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A LIOUVILLE TYPE THEOREM FOR HARMONIC MORPHISMS

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Abstract. Let M be a complete Riemannian manifold and let N be a Riemannian manifold of nonpositive scalar curvature. Let μ_0 be the least eigenvalue of the Laplacian acting on L^2-functions on M. We show that if $Ric^M \geq -\mu_0$ at all $x \in M$ and either $Ric^M > -\mu_0$ at some point x_0 or $\text{Vol}(M)$ is infinite, then every harmonic morphism $\phi : M \to N$ of finite energy is constant.

1. Introduction

Let (M, g) and (N, h) be smooth Riemannian manifolds and let $\phi : M \to N$ be a smooth map. For a compact domain $\Omega \subset M$, the *energy* E of ϕ over Ω is defined by

$$E(\phi;\Omega) = \frac{1}{2}\int_{\Omega} |d\phi|^2 \mu_M, \tag{1.1}$$

where the differential $d\phi$ is a section of the bundle $T^*M \otimes \phi^{-1}TN \to M$ and $\phi^{-1}TN$ denotes the pull-back bundle via the map ϕ. The bundle $T^*M \otimes \phi^{-1}TN \to M$ carries the connection ∇ induced by the Levi-Civita connections on M and N.

A map $\phi : M \to N$ is called *harmonic* if ϕ is a critical point of the energy functional defined by (1.1) on any compact domain $\Omega \subset M$, or equivalently the *tension field* $\tau(\phi) = \text{tr}_g \nabla d\phi$ is identically zero, where tr_g denote the trace with respect to the metric g. Several studies are given for harmonic maps ([3]). For these harmonic maps, there are Liouville type theorems, which states that a harmonic map ϕ is constant under some conditions. The classical Liouville theorem says that any bounded harmonic function defined on the whole plane must be constant. In 1975, S. T. Yau ([10]) generalized the Liouville theorem to harmonic functions on Riemannian manifolds of nonnegative Ricci curvature. In 1976, R. M. Schoen and S. T. Yau ([8]) proved the following theorem.

Theorem 1.1. ([8]) *Let $\phi : M \to N$ be a harmonic map from a complete, noncompact Riemannian manifold M with nonnegative Ricci curvature to a*

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complete Riemannian manifold N with nonpositive sectional curvature. If the energy of ϕ is finite, then ϕ is constant.

In 1997, S. D. Jung ([5]) improved Theorem 1.1 to harmonic maps on a complete Riemannian manifold M, where the Ricci curvature Ric^M is bounded from below by a negative constant. In fact, let μ_0 be the least eigenvalue of the Laplacian Δ^M acting on L^2-functions on the manifold M. Then

Theorem 1.2. ([5]) *Let $\phi : M \to N$ be a harmonic map from a complete Riemannian manifold M to a Riemannian manifold N with nonpositive sectional curvature. Assume $Ric^M \geq -\mu_0$ at all $x \in M$ and $Ric^M > -\mu_0$ at some point x_0. If the energy of ϕ is finite, then ϕ is constant.*

A C^0 map $\phi : M \to N$ is called a *harmonic morphism* if for any harmonic function $f : U \to \mathbb{R}$ on an open set $U \subset N$ such that $\phi^{-1}(U)$ is nonempty, the composition $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is also a harmonic function on $\phi^{-1}(U)$.

As a generalization of Riemannian submersions, a *horizontally weakly conformal* map is a map $\phi : (M, g) \to (N, h)$ with the property that for each $x \in M$ at which $d\phi_x \neq 0$, the restriction $d\phi_x|_{H_x} : H_x \to T_{\phi(x)}N$ is conformal and surjective, where H_x denotes the orthogonal complement of $V_x = \ker d\phi_x$ in T_xM. We call H_x the horizontal and V_x the vertical space of ϕ at x. Thus $T_xM = V_x \oplus H_x$. Let $C_\phi = \{x \in M | d\phi_x = 0\}$. Trivially, ϕ is horizontally weakly conformal if and only if there exists a function $\lambda : M \backslash C_\phi \to \mathbb{R}^+$ such that

$$h(d\phi(X), d\phi(Y)) = \lambda^2 g(X, Y) \quad \forall X, Y \in H_x. \tag{1.2}$$

Note that at the point $x \in C_\phi$ we can let $\lambda(x) = 0$ and obtain a continuous function $\lambda : M \to \mathbb{R}^+ \cup \{0\}$ which is called the *dilation* of a horizontally weakly conformal map ϕ.

