J. Korean Math. Soc. **44** (2007), No. 4, pp. 915–929

THE BEHAVIOR OF THE TWISTED p-ADIC (h,q)-L-FUNCTIONS AT $s=0$

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Abstract. The main result of this paper is to construct the derivative twisted p-adic (h,q)-L-functions at $s=0$. We obtain twisted version of Theorem 4 in [17]. We also obtain twisted (h,q)-extension of Proposition 1 in [3].

1. Introduction, definitions and notations

Throughout this paper, p will denote a prime number. $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote by the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $\mid p \mid_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we normally assume $\mid 1-q \mid_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $\mid x \mid_p \leq 1$. If $q \in \mathbb{C}$, then we normally assume$\mid q \mid < 1$ (see [5], [33], [9], [11]).

Kubota and Leopoldt proved the existence of meromorphic functions, which is defined over the p-adic number field as follows

$$L_p(s,\chi) = \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{\chi(n)}{n^s} = (1-\chi(p)p^{-s})L(s,\chi),$$

where $L(s,\chi)$ is the Dirichlet L-function cf. ([22], [3], [1], [5], [20], [7]).

$L_p(s,\chi)$ function interpolates generalized Bernoulli numbers $B_{n,\chi}$ at non-positive integers as follows [22]:

$$L_p(1-n,\chi) = -\frac{(1-\chi(p)p^{n-1})}{n} B_{n,\chi_n}, \text{ for } n \in \mathbb{Z}^+,$$

Received February 13, 2006.

2000 *Mathematics Subject Classification.* 11B68, 11S40, 11S80, 11B99, 28B99.

Key words and phrases. q-Bernoulli numbers and polynomials, twisted q -Bernoulli numbers and polynomials, q- zeta function, p-adic L-function, twisted q-zeta function, twisted q-L-functions, q-Volkenborn integral.

This paper was supported by the Scientific Research Project Administration Akdeniz University.

where $B_{n,\chi}$ denotes the nth generalized Bernoulli numbers associate with the primitive Dirichlet character χ, and $\chi_n = \chi w^{-n}$, where w is the Teichmüller character. w is the unique $\mathbb{Z}_p$-valued character of conductor p such that $w(a) \equiv a(\mathrm{mod}\, p\mathbb{Z}_p)$ for all $a \in \mathbb{Z}$ cf. [3].

Ferrero and Greenberg [3] found the formula for the derivative of the p-adic L-function at $s = 0$. They also gave some fundamental properties of the p-adic L-function. Proofs of the existence and fundamental properties of the p-adic L-function are given in cf. ([22], [5]) and also in cf. ([3], [1], [21], [20], [33], [7], [8], [34]).

Let $T_p = \cup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} C_{p^n}$, where $C_{p^n} = \left\{ \xi \mid \xi^{p^n} = 1 \right\}$ is the cyclic group of order p^n. For $\xi \in T_p$, we denote by $\phi_\xi : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \to \xi^x$, cf. ([6], [19]). The integer p^* is defined by $p^* = p$ if $p > 2$ and $p^* = 4$ if $p = 2$ cf. ([7], [34]).

Twisted two-variable (h, q)-L- function is defined by [30]:

Definition 1. Let $s \in \mathbb{C}$. Let χ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. We define

$$L_{\xi,q}^{(h)}(s, z, \chi) = \sum_{m=0}^{\infty} \frac{\chi(m)\xi^m q^{hm}}{(z+m)^s} - \frac{\log q^h}{s-1} \sum_{m=0}^{\infty} \frac{\chi(m)\xi^m q^{hm}}{(z+m)^{s-1}}. \tag{1}$$

The main result of this paper is to find the derivative of $L_{\xi,p,q}^{(h)}(s, t, \chi)$ at $s = 0$. We study on the behavior of twisted p-adic (h, q)-L-functions, at $s = 0$ in detail. We give relation between (h, q)-partial zeta function and $L_{\xi,p,q}^{(h)}(s, t, \chi)$. We obtain twisted version of Theorem 4 in [17]. We also obtain twisted (h, q)-extension of Proposition 1 in [3]. Our main theorem is given as follows:

