is impossible. Thus there is not any 3-atomic representing measure. Furthermore, we may construct $M(2)$ as a positive extension of $M(2,0)$ with rank $M(2) = 4$ as following

$$M(2,2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 2 & 1-i & -2i\\
1 & 0 & 0 & 1+i & 2 & 1-i\\
0 & 0 & 0 & 2i & 1+i & 2
\end{pmatrix}.$$

Observe that $\bar{Z}Z = 1 + \frac{1}{2}(1-i)Z^2$ and $\bar{Z}^2 = -iZ^2$, which implies that $\gamma_{32} = \frac{1}{2}(1-i)\gamma_{23}$. That is $\gamma_{23} = 0$. Therefore, $\gamma_{14} = \gamma_{05} = 0$, and so we obtain the flat extension $M(3,3)$ of $M(2,2)$

$$M(3) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 1+i & 2 & 1-i & -2i\\
0 & 0 & 1 & 0 & 0 & 0 & 2i & 1+i & 2 & 1-i\\
0 & 0 & 0 & 2 & 1-i & -2i & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 1+i & 2 & 1-i & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 2i & 1+i & 2 & 0 & 0 & 0 & 0\\
0 & 1-i & -2i & 0 & 0 & 0 & 6 & 4-4i & -6i & -4-4i\\
0 & 2 & 1-i & 0 & 0 & 0 & 4+4i & 6 & 4-4i & -6i\\
0 & 1+i & 2 & 0 & 0 & 0 & 6i & 4+4i & 6 & 4-4i\\
0 & 2i & 1+i & 0 & 0 & 0 & -4+4i & 6i & 4+4i & 6
\end{pmatrix}.$$

Thus there is a 4-atomic representing measure $\mu = \rho_0\delta_{z_0} + \rho_1\delta_{z_1} + \rho_2\delta_{z_2} + \rho_3\delta_{z_3}$. The four atoms are roots of $z^4 - 2z^2 + i\left(-2z^2 - 2\right) = 0$, i.e., $z_0 = 1 + \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}$, $z_1 = 1 - \frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}$, $z_2 = -1 + \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}$, $z_3 = -1 - \frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}$. By the Vandermonde equation

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ z_0 & z_1 & z_2 & z_3 \\ z_0^2 & z_1^2 & z_2^2 & z_3^2 \\ z_0^3 & z_1^3 & z_2^3 & z_3^3 \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \gamma_{02} \\ \gamma_{03} \end{pmatrix},$$

we obtain four densities $\rho_0 = \rho_3 = -\frac{1}{8}\sqrt{2} + \frac{1}{4}$ and $\rho_1 = \rho_2 = \frac{1}{4} + \frac{1}{8}\sqrt{2}$. Hence we obtain a required moment matrix. □

The moment matrix $M(2,2)$ was discussed in [1] and [2] and also $M(2,0)$ was discussed in [5]. So we need consider $M(2,1)$ here. First we recall

$$M(2,1) = \begin{array}{c} \begin{matrix} \mathit{1} & Z & \bar{Z} & Z^2 & \bar{Z}Z \end{matrix} \\ \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} \end{pmatrix} \end{array}.$$

By the above discussion, if $M(2,1) \geq 0$ and flat, i.e., rank $M(2,1)$ = rank $M(1,1)$, then it admits a flat extension of $M(3,1)$. Hence there is a rank $M(2,1)$-atomic representing measure. Thus we have $\mu = \rho_0\delta_{z_0} + \rho_1\delta_{z_1} + \rho_2\delta_{z_2}$. In fact, the flatness of $M(2,1)$ means that $Z^2 = a_1\mathit{1} + b_1Z + c_1\bar{Z}$, and $\bar{Z}Z = a_2\mathit{1} + b_2Z + c_2\bar{Z}$, and so we obtain $Z^3 = (c_1a_2 - c_2a_1)\mathit{1} + (a_1 + b_2c_1 - b_1c_2)Z + (b_1 + c_2)Z^2$. Hence, the atoms, z_0, z_1, z_2, are the roots of $z^3 - (b_1 + c_2)z^2 - (a_1 + b_2c_1 - b_1c_2)z - (c_1a_2 - c_2a_1) = 0$. And the densities ρ_0, ρ_1, and ρ_2 can be obtained by Vandermonde equation.