It is well-known ([4]) that a smooth map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is a harmonic morphism if any only if it is harmonic and horizontally weakly conformal. It is also well-known ([4]) that if $\dim(M) < \dim(N)$, then every harmonic morphism must be constant.

For the Liouville type theorem for harmonic morphisms in case of $\dim M \geq \dim N$, G. Choi and G. Yun ([2]) recently proved the following theorem.

Theorem 1.3. ([2]) *Let $\phi : M \to N$ be a harmonic morphism from a complete, noncompact Riemannian manifold M of nonnegative Ricci curvature to a complete Riemannian manifold N with nonpositive scalar curvature. If the energy of ϕ is finite, then ϕ is constant.*

In this paper, we give extension of Theorem 1.3 to manifolds, where the Ricci curvature of M is bounded from below by $-\mu_0$. That is, our main theorem is the following:

Theorem 1.4. *Let $\phi : M \to N$ be a harmonic morphism from a complete, noncompact Riemannian manifold M to a complete Riemannian manifold N*

with nonpositive scalar curvature. Assume that $Ric^M \geq -\mu_0$ at all $x \in M$ and either $Ric^M > -\mu_0$ at some point x_0 or $\mathrm{Vol}(M)$ is infinite. If the energy of ϕ is finite, then ϕ is constant.

2. The Weitzenböck formula

In this section, we review the Weitzenböck formula (see [7, 9]). Let (M^m, g) and (N^n, h) be Riemannian manifolds and let ∇^M and ∇^N be their Levi-Civita connections respectively. Let $\phi : M \to N$ be a smooth map and $E = \phi^{-1}TN$ be the induced bundle over M. Then E has a naturally induced metric connection $\nabla \equiv \phi^{-1}\nabla^N$ and $d\phi$ is a cross section of $Hom(TM, E)$ over M. Since $Hom(TM, E)$ is canonically identified with $T^*M \otimes E$, $d\phi$ is regarded as an E-valued 1-form. Let $d_\nabla : A^r(E) \to A^{r+1}(E)$ be an anti-derivation and δ_∇ the formal adjoint of d_∇, where $A^r(E)$ is the space of E-valued r-forms with an inner product $\langle \cdot, \cdot \rangle$ on M. Let $\{e_i\}_{i=1,\dots,m}$ and $\{v_a\}_{a=1,\dots,n}$ be local orthonormal frame fields on M and N respectively, and let $\{\omega^i\}$ and $\{\theta^a\}$ be their dual coframe fields respectively. Locally, the operators d_∇ and δ_∇ are expressed by

$$d_\nabla = \sum_{j=1}^{m} \omega^j \wedge \nabla_{e_j} \quad \text{and} \quad \delta_\nabla = -\sum_{j=1}^{m} i(e_j)\nabla_{e_j},$$

respectively, where $i(X)$ is the interior product. The Laplacian Δ on $A^*(E)$ is defined by

$$\Delta = d_\nabla \delta_\nabla + \delta_\nabla d_\nabla. \tag{2.1}$$

Then the Weitzenböck formula is given by

$$\Delta = -\sum_j \nabla^2_{e_j e_j} + \sum_{k,j} \omega^k \wedge i(e_j) R(e_j, e_k), \tag{2.2}$$

where $\nabla^2_{XY} = \nabla_X \nabla_Y - \nabla_{\nabla^M_X Y}$ and $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for any $X, Y \in TM$. From (2.2), we have that for any $\Phi \in A^r(E)$,

$$-\frac{1}{2}\Delta^M |\Phi|^2 = |\nabla\Phi|^2 + \langle \sum_j \nabla^2_{e_j e_j}\Phi, \Phi \rangle. \tag{2.3}$$