Theorem 2. *Let χ be the primitive Dirichlet character, and let F be a positive integral multiple of p^* and $f = f_{\chi_n}$. For $t \in \mathbb{C}_p$ with $\mid t \mid_p \leq 1$, $h \in \mathbb{Z}$, and $\xi \in T_p$, then we have*

$$\begin{aligned} &\frac{\partial}{\partial s} L_{\xi,p,q}^{(h)}(0, t, \chi) \\ = &\sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi_1(a) q^{ha} \xi^a G_{p,q^F,\xi^F}^{(h)}\left(\frac{a+p^*t}{F}\right) \\ &- L_{\xi,p,q}^{(h)}(0, t, \chi) \log_p F - \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi_1(a) q^{ha} \xi^a B_{1,\xi^F}^{(h)}(q^F). \end{aligned}$$

In [23], Shiratani and Yamamoto constructed a p-adic interpolation $G_p(s, u)$ of the Frobenius-Euler numbers $H_n(u)$ and as its application, they obtained an explicit formula for $L_p'(0, \chi)$ with any Dirichlet character χ. In [7], Kim presented q-Euler numbers occurring in the coefficients of some Stirling type series for p-adic analytic functions. He treated generalized Kummer congruences for q-Bernoulli numbers. He also studied on q- analogue of the p-adic L-function, $L_{p,q}(s,\chi)$. He found the value of $L_{p,q}(s,\chi)$ at $s=1$. In [8], Kim found interesting and useful results of $L_{p,q}(s,\chi)$ function. By p-adic q-integral, he also constructed generating function of Carlitz's q-Bernoulli number. Young [34] defined some p-adic integral representation for the two-variable p-adic L-functions, introduced by Fox [4]. For powers of the Teichmüller character, he used the integral representation to extend the L-function to the large domain, in which it is a meromorphic function in the first variable and an analytic element in the second. These integral representations imply systems of congruences for the generalized Bernoulli polynomials. In [16], Kim constructed the two-variable p-adic q-L-function, which interpolates the generalized q-Bernoulli polynomials. This function is the q-extension of the two-variable p-adic L-function. He gave a p-adic integral representation for this two-variable p-adic q-L-function. He also derived q-extension of the generalized formula of Diamond and Ferrero and Greenberg formula for the two variable p-adic L-function in terms of the p-adic gamma and log gamma functions.

In [17], Kim constructed new p-adic (h,q)-L-function, $L_{p,q}^{(h)}(s,t,\chi)$, which is generalized Leopoldt-Kubota p-adic L-function and Proposition 1 in [3]. This function interpolates the generalized new (h,q)-generalized Bernoulli polynomials cf. [15].

The p-adic q-integral (or q-Volkenborn integral) are originately constructed by Kim [9]. Kim indicated a connection between the q-Volkenborn integral and non-Archimedean combinatorial analysis. The q-Volkenborn integral is used in mathematical physics for example the functional equation of the q-zeta function, the q-Stirling numbers, and q-Mahler theory of integration with respect to a ring $\mathbb{Z}_p$ together with Iwasawa's p-adic q-L-function.

For $f \in UD(\mathbb{Z}_p,\mathbb{C}_p) = \{f \mid f : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable function$\}$, the p-adic q-integral (or q-Volkenborn integration) was defined by

$$I_q(f)=\int_{\mathbb{Z}_p} f(x)d\mu_q(x)=\lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} q^x f(x), \tag{2}$$

where $[x]_q=\frac{1-q^x}{1-q}$ ([9], [10], [11], [12], [13], [14], [31], [32]). For a fixed positive integer f with $(p,f)=1$, we set

$$\begin{aligned}\mathbb{X} &= \mathbb{X}_f=\lim_{\overleftarrow{N}}\mathbb{Z}/fp^N\mathbb{Z},\\ \mathbb{X}_1 &= \mathbb{Z}_p,\ \mathbb{X}^*=\bigcup_{\substack{0<a<fp\\(a,p)=1}} a+fp\mathbb{Z}_p\end{aligned}$$

and

$$a+fp^N\mathbb{Z}_p=\left\{x\in\mathbb{X}\mid x\equiv a(\operatorname{mod} fp^N)\right\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < fp^N$. For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, $\int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \int_{\mathbb{X}} f(x)d\mu_1(x)$,(for details see [9], [10], [13]).