Example 4.2. Let

$$M(2,1) = \begin{pmatrix} 1 & i & -i & i & 3 \\ -i & 3 & -i & 1+i & 1-i \\ i & i & 3 & 2i & 1+i \\ -i & 1-i & -2i & 6 & -2-11i \\ 3 & 1+i & 1-i & -2+11i & 6 \end{pmatrix}.$$

It is easy to check that rank $M(2,1)$ = rank $M(1,1) = 3$. By Theorem 3.5, $\Gamma_{2,1}$ admits a 3-atomic representing measure. Since $Z^2 = (-1+4i)\mathit{1} + (-\frac{3}{2} + \frac{1}{2}i)Z + (\frac{3}{2} + \frac{3}{2}i)\bar{Z}$, $\bar{Z}Z = 10\mathit{1} + (\frac{3}{2} + \frac{7}{2}i)Z + (\frac{3}{2} - \frac{7}{2}i)\bar{Z}$, we obtain $Z^3 = -3iZ^2 - \left(\frac{7}{2} - \frac{11}{2}i\right)Z + \left(\frac{5}{2} + \frac{11}{2}i\right)\mathit{1}$. The three atoms are the roots of $z^3 + 3iz^2 + \left(\frac{7}{2} - \frac{11}{2}i\right)z - \left(\frac{5}{2} + \frac{11}{2}i\right) = 0$, i.e., $z_0 \cong -1.0128 - 4.3466i$, $z_1 \cong -0.57411 +$

$0.48079i$, and $z_2 \cong 1.5869 + 0.86579i$. From

$$\begin{pmatrix} 1 & 1 & 1 \\ z_0 & z_1 & z_2 \\ z_0^2 & z_1^2 & z_2^2 \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \gamma_{02} \end{pmatrix},$$

we have $\rho_0 \cong 0.00413$, $\rho_1 \cong 0.67747$, and $\rho_2 \cong 0.31840$. Thus we obtain the representing measure $\mu = \rho_0\delta_{z_0} + \rho_1\delta_{z_1} + \rho_2\delta_{z_2}$.

If $M(2,1)$ is not flat, i.e., rank $M(2,1) > 3$, then we have two ideas. The one is to find double flat extension $M(4,1)$ and the other one is to find flat extension $M(2,2)$ first, and using the results of [7].

Lemma 4.3. *The positivity of $M(2,1)$ is not sufficient for the existence of representing measure.*

Proof. In fact, we let

$$M(2,1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Then $M(2,1)$ is positive and the rank $M(2,1) = 4$. But, there is no representing measure at all. Since $M(2,1)$ has unique positive extension $M(2,2)$ as the following

$$M(2,2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 \end{pmatrix},$$

which has no representing measure ([7, Example 2.4]). □

Proposition 4.4. *Let*

$$M(2,1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \gamma_{03} & 0 \\ 0 & 0 & \gamma_{30} & \gamma_{22} & \gamma_{31} \\ 1 & 0 & 0 & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

If $M(2,1)$ is positive with rank $M(2,1) = 4$ *and* $|\gamma_{03}| \neq 1$, *then there is* 4*-atomic representing measure.*

Proof. Since $\bar{Z}Z = A1 + BZ + C\bar{Z} + DZ^2$, where

$$A = 1, \quad B = 0, \quad C = -\frac{\gamma_{03}\gamma_{31}}{\gamma_{22} - \gamma_{03}\gamma_{30}}, \quad D = \frac{\gamma_{31}}{\gamma_{22} - \gamma_{03}\gamma_{30}}.$$

By [3, Theorem 3.1] we know that $M(2,1)$ admits a 4-atomic representing measure if and only if there exists $\gamma_{23} \in \mathbb{C}$ such that $\gamma_{23}\gamma_{31} + \left(|\gamma_{03}|^2 - \gamma_{22}\right)\gamma_{32} = \gamma_{03}\gamma_{31}^2$. Since

$$\det M(2,1) = \gamma_{22}^2 + (1-\gamma_{22})|\gamma_{03}|^2 - \gamma_{22} - |\gamma_{13}|^2 = 0,$$

we have that $0 \neq |\gamma_{13}|^2 - (|\gamma_{03}|^2 - \gamma_{22})^2 = (|\gamma_{03}|^2 - \gamma_{22})(1 - |\gamma_{03}|^2)$. Since $\gamma_{22} - |\gamma_{03}|^2 > 0$, we have our conclusion. □

In case of $|\gamma_{03}| = 1$, in the proof of Lemma 4.2, we showed there is a case in which there exists no representing measure.

Lemma 4.5 ([6, Lemma 2.13]). *The equation $A|z|^2 + 2Re(Cz) = B$, $(A > 0,\ C \in \mathbb{C},\ B \in \mathbb{R})$ has a solution if and only if $AB + |C|^2 \geq 0$.*

Proof. In fact, we have $\left|\bar{z} + \frac{C}{A}\right|^2 = \frac{AB+|C|^2}{A^2}$. □

Proposition 4.6. *Let $M(1,0) \geq 0$. Then $M(1,0)$ has a positive flat extension $M(1,1)$, that is,* $\operatorname{rank} M(1,0) = \operatorname{rank} M(1,1)$.