Equivalently,

$$-\frac{1}{2}\Delta^M |\Phi|^2 = |\nabla\Phi|^2 - \langle \Delta\Phi, \Phi \rangle + \sum_{k,j} \langle \omega^k \wedge i(e_j) R(e_j, e_k)\Phi, \Phi \rangle. \tag{2.4}$$

Let R^E be the curvature tensor of ∇ on E. Then R^E is related to the curvature tensor R^N of ∇^N in the following way: let $X, Y \in T_xM$ and $s \in \Gamma E$, then

$$R^E(X,Y)s = R^N(d\phi_x(X), d\phi_x(Y))s. \tag{2.5}$$

When a function f is given on N, we shall identify it throughout this paper with the function $f \circ \phi$ induced on M. Let $f^a \equiv \phi^*\theta^a$. Then $d\phi$ is expressed by

$$d\phi = \sum_{a=1}^{n} f^a \otimes v_a. \tag{2.6}$$

Since a direct calculation gives

$$R(e_j, e_k)d\phi = \sum_a R^M(e_j, e_k)f^a \otimes v_a + \sum_a f^a \otimes R^E(e_j, e_k)v_a, \tag{2.7}$$

we have

$$\begin{aligned}\sum_{k,j}\langle \omega^k \wedge i(e_j)R(e_j, e_k)d\phi, d\phi\rangle = &\sum_{k,j,a,b} \langle \omega^k \wedge i(e_j)R^M(e_j, e_k)f^a \otimes v_a, f^b \otimes v_b\rangle \\ &+ \sum_{k,j,a,b} g(\omega^k \wedge i(e_j)f^a, f^b)h(R^E(e_j, e_k)v_a, v_b).\end{aligned}$$

Since $d\phi(e_l) = \sum_a f^a(e_l)v_a$, we have

$$\sum_{k,j,a} g(\omega^k \wedge i(e_j)R^M(e_j, e_k)f^a, f^a) = \sum_k h(d\phi(Ric^M(e_k)), d\phi(e_k)). \tag{2.8}$$

From (2.5) and (2.8), we have

(2.9)
$$\begin{aligned}\sum_{k,j}\langle \omega^k \wedge i(e_j)R(e_j, e_k)d\phi, d\phi\rangle =&\sum_k h(d\phi(Ric^M(e_k)), d\phi(e_k)) \\ &+\sum_{k,j} h(R^N(d\phi(e_j), d\phi(e_k))d\phi(e_j), d\phi(e_k)).\end{aligned}$$

Hence we have the following lemma.

Lemma 2.1. ([7]) *Let $\phi : (M, g) \to (N, h)$ be an arbitrary smooth map. Then the Weitzenböck formula is given by*

$$-\frac{1}{2}\Delta^M|d\phi|^2 = |\nabla d\phi|^2 - \langle d\phi, \Delta d\phi\rangle + F(\phi), \tag{2.10}$$

where

$$\begin{aligned}F(\phi) =&\sum_{k=1}^{m} h(d\phi(Ric^M(e_k)), d\phi(e_k)) \\ &- \sum_{k,j=1}^{m} h(R^N(d\phi(e_j), d\phi(e_k))d\phi(e_k), d\phi(e_j)).\end{aligned} \tag{2.11}$$

3. Proof of Theorem 1.4

Assume that $\dim M = m \geq n = \dim N$. Let $\phi : M \to N$ be a harmonic morphism and λ the dilation of ϕ. Let $\{e_i\}_{i=1,\dots,m}$ be a local orthonormal frame field on M such that $\{e_i\} \in H_x$ $(i = 1, \dots, n)$ and $\{e_{n+i}\} \in V_x$ $(i = 1, \dots, m-n)$. Note that for any harmonic map, $d_\nabla(d\phi) = \delta_\nabla(d\phi) = 0$ ([3]).

