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HOLOMORPHIC FUNCTIONS SATISFYING MEAN LIPSCHITZ CONDITION IN THE BALL

Ern Gun Kwon*, Hyung Woon Koo, and Hong Rae Cho

Abstract. Holomorphic mean Lipschitz space is defined in the unit ball of $\mathbb{C}^n$. The membership of the space is expressed in terms of the growth of radial derivatives, which reduced to a classical result of Hardy and Littlewood when $n = 1$. The membership is also expressed in terms of the growth of tangential derivatives when $n \geq 2$.

I. Introduction

Let $B = B_n$ be the open unit ball of $\mathbb{C}^n$ which will be denoted by D when $n = 1$. Let S be the boundary of B, and ν and σ denote the Lebesgue measure on $\mathbb{C}^n$ and the surface area measure on S respectively, normalized to be $\nu(B) = 1$ and $\sigma(S) = 1$.

For $0 < \epsilon < 1$, we say that $f \in Lip_\epsilon(B)$ if f is holomorphic in B, continuous in $\bar{B}$, and satisfies the Lipschitz condition of order ϵ : $|f(z) - f(w)| = O(|z - w|^\epsilon)$, $z, w \in \bar{B}$.

Let $H^p(B)$, $1 \leq p < \infty$, denote the Hardy space in B. It consists of holomorphic f in B for which $\|f\|_{H^p} = \lim_{r\to 1} M_p(r, f) < \infty$, where

$$M_p(r,f) = \left(\int_S |f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p}.$$

When $n = 1$, the holomorphic mean Lipschitz space $Lip_\epsilon^p(D)$, $1 \leq p < \infty$, $0 < \epsilon < 1$, is defined to consist of $f \in H^p(D)$ satisfying the mean Lipschitz condition :

$$\left(\int_0^{2\pi} |f(e^{i\theta}) - f(e^{i(\theta+h)})|^p \frac{d\theta}{2\pi} \right)^{1/p} = O(|h|^\epsilon). \tag{1.1}$$

We in this note extend the definition of the holomorphic mean Lipschitz space to the unit ball of $\mathbb{C}^n$. Then we give a characterization of the membership of the space in terms of the mean growth of the concerning derivatives, which

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generalizes a classical result of Hardy and Littlewood. We adapt in this paper the definition of $Lip_\epsilon^p(B)$ as follows.

Definition. For $0 < \epsilon < 1$ and $1 \le p < \infty$, we say that $f \in Lip_\epsilon{}^p(B)$ if $f \in H^p(B)$ and

$$\left(\int_S |f(U\zeta) - f(\zeta)|^p \, d\sigma(\zeta)\right)^{1/p} = O(|h|^\epsilon) \tag{1.2}$$

for all unitary operators U of $\mathbb{C}^n$, where $h = (h_1, h_2, \dots, h_n) \in \mathbb{R}^n$ is determined to be that $e^{ih_1}, e^{ih_2}, \dots, e^{ih_n}$ are the eigenvalues of U.

Note that in the right side of (1.2) $|h|^\epsilon = \left(\sum_{j=1}^n |h_j|^2\right)^{\epsilon/2}$. When $n = 1$, this definition reduces to (1.1). $Lip_\epsilon^p(B)$ is a Banach space with the norm

$$\|f\|_{H^p} + \sup \frac{1}{|h|^\epsilon}\left(\int_S |f(U\zeta) - f(\zeta)|^p \, d\sigma(\zeta)\right)^{1/p},$$

where the supremum is taken with respect to the unitary operators U and h is determined as in (1.2).