Proof. Since

$$M(1,1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix},$$

if we let $z = \gamma_{02}$, then from $\det M(1,1) = 0$, we can take z as the roots of the equation $\left|z - \gamma_{10}^2/\gamma_{00}\right| = (\gamma_{00}\gamma_{11} - |\gamma_{10}|^2)/\gamma_{00}$. □

Proposition 4.7. *Let $M(2,1) \geq 0$. Then $M(2,1)$ has a flat extension $M(2,2)$. In particular, if $M(2,1)$ is singular, then has a unique flat extension $M(2,2)$; if $M(2,1)$ is nonsingular, then has infinitely many flat extension $M(2,2)$.*

Proof. Put $z = \gamma_{04}$. Then

$$M(2,2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \bar{z} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & z & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

Let $\phi(z) = \det M(2,2) = A|z|^2 + 2\mathrm{Re}(Cz) - B$, for some $A, B \in \mathbb{R}, C \in \mathbb{C}$. Using Mathematica ([8]), we can show that $AB + |C|^2 = (\det M(2,1))^2 \geq 0$. According to Lemma 4.4, we know that $\phi(z) = 0$ has roots, so $M(2,1)$ has a flat extension $M(2,2)$. □

Example 4.8. Let

$$M(2,1)=\begin{pmatrix} 1 & 0 & 0 & i & 2 \\ 0 & 2 & -i & 1+i & 1-i \\ 0 & i & 2 & 2i & 1+i \\ -i & 1-i & -2i & 9 & 2-\frac{10-4\sqrt{10}}{3}i \\ 2 & 1+i & 1-i & 2+\frac{10-4\sqrt{10}}{3}i & 9 \end{pmatrix},$$

then it is easy to see that $M(2,1)$ is positive and rank $M(2,1)=4$. The flat extension $M(2,2)$ is the form

$$M(2,2)=\begin{pmatrix} 1 & 0 & 0 & i & 2 & -i \\ 0 & 2 & -i & 1+i & 1-i & -2i \\ 0 & i & 2 & 2i & 1+i & 1-i \\ -i & 1-i & -2i & 9 & 2-\frac{10-4\sqrt{10}}{3}i & \gamma_{40} \\ 2 & 1+i & 1-i & 2+\frac{10-4\sqrt{10}}{3}i & 9 & 2-\frac{10-4\sqrt{10}}{3}i \\ i & 2i & 1+i & \gamma_{04} & 2+\frac{10-4\sqrt{10}}{3}i & 9 \end{pmatrix},$$

put $z=\gamma_{04}$, then $\det M(2,2)=-A|z|^2-2\mathrm{Re}(Cz)+B$, where $A=11$, $B=\frac{256}{9}\sqrt{10}-\frac{6475}{9}$, and $C=81+\left(\frac{44}{3}-\frac{32}{3}\sqrt{10}\right)i$. Since $AB+|C|^2=0$, and $\bar{z}=-\frac{C}{A}=-\left(\frac{4}{3}-\frac{32}{33}\sqrt{10}\right)i-\frac{81}{11}$, by Lemma 4.4, $M(2,1)$ has a positive flat extension $M(2,2)$ if and only if

$$\gamma_{04}=\left(\frac{4}{3}-\frac{32}{33}\sqrt{10}\right)i-\frac{81}{11}.$$

On the other hand, since $\bar{Z}Z=a\mathit{1}+bZ+c\bar{Z}+dZ^2$, and $\bar{Z}^2=a'\mathit{1}+b'Z+c'\bar{Z}+d'Z^2$, where $a=\frac{1}{4}\sqrt{10}+\left(2-\frac{1}{4}i\right)$, $b=\frac{1}{6}\sqrt{10}+\left(\frac{1}{3}-\frac{1}{2}i\right)$, $c=\left(\frac{1}{4}-\frac{1}{12}i\right)\sqrt{10}+\left(\frac{1}{4}+\frac{1}{12}i\right)$, and $d=\frac{1}{4}i\sqrt{10}+\frac{1}{4}$, and $a'=\frac{2}{11}\sqrt{10}-\frac{2}{11}i$, $b'=\frac{4}{33}\sqrt{10}+\left(\frac{1}{3}-\frac{5}{11}i\right)$, $c'=\left(\frac{2}{11}-\frac{2}{33}i\right)\sqrt{10}+\left(\frac{3}{11}+\frac{5}{33}i\right)$, and $d'=\frac{2}{11}i\sqrt{10}-\frac{9}{11}$. Moreover, the matrix $M(2,1)$ has the following flat extension of the form