From (1.2) and (2.10), we have the following lemma.

Lemma 3.1. ([6]) *If $\phi : M \to N$ is a harmonic morphism, then*

$$-\frac{n}{2}\Delta^M \lambda^2 = |\nabla d\phi|^2 + \lambda^2 \mathrm{tr}Ric^M|_{\mathcal{H}} - \lambda^4 r_N \circ \phi, \tag{3.1}$$

where λ denotes the dilation, $\mathrm{tr}Ric^M|_{\mathcal{H}}$ the trace of the Ricci tensor of M on the horizontal distribution $\mathcal{H}$, and r_N the scalar curvature of N.

Let μ_0 be the least eigenvalue of Δ^M acting on L^2-functions on M. Then we have the following lemma.

Lemma 3.2. *Let M be a complete Riemannian manifold such that $Ric^M \geq -\mu_0$ at all $x \in M$ and let N be a Riemannian manifold of nonpositive scalar curvature. If $\phi : M \to N$ is a harmonic morphism, then*

$$n\Delta^M \lambda \leq -\lambda \mathrm{tr}Ric^M|_{\mathcal{H}} \leq n\mu_0 \lambda. \tag{3.2}$$

Proof. Since $\Delta^M \lambda^2 = 2\lambda\Delta^M\lambda - 2|\nabla^M \lambda|^2$, we have from (3.1),

$$n\lambda\Delta^M\lambda = n|\nabla^M\lambda|^2 - |\nabla d\phi|^2 - \lambda^2 \mathrm{tr}Ric^M|_{\mathcal{H}} + \lambda^4 r_N \circ \phi. \tag{3.3}$$

Since $|d\phi|^2 = n\lambda^2$, we have $|d\phi|\nabla^M|d\phi| = n\lambda\nabla^M\lambda$ and

$$|\nabla^M|d\phi||^2 = n|\nabla^M\lambda|^2. \tag{3.4}$$

By the first Kato's inequality ([1]), i.e., $|\nabla^M|d\phi||^2 \leq |\nabla d\phi|^2$, (3.4) yields

$$n|\nabla^M\lambda|^2 \leq |\nabla d\phi|^2. \tag{3.5}$$

Since the scalar curvature r_N of N is nonpositive, the first inequality of (3.2) follows from (3.3) and (3.5). The second inequality of (3.2) is trivial from $Ric^M \geq -\mu_0$. □

Proof of Theorem 1.4. We choose a Lipschitz continuous function ω_ℓ on M such that $\omega_\ell \in C_0^\infty(M)$ and $\omega_\ell \equiv 1$ on $B(x_0, \ell)$, $\lim_{\ell\to\infty} \omega_\ell = 1$, supp $\omega_\ell \subset B(x_0, 2\ell)$ and $|d\omega_\ell| \leq C/\ell$ for some constant C, where $\ell \in \mathbb{R}^+$ and $B(x_0, \ell)$ is the Riemannian open ball with radius ℓ.

Multiplying (3.2) by $\omega_\ell^2\lambda$ and integrating by parts, we obtain

$$n\int_M \langle d\lambda, d(\omega_\ell^2\lambda)\rangle \leq -\int_M \omega_\ell^2\lambda^2 \mathrm{tr}Ric^M|_{\mathcal{H}} \leq n\mu_0 \int_M (\omega_\ell\lambda)^2. \tag{3.6}$$

By a direct calculation, we have

$$\begin{aligned} \langle d\lambda, d(\omega_\ell^2\lambda)\rangle &= 2\omega_\ell\lambda\langle d\lambda, d\omega_\ell\rangle + |\omega_\ell d\lambda|^2 \\ &= |d(\omega_\ell\lambda)|^2 - \lambda^2|d\omega_\ell|^2. \end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we have

$$\int_M |d(\omega_\ell\lambda)|^2 \le -\frac{1}{n}\int_M \omega_\ell^2\lambda^2 \mathrm{tr}Ric^M|_{\mathcal{H}} + \int_M \lambda^2|d\omega_\ell|^2 \le \mu_0\int_M(\omega_\ell\lambda)^2 + \int_M \lambda^2|d\omega_\ell|^2. \tag{3.8}$$