In [6], Kim defined analogue of Bernoulli numbers, which is called twisted Bernoulli numbers in this paper. He gave relation between these numbers and Frobenious-Euler numbers. Kim, Jang, Rim and Pak [19] defined twisted q-Bernoulli numbers by using p-adic invariant integrals on $\mathbb{Z}_p$. They gave twisted q-zeta function and q-L-series which interpolate twisted q-Bernoulli numbers. By using the q-Volkenborn integral on $\mathbb{Z}_p$, we [28], [30] constructed new generating functions of the twisted (h,q)-Bernoulli polynomials and numbers. By applying the Mellin transformation to the generating functions, we constructed integral representation of the twisted (h,q)-Hurwitz function and twisted (h,q)-two-variable L-function. By using these functions, we [29] constructed p-adic twisted (h,q)-L-function, $L^{(h)}_{\xi,p,q}(s,t,\chi)$ which is twisted version of generalized Leopoldt-Kubota p-adic L-function and Kim's p-adic (h,q)-L-function [17]. This function interpolates the twisted (h,q)-generalized Bernoulli polynomials and numbers. In [26], we constructed generating functions of q-generalized Euler numbers and polynomials. The author also constructed a complex analytic twisted l-series, which is interpolated twisted q-Euler numbers at non-positive integers. In [27], by using generating functions of the q-Benoulli numbers and Mellin transform, we constructed q-L-functions, two-variable q-L-functions and q-Dedekind type sums. We alse gave some new relations related to q-Dedekind type sums and q-L-functions.

By using q-Volkenborn integration, we [28] constructed generating function of the twisted (h,q)-extension of Bernoulli numbers, $B^{(h)}_{n,\xi}(q)$ and polynomials, $B^{(h)}_{n,\xi}(z,q)$ by means of the following generating functions

$$F^{(h)}_{w,q}(t) = \frac{\log q^h + t}{\xi q^h e^t - 1} = \sum_{n=0}^{\infty} B^{(h)}_{n,\xi}(q)\frac{t^n}{n!}.$$

$$F^{(h)}_{\xi,q}(t,z) = \frac{(t+\log q^h)e^{tz}}{\xi q^h e^t - 1} = \sum_{n=0}^{\infty} B^{(h)}_{n,\xi}(z,q)\frac{t^n}{n!}. \tag{3}$$

We note that since $F^{(h)}_{\xi,q}(t,0) = F^{(h)}_{\xi,q}(t,z)$, we set $B^{(h)}_{n,\xi}(0,q) = B^{(h)}_{n,\xi}(q)$. If $\xi \to 1$, then $B^{(h)}_{n,\xi}(q) \to B^{(h)}_{n}(q)$ and $F^{(h)}_{\xi,q}(t) \to F^{(h)}_{q}(t) = \frac{h\log q + t}{q^h e^t - 1}$ this generating function was defined by Kim [15]. If $\xi \to 1$ and $q \to 1$, then $F_{\xi,q}(t) \to F(t) = \frac{t}{e^t-1}$ and $B_{n,\xi}(q) \to B_n$ are the usual Bernoulli numbers (see [5], [20], [33], [9], [10], [11], [12], [13], [32]). The generalized twisted new (h,q)-extension of Bernoulli polynomials $B^{(h)}_{n,\chi,\xi}(z,q)$ are defined by means of the generating function ([28], [30])

$$F^{(h)}_{\chi,\xi,q}(t,z) = \sum_{a=1}^{f} \frac{\chi(a)\phi_\xi(a)q^{ha}e^{(z+a)t}(t+\log q^h)}{\xi^f q^{hf} e^{ft} - 1} = \sum_{n=0}^{\infty} B^{(h)}_{n,\chi,\xi}(z,q)\frac{t^n}{n!},$$

where

$$B^{(h)}_{n,\chi,w}(z,q)=\sum_{k=0}^{n}\binom{n}{k}z^{n-k}B^{(h)}_{k,\chi,\xi}(q). \tag{4}$$