$$M(2,2)=\begin{pmatrix} 1 & 0 & 0 & i & 2 & -i \\ 0 & 2 & -i & 1+i & 1-i & -2i \\ 0 & i & 2 & 2i & 1+i & 1-i \\ -i & 1-i & -2i & 9 & 2-\frac{10-4\sqrt{10}}{3}i & (\frac{32}{33}\sqrt{10}-\frac{4}{3})i-\frac{81}{11} \\ 2 & 1+i & 1-i & 2+\frac{10-4\sqrt{10}}{3}i & 9 & 2-\frac{10-4\sqrt{10}}{3}i \\ i & 2i & 1+i & (\frac{4}{3}-\frac{32}{33}\sqrt{10})i-\frac{81}{11} & 2+\frac{10-4\sqrt{10}}{3}i & 9 \end{pmatrix}.$$

By [1, Theorem 3.1] we know that $M(2,2)$ admits a flat extension and $\Gamma_{2,2}$ admits a 4-atomic representing measure. The atoms are the roots of

$$\begin{aligned} 0 &= \left(\frac{103}{99}-\frac{119}{792}i\right)\sqrt{10}+\left(\frac{8815}{1584}-\frac{1319}{792}i\right) \\ &+\left(\left(\frac{1487}{792}-\frac{205}{88}i\right)+\left(\frac{34}{99}-\frac{41}{99}i\right)\sqrt{10}\right)z \\ &+\left(\left(\frac{1253}{792}+\frac{35}{33}i\right)+\left(\frac{5}{99}+\frac{59}{99}i\right)\sqrt{10}\right)z^2 \\ &+\left(\left(\frac{43}{132}+\frac{35}{22}i\right)-\left(\frac{23}{132}-\frac{7}{44}i\right)\sqrt{10}\right)z^3+\left(\frac{45}{176}-\frac{5}{88}i\sqrt{10}\right)z^4. \end{aligned}$$

In fact, the roots are $z_0 \cong -1.8218+0.88716i$, $z_1 \cong -0.56381-0.87851i$, $z_2 \cong 1.3177+1.034i$, and $z_3 \cong 5.5103-6.1107i$ and the densities are $\rho_0 \cong 0.09659$, $\rho_1 \cong 0.57237$, $\rho_2 \cong 0.33058$, $\rho_3 \cong 0.00046$.

Example 4.9. Let

$$M(2,1)=\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Since $\det[M(2,1)]_4 = 2, \det M(2,1) = 1$, $M(2,1)$ is positive and nonsingular. If we let $\gamma_{04} = x+yi$, then $M(2,1)$ has a flat extension of $M(2,2)$ if and only if $x^2+y^2 = 2x$. Thus, in particular, if we take $x = 2, y = 0$, then one of the flat extension is

$$M(2,2)=\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 1 & 2 \end{pmatrix}.$$

It is easy to see that $\bar{Z}^2 = Z^2$. Let $\gamma_{23} = s+ti$, by [3, Theorem 4.1], $\gamma_{41} = \gamma_{23}$, and $\Gamma_{2,2}$ admits a 5-atomic representing measure if and only if $s = 0$ or $t = 0$. In particular, if we take $\gamma_{23} = i$, then

$$\begin{aligned} Z^3 &= 3i\mathit{1}+1Z+2\bar{Z}+2iZ^2-3i\bar{Z}Z, \\ \bar{Z}Z^2 &= -3i\mathit{1}+2Z+\bar{Z}-2iZ^2+3i\bar{Z}Z, \\ Z^4 &= -15\mathit{1}-iZ+i\bar{Z}-9Z^2+17\bar{Z}Z. \end{aligned}$$

By some transformation, we obtain $Z^5 = -9i\mathit{1} + 3Z - 12iZ^2 + 2Z^3 + 5iZ^4$. Thus, the five atoms and the densities are

$$\begin{aligned} z_0 &= -\sqrt{3}, & \rho_0 &\cong 0.08333, \\ z_1 &= \sqrt{3}, & \rho_1 &\cong 0.08333, \\ z_2 &\cong -0.809\,79i, & \rho_2 &\cong 0.39572, \\ z_3 &\cong 0.729\,18i, & \rho_3 &\cong 0.43731, \\ z_4 &\cong 5.080\,6i, & \rho_4 &\cong 0.00031. \end{aligned}$$

Therefore, the 5-atomic representing measure is $\mu = \rho_0\delta_{z_0} + \rho_1\delta_{z_1} + \rho_2\delta_{z_2} + \rho_3\delta_{z_3} + \rho_4\delta_{z_4}$.