As far as the Lipschitz functions are concerned, the most interesting and dominant result may be the relationship with the growth of their derivatives. As is well-known, there are results of Hardy and Littlewood expressing the membership of $Lip_\epsilon(D)$ and $Lip_\epsilon^p(D)$ in terms of the growth of the derivatives. That is,

$$f \in Lip_\epsilon(D) \iff M_\infty(r, f') = O\left((1-r)^{\epsilon-1}\right) \tag{1.3}$$

and

$$f \in Lip_\epsilon^p(D) \iff M_p(r, f') = O\left((1-r)^{\epsilon-1}\right) \tag{1.4}$$

for holomorphic f in D. See [2, Theorem 5.1] and [2, Theorem 5.4]. It also is known that (1.3) has an extension to $n > 1$, with $\mathcal{R}f$ in place of f' :

$$f \in Lip_\epsilon(B) \iff M_\infty(r, \mathcal{R}f) = O\left((1-r)^{\epsilon-1}\right),$$

where $\mathcal{R}f$, the radial derivative of f, is defined by $\mathcal{R}f = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$. See [4, Theorem 6.4.9 and Theorem 6.4.10]. Using $\mathcal{R}f$ in place of f', it is naturally called for to extend (1.4) to the case of $n \ge 1$, and our first result says, as supposed to be,

Theorem 1. *Let $0 < \epsilon < 1$ and $1 \le p < \infty$. If f is holomorphic in B, then*

$$f \in Lip_\epsilon^p(B) \iff M_p(r, \mathcal{R}f) = O\left((1-r)^{\epsilon-1}\right).$$

Next, we proceed to handle the mean p growth of multiple derivatives and of tangential derivatives. For $1 \le i < j \le n$, we define $\mathcal{T}_{ij}$ and $\overline{\mathcal{T}}_{ij}$ by

$$\mathcal{T}_{ij} = \bar{z}_i \partial_j - \bar{z}_j \partial_i, \qquad \overline{\mathcal{T}}_{ij} = z_i \overline{\partial}_j - z_j \overline{\partial}_i,$$

where $\partial_j = \frac{\partial}{\partial z_j}$ and $\overline{\partial}_j = \frac{\partial}{\partial \bar{z}_j}$. Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, we abuse the notation T^α to mean $T_{i_1 j_1}^{\alpha_1} \cdots T_{i_n j_n}^{\alpha_n}$ for some choice of $i_1, \dots, i_n$ and

$j_1, \dots, j_n$, where T_{ij} is either $\mathcal{T}_{ij}$ or $\overline{\mathcal{T}}_{ij}$. With these conventions, we can generalize Theorem 1 to the following

Theorem 2. *Let f be holomorphic in B and $0 < \epsilon < 1$. Then each of the following is equivalent to the membership $f \in Lip^p_\epsilon(B)$.*

1. *For some $k \geq 1$, $M_p(r, \mathcal{R}^k f) = O((1-r)^{\epsilon - k})$.*
2. *For all $k \geq 1$, $M_p(r, \mathcal{R}^k f) = O((1-r)^{\epsilon - k})$.*
3. *For some $k \geq 2$, $M_p(r, T^\alpha f) = O((1-r)^{\epsilon - k/2})$ for all multi-index α with $|\alpha| = k$.*
4. *For all $k \geq 2$, $M_p(r, T^\alpha f) = O((1-r)^{\epsilon - k/2})$ for all multi-index α with $|\alpha| = k$.*

Moreover, if $0 < \epsilon < \frac{1}{2}$, then these are equivalent to

(5) *$M_p(r, T^\alpha f) = O((1-r)^{\epsilon - 1/2})$ for all multi-index α with $|\alpha| = 1$.*

II. Proof of Theorem 1

Suppose $M_p(r, \mathcal{R}f) = O\left((1-r)^{\epsilon - 1}\right)$. By applying Minkowski's inequality to the relation

$$\begin{aligned} |f(r\zeta)| &\leq |f(0)| + \int_0^r \left| \frac{d}{ds}(f(s\zeta)) \right| ds \\ &\leq |f(0)| + \sup_{s \leq 1/2} \left| \frac{d}{ds}(f(s\zeta)) \right| + 2\int_{1/2}^1 |\mathcal{R}f(s\zeta)|\, ds, \end{aligned}$$

it is easy to see that $f \in H^p(B)$. In particular, $f(\zeta) = \lim_{\rho \to 1^-} f(\rho\zeta)$ exists for almost every $\zeta \in S$.