Note that $B^{(h)}_{n,\chi,\xi}(0,q)=B^{(h)}_{n,\chi,\xi}(q)$, $\lim_{q\to1}B^{(h)}_{n,\chi,\xi}(q)=B^{(h)}_{n,\chi,\xi}$, where $B^{(h)}_{n,\chi,\xi}$ are the twisted Bernoulli numbers (see [28]). If $w\to1$ and $q\to1$, then $B_{n,\chi,\xi}(q)\to B_{n,\chi}$ are the usual generalized Bernoulli numbers, and $B_{n,\chi,\xi}(z,q)\to B_{n,\chi}(z)$ are the usual generalized Bernoulli polynomials (see [5], [22], [33], [20], [21], [4], [1], [2], [6], [12], [13], [15], [16], [18], [19], [17], [25], [26], [24], [32]).

2. (h,q)-partial zeta function and the derivative of $L^{(h)}_{\xi,p,q}(s,t,\chi)$ function at $s=0$

The aim of this section is to evaluate $\frac{\partial}{\partial s}L^{(h)}_{\xi,p,q}(s,t,\chi)$ and prove

$$\frac{\partial}{\partial s}L^{(h)}_{\xi,p,q}(s,t,\chi)\mid_{s=0}=\frac{\partial}{\partial s}L^{(h)}_{\xi,p,q}(0,t,\chi).$$

Therefore, we need the following definitions and theorems.

Twisted (h,q)-extension of Hurwitz zeta function, $\zeta^{(h)}_{\xi,q}(s,z)$ is defined by ([28], [30])

$$\zeta^{(h)}_{\xi,q}(s,x)=\sum_{n=0}^{\infty}\frac{\xi^nq^{nh}}{(n+x)^s}-\frac{h\log q}{s-1}\sum_{n=0}^{\infty}\frac{\xi^nq^{nh}}{(n+x)^{s-1}},$$

where $s\in\mathbb{C}$, $x\in\mathbb{R}^+$. Relation between $\zeta^{(h)}_{\xi,q}(s,z)$ and $L^{(h)}_{\xi,q}(s,z,\chi)$ is given by

$$L^{(h)}_{\xi,q}(s,z,\chi)=\frac{1}{f^s}\sum_{a=1}^{f}q^{ha}\xi^a\chi(a)\zeta^{(h)}_{\xi^f,q^f}(s,\frac{a+z}{f})\ \text{([28], [30])}. \tag{5}$$

Theorem 3. ([29], [30]) *Let χ be a Dirichlet character of conductor $f\in\mathbb{Z}^+$. Let $n\in\mathbb{Z}^+$. We have*

$$L^{(h)}_{\xi,q}(1-n,z,\chi)=-\frac{B^{(h)}_{n,\chi,\xi}(z,q)}{n}. \tag{6}$$

Remark 4. Observe that if $\xi\to1$, then $L^{(h)}_{\xi,q}(s,z,\chi)$ is reduced to $L^{(h)}_q(s,z,\chi)$ cf. ([15], [17]). If $q\to1$ and $h=1$, (1) is reduced to twisted two-variable L-function:

$$L^{(h)}_{\xi}(s,z,\chi)=\sum_{m=0}^{\infty}\frac{\chi(m)\xi^m}{(z+m)^s}.$$

Substituting $z=1$ in the above, we have the twisted L-functions

$$L_{\xi}(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)\xi^n}{n^s},$$

where $r \in \mathbb{Z}^+$, set of positive integers, χ is a Dirichlet character of conductor $f \in \mathbb{Z}^+$, and let $\xi^r = 1$, $\xi \neq 1$ [21]. Since the function $n \to \chi(n)\xi^n$ has period fr, this is a special case of the Dirichlet L-functions. Koblitz [21] and the author gave relation between $L(s, \chi, \xi)$ and twisted Bernoulli numbers, $B_{n,\chi,\xi}$ at non-positive integers(see [20], [21], [25], [24]). In [18], Kim and Rim constructed two-variable L-function, $L(s, x \mid \chi)$. They showed that this function interpolates the generalized Bernoulli polynomials associated with χ. By the Mellin transforms, they gave the complex integral representation for the two-variable Dirichlet L-function. They also found some properties of the two-variable Dirichlet L-function. In [31], Simsek, D. Kim and Rim defined q-analogue two-variable L-function.

Let s be a complex variable, a and $f \in \mathbb{Z}^+$ with $0 < a < f$. Twisted (h, q)-partial zeta function is defined by [29]:

Definition 5. Let $s \in \mathbb{C}$.

$$H_{\xi,q}^{(h)}(s, a : f) = \sum_{\substack{n \equiv a (\mathrm{mod}\, f) \\ n > 0}}^{\infty} \frac{q^{nh}\xi^n}{n^s} - \frac{\log q^h}{s-1} \sum_{\substack{n \equiv a (\mathrm{mod}\, f) \\ n > 0}}^{\infty} \frac{q^{nh}\xi^n}{n^{s-1}}.$$

Relation between $H_{\xi,q}^{(h)}(s, a : f)$ and $\zeta_{\xi,q}^{(h)}(s, x)$ are given by [29]

$$H_{\xi,q}^{(h)}(s, a : f) = q^{ha}\xi^a f^{-s} \zeta_{\xi^f, q^f}^{(h)}(s, \frac{a}{f}). \tag{7}$$

Observe that, the function $H_{\xi,q}^{(h)}(s, a : f)$ is meromorphic function for $s \in \mathbb{C}$ with simple pole at $s = 1$, having residue $\frac{q^{ha}\xi^a \log q^h}{q^{hf}\xi^f - 1}$. $H_{\xi,q}^{(h)}(s, a : f)$ interpolates generalized twisted (h, q)-Bernoulli polynomials at non-positive integer.

Corollary 6. ([29]) *Let $n \in \mathbb{Z}^+$. We have*

$$H_{\xi,q}^{(h)}(1-n, a : f) = -\frac{q^{ha}\xi^a f^{n-1} B_{n,\chi,\xi^f}^{(h)}(\frac{a}{f}, q^f)}{n}. \tag{8}$$

By using (8) and (4), after some elementary calculations, we arrive at the following theorem.

Theorem 7. *Let $n \in \mathbb{Z}^+$. We have*

$$\begin{aligned} & H_{\xi,q}^{(h)}(1-n, x+a : f) \\ = & -\frac{q^{ha}\xi^a f^{n-1}}{n} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x+a}{f}\right)^{n-k} B_{k,\xi^f}^{(h)}(\frac{x+a}{f}, q^f). \end{aligned} \tag{9}$$

We modify the twisted (h, q)-extension of the partial zeta function as follows:

Corollary 8. ([29]) *Let $s \in \mathbb{C}$. We have*

$$H_{\xi,q}^{(h)}(s,a:f) = \frac{a^{s-1}q^{ha}\xi^a}{(s-1)f}\sum_{k=0}^{\infty}\binom{1-s}{k}\left(\frac{f}{a}\right)^k B_{k,\xi^f}^{(h)}(q^f). \tag{10}$$

Observe that if $\xi = 1$, then $H_{\xi,q}^{(h)}(s,a:f)$ is reduced to $H_q^{(h)}(s,a:f)$ cf. (for detail see [17]).