5. Nonsingular quartic moment matrices

We introduce an algorithm for finding moment measure of the nonsingular matrix $M(2,0)$. The case of $M(2,1)$ and $M(2,2)$ can be discussed similarly and so we leave them to interesting readers. For brevity, we write $M(3) := M(3,3)$ and $E(3) := M(3,0)$ as usual.

Recall that if $M(2)$ has a flat extension $M(3)$, then there exists an associated moment measure ([1]). However the nonsingular quartic moment problem of $M(2)$ is open still ([7]). In this section we discuss the nonsingular quartic moment problem of $E(2)$. Because the double flatness is required on the case of $E(2)$, we assume that $E(2)$ is positive and invertible and has a double flat extension $E(4)$ for being time. Then we obtain the corresponding finite sequence

$$\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \gamma_{12}, \gamma_{21}, \gamma_{13}, \gamma_{22}, \gamma_{31}, \dots, \gamma_{44}, \gamma_{53}, \gamma_{62}.$$

Since $E(2)$ is positive and invertible, $\Delta_d > 0$ for all $d = 1, 2, 3$ and 4. In this case, there exist c_0, c_1, c_2, c_3 and d_0, d_1, d_2, d_3 such that

$$Z^3 = c_0\mathit{1} + c_1 Z + c_2 Z^2 + c_3 \bar{Z}Z, \tag{5.1a}$$

$$\bar{Z}Z^2 = d_0\mathit{1} + d_1 Z + d_2 Z^2 + d_3 \bar{Z}Z, \tag{5.1b}$$

where

$$c_0 = \frac{1}{\Delta_4}\begin{vmatrix} \gamma_{03} & \gamma_{01} & \gamma_{02} & \gamma_{11} \\ \gamma_{13} & \gamma_{11} & \gamma_{12} & \gamma_{21} \\ \gamma_{23} & \gamma_{21} & \gamma_{22} & \gamma_{31} \\ \gamma_{14} & \gamma_{12} & \gamma_{13} & \gamma_{22} \end{vmatrix}, \quad c_1 = \frac{1}{\Delta_4}\begin{vmatrix} \gamma_{00} & \gamma_{03} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{13} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{23} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{14} & \gamma_{13} & \gamma_{22} \end{vmatrix},$$

$$c_2 = \frac{1}{\Delta_4}\begin{vmatrix} \gamma_{00} & \gamma_{01} & \gamma_{03} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{13} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{23} & \gamma_{31} \\ \gamma_{11} & \gamma_{12} & \gamma_{14} & \gamma_{22} \end{vmatrix}, \quad c_3 = \frac{1}{\Delta_4}\begin{vmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{03} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} \end{vmatrix},$$

and

$$d_0 = \frac{1}{\Delta_4}\begin{vmatrix} \gamma_{12} & \gamma_{01} & \gamma_{02} & \gamma_{11} \\ \gamma_{22} & \gamma_{11} & \gamma_{12} & \gamma_{21} \\ \gamma_{32} & \gamma_{21} & \gamma_{22} & \gamma_{31} \\ \gamma_{23} & \gamma_{12} & \gamma_{13} & \gamma_{22} \end{vmatrix}, \quad d_1 = \frac{1}{\Delta_4}\begin{vmatrix} \gamma_{00} & \gamma_{12} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{22} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{32} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{23} & \gamma_{13} & \gamma_{22} \end{vmatrix},$$

$$d_2 = \frac{1}{\Delta_4}\begin{vmatrix} \gamma_{00} & \gamma_{01} & \gamma_{12} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{22} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{32} & \gamma_{31} \\ \gamma_{11} & \gamma_{12} & \gamma_{23} & \gamma_{22} \end{vmatrix}, \quad d_3 = \frac{1}{\Delta_4}\begin{vmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{12} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{22} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{32} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{23} \end{vmatrix}.$$

We note that c_i $(i = 0,1,2,3)$ depends on γ_{03}, γ_{23}, and γ_{14}; while, d_i $(i = 0,1,2,3)$ depends on γ_{23} only. Since $Z^4 = c_0 Z + c_1 Z^2 + c_2 Z^3 + c_3 \bar{Z} Z^2$ and $\bar{Z} Z^3 = d_0 Z + d_1 Z^2 + d_2 Z^3 + d_3 \bar{Z} Z^2$, we have

$$\gamma_{04} := c_0\gamma_{01} + c_1\gamma_{02} + c_2\gamma_{03} + c_3\gamma_{12}, \tag{5.2a}$$

$$\gamma_{34} := c_0\gamma_{31} + c_1\gamma_{32} + c_2\gamma_{33} + c_3\gamma_{42}, \tag{5.2b}$$

$$\gamma_{25} := c_0\gamma_{22} + c_1\gamma_{23} + c_2\gamma_{24} + c_3\gamma_{33}. \tag{5.2c}$$