To prove mean Lipschitz condition, let U be a unitary operator of $\mathbb{C}^n$. Then there is another unitary operator V of $\mathbb{C}^n$ such that $V^{-1}UV = D$, where D is the diagonal matrix consisting of eigenvalues of U. So, by the unitary invariance of $d\sigma$

$$\begin{aligned} &\int_S |f(U\zeta) - f(\zeta)|^p\, d\sigma(\zeta) = \int_S |f(UV\zeta) - f(V\zeta)|^p\, d\sigma(\zeta) \\ &= \int_S |f(VD\zeta) - f(V\zeta)|^p\, d\sigma(\zeta) = \int_S |f \circ V(e^{ih}\zeta) - f \circ V(\zeta)|^p\, d\sigma(\zeta), \end{aligned}$$

where $e^{ih_j}, 1 \leq j \leq n$, are the eigenvalues of U and

$$e^{ih}\zeta = \left(e^{ih_1}\zeta_1, e^{ih_2}\zeta_2, \dots, e^{ih_n}\zeta_n\right).$$

If $|h| > \frac{1}{2}$, then it follows directly that

$$\left(\int_S |f(U\zeta) - f(\zeta)|^p\, d\sigma(\zeta)\right)^{1/p} \leq C\,|h|^\epsilon$$

by simply taking $C = 2^{1+\epsilon}\|f\|_{H^p}$.

Assume otherwise that $0 < |h| < \frac{1}{2}$. Taking $r = 1 - |h|$, Minkowski's inequality gives

$$\begin{aligned} &\left\{\int_S |f \circ V(e^{ih}\zeta) - f \circ V(\zeta)|^p \, d\sigma(\zeta)\right\}^{1/p} \\ &\leq \left\{\int_S |f \circ V(e^{ih}\zeta) - f \circ V(re^{ih}\zeta)|^p \, d\sigma(\zeta)\right\}^{1/p} \\ &\quad + \left\{\int_S |f \circ V(re^{ih}\zeta) - f \circ V(r\zeta)|^p d\sigma(\zeta)\right\}^{1/p} \\ &\quad + \left\{\int_S |f \circ V(r\zeta) - f \circ V(\zeta)|^p d\sigma(\zeta)\right\}^{1/p}, \end{aligned}$$

and the right hand side quantity of this inequality equals

$$\begin{aligned} &2\left\{\int_S |f(r\zeta) - f(\zeta)|^p \, d\sigma(\zeta)\right\}^{1/p} \\ &+ \left\{\int_S |f \circ V(re^{ih}\zeta) - f \circ V(r\zeta)|^p \, d\sigma(\zeta)\right\}^{1/p} \end{aligned} \tag{2.1}$$

by the unitary invariance of $d\sigma$ again. If we denote the slice function of f on $\zeta \in S$ by $f_\zeta(\lambda) = f(\lambda\zeta), \lambda \in D$, then since $\mathcal{R}f(\lambda\zeta) = \lambda f'_\zeta(\lambda)$ (see [4, 6.4.4]), it follows that

$$\left\{\int_S |f(r\zeta) - f(\zeta)|^p \, d\sigma(\zeta)\right\}^{1/p} = \left\{\int_S \lim_{\delta \to 1-} \left|\int_r^\delta f'_\zeta(\lambda) \, d\lambda\right|^p \, d\sigma(\zeta)\right\}^{1/p},$$

which is, by use of the continuous form of Minkowski's inequality, bounded by

$$\int_r^1 M_p(\rho, \mathcal{R}f) \frac{d\rho}{\rho}.$$

By the hypothesis on $\mathcal{R}f$, this quantity is bounded by a constant times $|h|^\epsilon$.