Theorem 9. ([29]) *Let $s \in \mathbb{C}$ and let χ ($\chi \neq 1$) be a Dirichlet character of conductor $f \in \mathbb{Z}^+$.*

$$\begin{aligned} &L_{\xi,q}^{(h)}(s,\chi) \\ &= \sum_{a=1}^{f}\chi(a)H_{\xi,q}^{(h)}(s,a:f) \\ &= \frac{1}{(s-1)f}\sum_{a=1}^{f}\chi(a)a^{s-1}q^{ha}\xi^a\sum_{k=0}^{\infty}\binom{1-s}{k}\left(\frac{f}{a}\right)^k B_{k,\xi^f}^{(h)}(q^f). \end{aligned} \tag{11}$$

Twisted (h,q)-partial Hurwitz zeta function is defined by [29].

Definition 10. Let $s \in \mathbb{C}$.

$$\begin{aligned} &H_{\xi,q}^{(h)}(s,x+a:f) \\ &= \sum_{\substack{n \equiv a (\mathrm{mod}\, f) \\ n \geq 0}}^{\infty}\frac{q^{nh}\xi^n}{(x+n)^s} - \frac{\log q^h}{s-1}\sum_{\substack{n \equiv a (\mathrm{mod}\, f) \\ n \geq 0}}^{\infty}\frac{q^{nh}\xi^n}{(x+n)^{s-1}}. \end{aligned}$$

Thus, by the above equation, we obtain

$$H_{\xi,q}^{(h)}(s,x+a:f) = \frac{(x+a)^{1-s}q^{ha}\xi^a}{(s-1)f}\sum_{k=0}^{\infty}\binom{1-s}{k}\left(\frac{f}{x+a}\right)^k B_{k,\xi^f}^{(h)}(q^f).$$

By (10), we have the following relation:

Let s be a complex variable, a and f be integers with $0 < a < f$, $x \in \mathbb{R}$ with $0 < x < 1$, we have

$$\begin{aligned} &L_{\xi,q}^{(h)}(s,x,\chi) \\ &= \sum_{a=1}^{f}\chi(a)H_{\xi,q}^{(h)}(s,x+a:f) \\ &= \frac{1}{(s-1)f}\sum_{a=1}^{f}\chi(a)(x+a)^{1-s}q^{ha}\xi^a \\ &\quad \times\sum_{k=0}^{\infty}\binom{1-s}{k}\left(\frac{f}{x+a}\right)^k B_{k,\xi^f}^{(h)}(q^f). \end{aligned} \tag{12}$$

By the above equation, $L_{\xi,q}^{(h)}(s,x,\chi)$ is an analytic for $x \in \mathbb{R}$ with $0 < x < 1$ and $s \in \mathbb{C}$ except $s = 1$. Observe that if $\xi \to 1$, then $L_{\xi,q}^{(h)}(s,x,\chi)$ is reduced to $L_q^{(h)}(s,x,\chi)$ cf. (for detail see [17]).

Here we can use some notations, which are due to Kim [17]. Let w be the Teichmüller character, having conductor $f_w = p^*$. For an arbitrary character χ, we define $\chi_n = \chi w^{-n}$, where $n \in \mathbb{Z}$, in the sense of the product of characters. In this section, if $q \in \mathbb{C}_p$, then we assume $\mid 1 - q \mid_p < p^{-\frac{1}{p-1}}$. Let $\langle a \rangle = w^{-1}(a)a = \frac{a}{w(a)}$. We note that $\langle a \rangle \equiv 1(\mathrm{mod}\, p^*\mathbb{Z}_p)$. Thus, we easily see that

$$\begin{aligned} \langle a + p^*t \rangle &= w^{-1}(a+p^*t)(a+p^*t) \\ &= w^{-1}(a)a + w^{-1}(a)(p^*t) \equiv 1(\mathrm{mod}\, p^*\mathbb{Z}_p[t]), \end{aligned}$$

where $t \in \mathbb{C}_p$ with $\mid t \mid_p \leq 1$, $(a,p) = 1$. The p-adic logarithm function, $\log_p$, is the unique function $\mathbb{C}_p^x \to \mathbb{C}_p$ that satisfies the following conditions:

i)

$$\log_p(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}, \quad \mid x \mid_p < 1,$$

ii) $\log_p(xy) = \log_p x + \log_p y$, $\forall x, y \in \mathbb{C}_p^x$, and *iii)* $\log_p p = 0$.