Suppose

$$\bar{Z}^2 Z^2 = e_0 \mathit{1} + e_1 Z + e_2 Z^2 + e_3 \bar{Z} Z \quad \text{for some} \quad e_i \in \mathbb{C}. \tag{5.3}$$

Then

$$e_0 = \frac{1}{\Delta_4}\begin{vmatrix} \gamma_{22} & \gamma_{01} & \gamma_{02} & \gamma_{11} \\ \gamma_{32} & \gamma_{11} & \gamma_{12} & \gamma_{21} \\ \gamma_{42} & \gamma_{21} & \gamma_{22} & \gamma_{31} \\ \gamma_{33} & \gamma_{12} & \gamma_{13} & \gamma_{22} \end{vmatrix}, \quad e_1 = \frac{1}{\Delta_4}\begin{vmatrix} \gamma_{00} & \gamma_{22} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{32} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{42} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{33} & \gamma_{13} & \gamma_{22} \end{vmatrix},$$

$$e_2 = \frac{1}{\Delta_4}\begin{vmatrix} \gamma_{00} & \gamma_{01} & \gamma_{22} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{32} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{42} & \gamma_{31} \\ \gamma_{11} & \gamma_{12} & \gamma_{33} & \gamma_{22} \end{vmatrix}, \quad e_3 = \frac{1}{\Delta_4}\begin{vmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{22} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{32} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{42} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{33} \end{vmatrix}.$$

Note that e_i $(i = 0,1,2,3)$ depends on γ_{23}, γ_{24}, and γ_{33}.

Comparing the columns of $E(4)$, we have the following.

Proposition 5.1. *Assume that $E(2)$ is positive and invertible. Then $E(2)$ has a double flat extension $E(4)$ if and only if*

$$\begin{aligned} c_0\gamma_{30} + c_1\gamma_{31} + c_2\gamma_{32} + c_3\gamma_{41} &= d_0\gamma_{21} + d_1\gamma_{22} + d_2\gamma_{23} + d_3\gamma_{32}, \\ c_0\gamma_{41} + c_1\gamma_{42} + c_2\gamma_{43} + c_3\gamma_{52} &= d_0\gamma_{32} + d_1\gamma_{33} + d_2\gamma_{34} + d_3\gamma_{43}, \\ d_0\gamma_{32} + d_1\gamma_{33} + d_2\gamma_{34} + d_3\gamma_{43} &= e_0\gamma_{22} + e_1\gamma_{23} + e_2\gamma_{24} + e_3\gamma_{33}, \\ c_0\gamma_{32} + c_1\gamma_{33} + c_2\gamma_{34} + c_3\gamma_{43} &= d_0\gamma_{23} + d_1\gamma_{24} + d_2\gamma_{25} + d_3\gamma_{34}. \end{aligned} \tag{5.4}$$

Corollary 5.2. *Let*

$$E(2) = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ a & 0 & 0 & b \end{pmatrix} \qquad a > 0, \ \ b > a^2.$$

Then $E(2)$ is positive and invertible and has a double flat extension $E(4)$.

Proof. For $E(4)$, if we let $\gamma_{23} = 0, \gamma_{03} = x + yi$ and $\gamma_{14} = s + ti$, then we can calculate

$$\begin{aligned} c_0 &= \frac{as - xb + i(at - yb)}{-b + a^2}, \ \ c_1 = c_2 = 0, \ \ c_3 = \frac{(ax - s) + i(ay + t)}{-b + a^2}, \\ d_0 &= d_2 = d_3 = 0, \ \ \text{and} \ \ d_1 = \frac{1}{a} b. \end{aligned}$$

Define $\gamma_{24} = 0, \gamma_{33} = \frac{1}{a} b^2$, so $e_0 = e_1 = e_2 = 0, e_3 = \frac{1}{a} b$. Define $\gamma_{34} = 0, \gamma_{25} = (s + it)\frac{b}{a}$. Then it follows from (5.4) that $(x - \frac{as}{b})^2 + (y - \frac{at}{b})^2 = 0$ and $s^2 + t^2 = \frac{b^3}{a}$. Therefore (5.4) has a solution $\gamma_{23} = 0, \ \gamma_{03} = \sqrt{ab} e^{i\theta}$, and $\gamma_{14} = \sqrt{\frac{b^3}{a}} e^{i\theta}$. Thus we have this corollary. □