To bound the second term of (2.1), note that $\mathcal{R}f$ is invariant under unitary composition if f is holomorphic in B, that is,

$$(\mathcal{R}f)(Vz) = (\mathcal{R}(f \circ V))(z), \quad z \in B, \tag{2.2}$$

for any unitary operator V of $\mathbb{C}^n$. This is easy to verify by direct computation. Fix V and let $f \circ V = F$ for simplicity. Then by (2.2) and the unitary invariance of $d\sigma$,

$$\int_S |\mathcal{R}F(r\zeta)|^p \, d\sigma(\zeta) = \int_S |(\mathcal{R}f)(rV\zeta)|^p \, d\sigma(\zeta) = \int_S |\mathcal{R}f(r\zeta)|^p \, d\sigma(\zeta). \tag{2.3}$$

From the obvious inequality

$$\begin{aligned} |F(re^{ih}\zeta) - F(r\zeta)| &= \left| \int_0^1 \frac{d}{dt} \left(F(re^{ith}\zeta) \right) dt \right| \\ &= \left| \int_0^1 \sum_{j=1}^n \frac{\partial}{\partial z_j} F(re^{ith}\zeta)\, re^{ith_j}\zeta_j\, ih_j\, dt \right| \\ &\leq r|h| \int_0^1 \left| \nabla F(re^{ith}\zeta) \right| dt, \end{aligned}$$

it follows by using Minkowski's inequality that

$$\begin{aligned} &\left\{ \int_S \left| F(re^{ih}\zeta) - F(r\zeta) \right|^p d\sigma(\zeta) \right\}^{1/p} \\ (2.4) \qquad &\leq r|h| \left\{ \int_S \left(\int_0^1 \left| \nabla F(re^{ith}\zeta) \right| dt \right)^p d\sigma(\zeta) \right\}^{1/p} \\ &\leq r|h| \int_0^1 \left\{ \int_S \left| \nabla F(re^{ith}\zeta) \right|^p d\sigma(\zeta) \right\}^{1/p} dt = r|h|\, M_p(r, \nabla F). \end{aligned}$$

Here ∇ denotes the complex gradient : $\nabla = (\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n})$. It is easy to see

$$|z|^2 |\nabla F(z)|^2 = |\mathcal{R}F(z)|^2 + \sum_{j<k} |\mathcal{T}_{jk} F(z)|^2 \tag{2.5}$$

(see [3, p. 1389] for example). By (2.4) and (2.5), we have

$$\left\{ \int_S \left| F(re^{ih}\zeta) - F(r\zeta) \right|^p d\sigma(\zeta) \right\}^{1/p} \leq |h|\, M_p(r, \mathcal{R}F) + \sum_{j<k} |h|\, M_p(r, \mathcal{T}_{jk}F).$$

But it is known that

$$M_p(r, \mathcal{T}_{jk}F) \leq C M_p(r, \mathcal{R}F)$$

(see [1, p.146] for example). Therefore, by (2.3) and the hypothesis on $\mathcal{R}f$, the second term of (2.1) is bounded by $C\,|h|(1-r)^{\epsilon-1} = C|h|^{\epsilon}$.

Conversely, suppose $f \in Lip_\epsilon^p(B)$. For $0 < r < 1$, slice integration([4, Proposition 1.4.7]) gives,

$$M_p(r, \mathcal{R}f)^p = \int_S |\mathcal{R}f(r\zeta)|^p\, d\sigma(\zeta) = \int_S d\sigma(\zeta) \int_0^{2\pi} |\mathcal{R}f(re^{i\theta}\zeta)|^p \frac{d\theta}{2\pi}.$$