Since μ_0 is the infimum of the spectrum of the Laplacian Δ^M acting on L^2-functions on M, the Rayleigh theorem implies

$$\int_M |d(\omega_\ell\lambda)|^2 \ge \mu_0\int_M(\omega_\ell\lambda)^2. \tag{3.9}$$

If we let $\ell \to +\infty$ in (3.8) with (3.9), then we have

$$\mu_0\int_M \lambda^2 \le -\frac{1}{n}\int_M \lambda^2\mathrm{tr}Ric^M|_{\mathcal{H}} \le \mu_0\int_M\lambda^2. \tag{3.10}$$

This means that

$$0 = \int_M (n\mu_0 + \mathrm{tr}Ric^M|_{\mathcal{H}})\lambda^2 = \frac{1}{n}\int_M (n\mu_0 + \mathrm{tr}Ric^M|_{\mathcal{H}})|d\phi|^2. \tag{3.11}$$

If $Ric^M \ge -\mu_0$ at all x and $Ric^M > -\mu_0$ at some x_0, then $n\mu_0 + \mathrm{tr}Ric^M|_{\mathcal{H}} \ge 0$ for all x and $n\mu_0 + \mathrm{tr}Ric^M|_{\mathcal{H}} > 0$ at some point x_0, respectively. The unique continuation property for sections implies $|d\phi| = 0$, i.e., ϕ is constant.

Now we study Theorem 1.4 under the assumption $Ric^M \ge -\mu_0$ and $\mathrm{Vol}(M) = \infty$. We first note that for any real number $\delta > 0$

$$|2\int_M \omega_\ell\lambda\langle d\lambda, d\omega_\ell\rangle| \le \delta^2\int_M \omega_\ell^2|d\lambda|^2 + \frac{1}{\delta^2}\int_M\lambda^2|d\omega_\ell|^2. \tag{3.12}$$

From (3.6), (3.7) and (3.12), we have

$$(1-\delta^2)\int_M \omega_\ell^2|d\lambda|^2 - \frac{1}{\delta^2}\int_M \lambda^2|d\omega_\ell|^2 \le -\frac{1}{n}\int_M \omega_\ell^2\lambda^2\mathrm{tr}Ric^M|_{\mathcal{H}} \le \mu_0\int_M(\omega_\ell\lambda)^2. \tag{3.13}$$

From (3.13), Fatou's lemma implies that $d\lambda$ is L^2-section. Hence if we choose $\delta = \frac{1}{\sqrt{\ell}}$ and let $\ell \to +\infty$, then

$$\int_M |d\lambda|^2 \le -\frac{1}{n}\int_M \lambda^2\mathrm{tr}Ric^M|_{\mathcal{H}} \le \mu_0\int_M\lambda^2. \tag{3.14}$$

On the other hand, from (3.7) and (3.12) we similarly obtain

$$(1+\delta^2)\int_M \omega_\ell^2|d\lambda|^2 \ge \int_M |d(\omega_\ell\lambda)|^2 - (1+\frac{1}{\delta^2})\int_M\lambda^2|d\omega_\ell|^2. \tag{3.15}$$

If we put $\delta = \frac{1}{\sqrt{\ell}}$ and let $\ell \to +\infty$, then we have from (3.9)

$$\int_M |d\lambda|^2 \ge \mu_0\int_M\lambda^2. \tag{3.16}$$

From (3.14) and (3.16), we have $\int_M (\Delta^M \lambda - \mu_0 \lambda)\lambda = 0$. Hence (3.2) implies that $\Delta^M \lambda = \mu_0 \lambda$. This means that λ is nonnegative L^2-subharmonic function. By the maximum principle ([11]), λ is constant. Since $\mathrm{Vol}(M) = \infty$, it is trivial that $\lambda = 0$, which yields that ϕ is constant. □

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