Algorithm 5.3. (I) Calculate $c_i, d_i, i = 0, 1, 2, 3$;

(II) Define γ_{24}, γ_{33} as

$$\begin{aligned} \gamma_{24} &:= c_0\gamma_{21} + c_1\gamma_{22} + c_2\gamma_{23} + c_3\gamma_{32}, \\ \gamma_{33} &:= c_0\gamma_{30} + c_1\gamma_{31} + c_2\gamma_{32} + c_3\gamma_{41}, \ \text{or} \\ &:= d_0\gamma_{21} + d_1\gamma_{22} + d_2\gamma_{23} + d_3\gamma_{32}; \end{aligned}$$

(III) Calculate $e_i, i = 0, 1, 2, 3$;

(IV) Define γ_{34}, γ_{25} as

$$\begin{aligned} \gamma_{34} &:= c_0\gamma_{31} + c_1\gamma_{32} + c_2\gamma_{33} + c_3\gamma_{42}, \\ \gamma_{25} &:= c_0\gamma_{22} + c_1\gamma_{23} + c_2\gamma_{24} + c_3\gamma_{33}; \end{aligned}$$

(V) Solve (5.4) with respect to $\gamma_{03}, \gamma_{23}, \gamma_{14}$;

(VI) If (5.4) has a solution, then go to the next step;

(VII) Define $\gamma_{04}, \gamma_{15}, \gamma_{44}, \gamma_{35}, \gamma_{26}$ as

$$\begin{aligned} \gamma_{04} &:= c_0\gamma_{01} + c_1\gamma_{02} + c_2\gamma_{03} + c_3\gamma_{12}, \\ \gamma_{15} &:= c_0\gamma_{12} + c_1\gamma_{13} + c_2\gamma_{14} + c_3\gamma_{23}, \\ \gamma_{44} &:= c_0\gamma_{41} + c_1\gamma_{42} + c_2\gamma_{43} + c_3\gamma_{52}, \\ \gamma_{35} &:= c_0\gamma_{32} + c_1\gamma_{33} + c_2\gamma_{34} + c_3\gamma_{43}, \\ \gamma_{26} &:= c_0\gamma_{23} + c_1\gamma_{24} + c_2\gamma_{25} + c_3\gamma_{34}; \end{aligned}$$

(VIII) We obtain a double flat extension $E(4)$ of $E(2)$.

Example 5.4. Let us consider a quartic moment matrix

$$E(2)=\begin{pmatrix}1&0&0&\frac{1}{2}\\0&\frac{1}{2}&0&0\\0&0&\frac{1}{2}&0\\\frac{1}{2}&0&0&\frac{1}{2}\end{pmatrix}$$

Since $\Delta_1=1,\Delta_2=\frac{1}{2},\Delta_3=\frac{1}{4}$, and $\Delta_4=\frac{1}{16}$, it is obvious that $E(2)$ is positive and invertible. We know $E(2)$ has a representing measure by Corollary 5.2. Here we find concrete representing measure according to Algorithm 5.3.

(I) $c_0=2\gamma_{03}-2\gamma_{14},c_1=0,c_2=2\gamma_{23},c_3=-2\gamma_{03}+4\gamma_{14}$, and $d_0=-2\gamma_{23},d_1=1,d_2=2\gamma_{32},d_3=4\gamma_{23}$.

(II) Define γ_{24},γ_{33} as $\gamma_{24}:=2\gamma_{23}^2+(-2\gamma_{03}+4\gamma_{14})\gamma_{32}$, $\gamma_{33}:=\frac{1}{2}+6\gamma_{32}\gamma_{23}$.

(III) $e_0=-12\gamma_{32}\gamma_{23}$, $e_1=2\gamma_{32}$, $e_2=4\gamma_{32}^2-4\gamma_{23}\gamma_{30}+8\gamma_{23}\gamma_{41}$, and $e_3=24\gamma_{32}\gamma_{23}+1$.