Since $f_\zeta \in H^1(D)$ almost every $\zeta \in S$, it follows by one variable Cauchy integral representation that

$$|\mathcal{R}f(re^{i\theta}\zeta)| \leq \left| f'_\zeta(re^{i\theta}) \right| \leq \int_0^{2\pi} \frac{|f(e^{it}\zeta) - f(e^{i\theta}\zeta)|}{|e^{it} - re^{i\theta}|^2} \frac{dt}{2\pi}$$

for almost every $\zeta \in S$. Here $e^{it}\zeta = \left(e^{it}\zeta_1, e^{it}\zeta_2, \dots, e^{it}\zeta_n\right)$. Hence

$$M_p(r, \mathcal{R}f) \le \left\{ \int_S \int_0^{2\pi} \left(\int_0^{2\pi} \frac{|f(e^{i(t+\theta)}\zeta) - f(e^{i\theta}\zeta)|}{|1 - re^{-it}|^2} \frac{dt}{2\pi} \right)^p \frac{d\theta}{2\pi} \, d\sigma(\zeta) \right\}^{1/p}.$$

The last quantity is bounded, by Minkowski's inequality, by

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|1 - re^{it}|^2} \left(\int_S \int_0^{2\pi} |f(e^{i(t+\theta)}\zeta) - f(e^{i\theta}\zeta)|^p \frac{d\theta}{2\pi} \, d\sigma(\zeta) \right)^{1/p}.$$

By the hypothesis,

$$\left(\int_S \left|f(e^{it}\zeta) - f(\zeta)\right|^p d\sigma(\zeta) \right)^{1/p} \le C\,|t|^\epsilon.$$

So if we use the fact that

$$\int_0^{2\pi} \frac{|t|^\epsilon}{|1 - re^{it}|^2} \, dt = O\left(\frac{1}{(1-r)^{1-\epsilon}} \right)$$

[4, P. 74], we obtain

$$M_p(r, \mathcal{R}f) = O\left(\frac{1}{(1-r)^{1-\epsilon}} \right).$$

III. Proof of Theorem 2

Let alone the role of Theorem 1, the equivalence of (1) $\sim$ (5) are somewhat known. We include a proof for the sake of completeness, which consists of a sequence of lemmas. We let, for simplicity, $\mathcal{R} = \sum_{j=1}^n z_j \partial_j$ and $\overline{\mathcal{R}} = \sum_{j=1}^n \bar{z}_j \overline{\partial}_j$.

Lemma 1. *For $1 \le i < j \le n$, let T_{ij} be either $\mathcal{T}_{ij}$ or $\overline{\mathcal{T}}_{ij}$. Then we have*

$$(\mathcal{R} + \overline{\mathcal{R}})T_{ij} = T_{ij}(\mathcal{R} + \overline{\mathcal{R}}).$$

Proof. By a simple calculation,

$$\mathcal{T}_{ij}\mathcal{R} = \mathcal{R}\mathcal{T}_{ij} + \mathcal{T}_{ij}, \qquad \overline{\mathcal{R}}\mathcal{T}_{ij} = \mathcal{T}_{ij}\overline{\mathcal{R}} + \mathcal{T}_{ij}.$$

Therefore,

$$(\mathcal{R} + \overline{\mathcal{R}})\mathcal{T}_{ij} = \mathcal{T}_{ij}(\mathcal{R} + \overline{\mathcal{R}}).$$

By taking conjugate we see that this holds with $\overline{\mathcal{T}}_{ij}$ in place of $\mathcal{T}_{ij}$. □

We will consider differential operators X appearing as composition

(3.1) $$Xf = X_1 \cdots X_k f,$$

where each X_i is $\mathcal{R}$ or a $\mathcal{T}_{ij}$ or a $\overline{\mathcal{T}}_{ij}$. For such an operator its weight is defined to be the sum of each weights of X_j, the weight of $\mathcal{R}$ being 1 and $\frac{1}{2}$ the weight of each $\mathcal{T}_{ij}$ and $\overline{\mathcal{T}}_{ij}$. The following is a weak version of Lemma 2.5 of [1], where the polydisc $P(z, \delta)$, $z \in B^n$, $\delta > 0$, is defined as follows.

If $z = r\zeta$, $0 \leq r < 1$, $\zeta \in S$, pick $\eta_2, \ldots, \eta_n$ so that $\{\zeta, \eta_2, \ldots, \eta_n\}$ is an orthonormal basis of $\mathbb{C}^n$. Then

$$P(z,\delta) = \left\{ w = z + \lambda\zeta + \sum_{j=2}^{n} \lambda_j \eta_j : \ |\lambda| < \delta, |\lambda_j| < \delta^{1/2}, j = 2, \ldots, n \right\}.$$

Lemma 2. *Let X and Y be the differential operators of the form (3.1) with the weight of X being m. Then there is a constant C such*

$$|XYf(z)|^p \leq \frac{C}{\delta^{n+1+mp}} \int_{P(z,\delta)} |Yf(w)|^p d\nu(w) \tag{3.2}$$

for all holomorphic function f in B and $P(z,\delta) \subset B$.

Lemma 3. *Let f be holomorphic in B and $l > 1$, then for all $s > 0$ there exists a compact subset K of B such that for all $k < l$ and $0 < r < 1$*

$$M_p(r, \mathcal{R}^k f) \leq C \sup_{z \in K} |f(z)| + s\, M_p(r, \mathcal{R}^l f).$$

If $0 < \epsilon < 1$ and f is holomorphic in B, then $M_p(r, \mathcal{R}^l f) = O((1-r)^{\epsilon - l})$ for some $l \geq 1$ if and only if $M_p(r, \mathcal{R}^k f) = O((1-r)^{\epsilon - k})$ for all $k \geq 1$.

Proof. For a positive integer m, a simple calculation using the homogeneous polynomial expansion gives us

$$f(z) = f(0) + \frac{1}{\Gamma(m)} \int_0^1 (-\log t)^{m-1} \mathcal{R}^m f(tz) \frac{dt}{t}. \tag{3.3}$$

Since $|\mathcal{R}^m f(z)| \leq C|z| \sum_{|\alpha| \leq m} |\partial^\alpha f(z)|$, by Cauchy integral formula we have

$$\begin{aligned} &\int_0^{1-1/N} (-\log t)^{m-1} |\mathcal{R}^m f(tz)| \frac{dt}{t} \\ &\leq C \sum_{|\alpha| \leq m} \sup_{|z| \leq 1-1/N} |\partial^\alpha f(z)| \int_0^{1/2} (-\log t)^{m-1} dt \leq C \sup_{|z| \leq 1-1/2N} |f(z)| \end{aligned}$$

for each positive integer N. Take $\mathcal{R}^k f$ in place of f and $(l-k)$ in place of m in (3.3), then, by the Minkowski's integral inequality together with the fact that $\mathcal{R}^{(l-k)} \mathcal{R}^k = \mathcal{R}^l$, we have

$$\begin{aligned} M_p(r, \mathcal{R}^k f) &\leq C \sup_{|z| \leq 1-1/2N} |f(z)| \\ &\quad + \frac{C}{\Gamma(l-k)} \int_{1-1/N}^1 (-\log t)^{(l-k)-1} M_p(tr, \mathcal{R}^l f) dt. \end{aligned} \tag{3.4}$$

Thus, for each positive integer N we have

$$
\begin{aligned}
&M_p(r,\mathcal{R}^k f)\\
&\le C \sup_{|z|<1-1/2N} |f(z)| + C\int_{1-1/N}^{1} (-\log t)^{(l-k)-1} M_p(tr,\mathcal{R}^l f)dt\\
&\le C \sup_{|z|<1-1/2N} |f(z)| + C\int_{1-1/N}^{1} (-\log t)^{(l-k)-1} M_p(r,\mathcal{R}^l f)dt\\
&\le C \sup_{|z|<1-1/2N} |f(z)| + CM_p(r,\mathcal{R}^l f)\int_{0}^{1/N} (2t)^{(l-k)-1} dt\\
&\le C \sup_{|z|<1-1/2N} |f(z)| + C\left(\frac{2}{N}\right)^{l-k} M_p(r,\mathcal{R}^l f).
\end{aligned}
\tag{3.5}
$$