(IV) Define γ_{34},γ_{25} as

$$\begin{aligned}\gamma_{34}&:=2\gamma_{23}(\frac{1}{2}+2\gamma_{32}\gamma_{23}+4\gamma_{23}\gamma_{32})\\&\quad+(-2\gamma_{03}+4\gamma_{14})(2\gamma_{32}^2+(-2\gamma_{30}+4\gamma_{41})\gamma_{23}),\\\gamma_{25}&:=\gamma_{03}-\gamma_{14}+2\gamma_{23}\left(2\gamma_{23}^2+(-2\gamma_{03}+4\gamma_{14})\gamma_{32}\right)\\&\quad+(-2\gamma_{03}+4\gamma_{14})\left(\frac{1}{2}+6\gamma_{32}\gamma_{23}\right).\end{aligned}$$

(V) By Proposition 5.1, $E(2)$ admits a double flat extension $E(4)$ if and only if

$$\gamma_{30}\gamma_{03}-\gamma_{30}\gamma_{14}-\gamma_{41}\gamma_{03}+2\gamma_{41}\gamma_{14}=\frac{1}{4}+2\gamma_{23}\gamma_{32}.$$

and $F_i(\gamma_{03},\gamma_{14},\gamma_{23})=0,i=1,2$, where

$$\begin{aligned}&F_1(\gamma_{03},\gamma_{14},\gamma_{23})\\=\;&-8\gamma_{32}\gamma_{23}+2\gamma_{41}\gamma_{14}-48\gamma_{32}^2\gamma_{23}^2+8\gamma_{30}\gamma_{23}^3-16\gamma_{41}\gamma_{23}^3\\&-\frac{1}{2}+16\gamma_{23}\gamma_{30}\gamma_{32}\gamma_{03}-32\gamma_{23}\gamma_{30}\gamma_{32}\gamma_{14}\\&-32\gamma_{23}\gamma_{41}\gamma_{32}\gamma_{03}-64\gamma_{23}\gamma_{41}\gamma_{32}\gamma_{14},\end{aligned}$$

and

$$\begin{aligned}&F_2(\gamma_{03},\gamma_{14},\gamma_{23})\\=\;&-4\gamma_{32}\gamma_{23}-80\gamma_{32}^2\gamma_{23}^2-8\gamma_{30}\gamma_{23}^3+16\gamma_{41}\gamma_{23}^3\\&+16\gamma_{23}\gamma_{30}\gamma_{32}\gamma_{03}-32\gamma_{23}\gamma_{30}\gamma_{32}\gamma_{14}\\&-32\gamma_{23}\gamma_{41}\gamma_{32}\gamma_{03}+64\gamma_{23}\gamma_{41}\gamma_{32}\gamma_{14}.\end{aligned}$$

If $\gamma_{23}=0$, then $F_1(\gamma_{03},\gamma_{14},\gamma_{23})=2\gamma_{41}\gamma_{14}-\frac{1}{2}$, and $F_2(\gamma_{03},\gamma_{14},\gamma_{23})=0$. Thus $|\gamma_{14}|=\frac{1}{2}$ and $\gamma_{30}\gamma_{03}-\gamma_{30}\gamma_{14}-\gamma_{41}\gamma_{03}=-\frac{1}{4}$. If we let $\gamma_{14}=\frac{1}{2}i$, then we have $\gamma_{03} = \frac{1}{2}i$. Thus we obtain a solution of (5.4), $\gamma_{23} = 0$ and $\gamma_{14} = \gamma_{03} = \frac{1}{2}i$.

(VI) Define $\gamma_{04} = 0, \gamma_{15} = 0, \gamma_{44} = \frac{1}{2}, \gamma_{35} = 0$, and $\gamma_{26} = 0$;

(VII) Finally, we obtain a double flat extension of $E(2)$ as following

$$DF = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}i & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2}i & 0 & 0 & -\frac{1}{2}i & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2}i \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}i & 0 & 0 \\ 0 & -\frac{1}{2}i & 0 & 0 & 0 & -\frac{1}{2}i & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

(VIII) The four atoms are $z_0 = 0, z_1 = \frac{1}{2}i + \frac{1}{2}\sqrt{3}, z_2 = \frac{1}{2}i - \frac{1}{2}\sqrt{3}$, and $z_3 = -i$. We know that z_0 is the center of unit disc $\mathbb{D}$ and z_1, z_2, z_3 are on the unit circle $\mathbb{T}$. Since $\gamma_{03} = \frac{1}{2}i$, then $\rho_0 = \frac{1}{2}, \rho_1 = \rho_2 = \rho_3 = \frac{1}{6}$. Therefore, one of the representing measure is

$$\mu = \frac{1}{2}\delta_0 + \frac{1}{6}(\delta_{\frac{1}{2}i+\frac{1}{2}\sqrt{3}} + \delta_{\frac{1}{2}i-\frac{1}{2}\sqrt{3}} + \delta_{-i}).$$

We close this article as the following open problem.

Problem 5.5. *If $E(2)$ is positive and invertible, does it have a double flat extension $E(4)$?*

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