Here, on the second inequality we used the fact that $M_p(r,g)$ is an increasing function of r for any holomorphic function g in B. For a given $s>0$ if we take N large enough to have the last constant $C(2/N)^{l-k}$ of (3.5) less than s and take $K=\{z:|z|\le 1-\frac{1}{2N}\}$, (3.5) therefore proves the first statement.

From (3.4), it is straight forward that if $M_p(r,\mathcal{R}^l f)=O((1-r)^{\epsilon-l})$ then $M_p(r,\mathcal{R}^k f)=O((1-r)^{\epsilon-k})$ for all $k=1,\dots,l$.

For $r\in(0,1)$ let $\delta=1-r$ and $r'=r+\delta/2$. By (3.2), for $k=l+j$

$$
\begin{aligned}
&M_p(r,\mathcal{R}^k f)^p\\
&\le C\int_S \frac{1}{\delta^{pj+n+1}}\int_{P(r\zeta,\delta/4)} |\mathcal{R}^l f(w)|^p \, d\nu(w)d\sigma(\zeta)\\
&\le C\int_{\delta/2<1-|w|<2\delta} \frac{1}{\delta^{pj+n+1}}\int_S |\mathcal{R}^l f(w)|^p \mathcal{X}_{P(r\zeta,\delta/4)}(w)\, d\sigma(\zeta)d\nu(w)\\
&\le C\frac{1}{\delta^{pj+1}}\int_{\delta/2<1-|w|<2\delta} |\mathcal{R}^l f(w)|^p \, d\nu(w)\\
&\le C\frac{M_p(r',\mathcal{R}^l f)^p}{\delta^{pj}}.
\end{aligned}
$$

From this it is straight forward that if $M_p(r,\mathcal{R}^l f)=O((1-r)^{\epsilon-l})$ then $M_p(r,\mathcal{R}^k f)=O((1-r)^{\epsilon-k})$ for all $k>l$. □

Lemma 4. *Let* $0<\epsilon<1$. *Suppose* f *is holomorphic in* B *and* $M_p(r,\mathcal{R}^k f)=O((1-r)^{\epsilon-k})$ *for some* $k\ge 1$. *Then, for each* $l\ge 2$ *we have* $M_p(r,T^\alpha f)=O((1-r)^{\epsilon-l/2})$ *for all multi-index* α *with* $|\alpha|=l$. *Moreover, when* $0<\epsilon<\frac{1}{2}$, *this holds with* $l=1$.

Proof. We assume $\epsilon\ge\frac{1}{2}$ and proof for the case $\epsilon<\frac{1}{2}$ can easily be modified from that of $\epsilon\ge\frac{1}{2}$.

Fix a multi-index α with $|\alpha| = l$. By Lemma 3, we may assume $k = 1$ and $M_p(r, \mathcal{R}f) = O((1-r)^{\epsilon-1})$. By the fundamental theorem of calculus, we have

$$T^\alpha f(z) = T^\alpha f(0) + \int_0^1 \frac{d[T^\alpha f(tz)]}{dt} dt = \int_0^1 [(\mathcal{R} + \overline{\mathcal{R}})T^\alpha f](tz)\frac{dt}{t}.$$

Since f is holomorphic, by Lemma 1 we have,

$$(\mathcal{R} + \overline{\mathcal{R}})T^\alpha f = T^\alpha(\mathcal{R} + \overline{\mathcal{R}})f = T^\alpha \mathcal{R} f$$

and

$$|T^\alpha \mathcal{R} f(z)| \leq C|z| \sum_{|\beta|=|\alpha|+1} |\partial^\beta f(z)|.$$

Therefore, by Minkowski's integral inequality we have

$$M_p(r, T^\alpha f) \leq C \sup_{|z|<3/4} |f(z)| + \int_{1/2}^1 M_p(tr, T^\alpha \mathcal{R} f) dt.$$

For $r \in (0,1)$, let $\delta = 1 - r$ and $r' = r + \frac{\delta}{2}$. Then by (3.2), we have

$$\begin{aligned} M_p(r, T^\alpha \mathcal{R} f)^p &\leq C \int_S \frac{1}{\delta^{pl/2+n+1}} \int_{P(r\zeta, \delta/4)} |\mathcal{R} f(w)|^p d\nu(w) d\sigma(\zeta) \\ &\leq C \int_{\delta/2<1-|w|<2\delta} \frac{1}{\delta^{pl/2+n+1}} \int_S |\mathcal{R} f(w)|^p \mathcal{X}_{P(r\zeta,\delta/4)}(w) d\sigma(\zeta) d\nu(w) \\ &\leq C \frac{1}{\delta^{pl/2+1}} \int_{\delta/2<1-|w|<2\delta} |\mathcal{R} f(w)|^p d\nu(w) \\ &\leq C \frac{M_p(r', \mathcal{R} f)^p}{\delta^{pl/2}}. \end{aligned}$$

Here, on the last inequality we used the fact that $M_p(r, \mathcal{R}f)$ is an increasing function of r. Therefore, we have

$$\begin{aligned} M_p(r, T^\alpha f) &\leq C \sup_{|z|<3/4} |f(z)| + \int_{1/2}^1 M_p(tr, T^\alpha \mathcal{R} f) dt \\ &\leq C \sup_{|z|<3/4} |f(z)| + C \int_0^1 (1 - tr)^{(\epsilon-1)-l/2} dt \\ &= O((1-r)^{\epsilon - l/2}). \end{aligned}$$

□

Lemma 5. *Let f be holomorphic in B. Suppose, either* (i) $0 < \epsilon < 1$ *and* $M_p(r, T^\alpha f) = O((1-r)^{\epsilon-l/2})$ *for some* $l \geq 2$ *and for all multi-index* α *with* $|\alpha| = l$, *or* (ii) $0 < \epsilon < 1/2$ *and* $M_p(r, T^\alpha f) = O((1-r)^{\epsilon-l/2})$ *for all multi-index* α *with* $|\alpha| = 1$. *Then, we have* $M_p(r, \mathcal{R}^k f) = O((1-r)^{\epsilon-k})$ *for all* $k \geq 1$.

Proof. Let $l \leq 2m$. A simple calculation shows that

$$-\sum_{i \neq j} \overline{T}_{ij} T_{ij} f(z) = 2(n-1)\mathcal{R} f(z).$$

Therefore, we have

$$M_p(r, \mathcal{R}^m f) \le C \sum_{|\beta|=2m} M_p(r, T^\beta f).$$

Let $|\beta| = 2m$, then following exactly the same argument as in the proof of Lemma 4, we have

$$M_p(r, T^\beta f) = O((1-r)^{(l-2m)/2}) \sum_{|\alpha|=l} M_p(r, T^\alpha f) = O((1-r)^{\epsilon - m}).$$

Therefore, we have

$$M_p(r, \mathcal{R}^m f) = O((1-r)^{\epsilon - m}).$$

And again by Lemma 3, this holds for all positive integer m. □

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E. G. Kwon
Department of Mathematics Education
Andong National University
Andong 760-749, Korea
E-mail address: egkwon@andong.ac.kr

Hyung Woon Koo
Department of Mathematics
Korea University
Seoul 136–701, Korea
E-mail address: koohw@math.korea.ac.kr

Hong Rae Cho
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail address: chohr@pusan.ac.kr