Let

$$A_j(x) = \sum_{n=0}^{\infty} a_{n,j} x^n,$$

where $a_{n,j} \in \mathbb{C}_p$, $j = 0, 1, 2, \ldots$, be a sequence of power series, each of which converges in a fixed subset

$$D = \left\{ s \in \mathbb{C}_p : \mid s \mid_p \leq \mid p^* \mid^{-1} p^{-\frac{1}{p-1}} \right\}$$

of $\mathbb{C}_p$ such that

1) $a_{n,j} \to a_{n,0}$ as $j \to \infty$, for $\forall n$,

2) for each $s \in D$ and $\epsilon > 0$, there exists $n_0 = n_0(s, \epsilon)$ such that $\mid \sum_{n \geq n_0} a_{n,j} s^n \mid_p < \epsilon$ for $\forall j$. Then $\lim_{j \to \infty} A_j(s) = A_0(s)$ for all $s \in D$. This is used by Washington [33] and Kim [17] to show that each of the function $w^{-s}(a)a^s$ and

$$\sum_{k=0}^{\infty} \binom{s}{k} \left(\frac{F}{a} \right)^k B_k,$$

where F is the multiple of p^* and $f = f_\chi$, is analytic in D. We consider the *twisted* p-adic analogs of the twisted two variable q-L-functions, $L_{\xi,q}^{(h)}(s,t,\chi)$. These functions are the q-analogs of the p-adic interpolation functions for the generalized twisted Bernoulli polynomials attached to χ. Let F be a positive integral multiple of p^* and $f = f_\chi$, and let

$$L^{(h)}_{\xi,p,q}(s,t,\chi) = \frac{1}{(s-1)F} \sum_{\substack{a=1\\(a,p)=1}}^{F} \chi(a)\langle a+p^*t\rangle^{1-s} q^{ha}\xi^{a}$$

$$\times \sum_{k=0}^{\infty} \binom{1-s}{k} \left(\frac{F}{a+p^*t}\right)^k B^{(h)}_{k,\xi^F}(q^F). \tag{13}$$

Then $L^{(h)}_{\xi,p,q}(s,t,\chi)$ is analytic for $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, provided $s \in D$, except $s = 1$ when $\chi \neq 1$. For $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, we see that

$$\sum_{k=0}^{\infty} \binom{1-s}{k} \left(\frac{F}{a+p^*t}\right)^k B^{(h)}_{k,\xi^F}(q^F)$$

is analytic for $s \in D$. By definition of $\langle a+p^*t\rangle$, it is readily follows that $\langle a+p^*t\rangle^s = \langle a\rangle^s \sum_{k=0}^{\infty} \binom{s}{k} (a^{-1}p^*t)^k$ is analytic for $t \in \mathbb{C}_p$ with $|t|_p \leq 1$ when $s \in D$. Thus, since $(s-1)L^{(h)}_{\xi,p,q}(s,t,\chi)$ is a finite sum of products of these two functions, it must also be analytic for $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, whenever $s \in D$.

Theorem 11. ([29]) *Let F be a positive integral multiple of p^* and $f = f_{\chi_n}$, and let*

$$L^{(h)}_{\xi,p,q}(s,t,\chi) = \frac{1}{(s-1)F} \sum_{\substack{a=1\\(a,p)=1}}^{F} \chi(a)\langle a+p^*t\rangle^{1-s} q^{ha}\xi^{a} \times \sum_{k=0}^{\infty} \binom{1-s}{k} \left(\frac{F}{a+p^*t}\right)^k B^{(h)}_{k,\xi^F}(q^F).$$

Then $L^{(h)}_{\xi,p,q}(s,t,\chi)$ is analytic for $h \in \mathbb{Z}^+$ and $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, provided $s \in D$, except $s = 1$. Also, if $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, this function is analytic for $s \in D$ when $\chi \neq 1$, and meromorphic for $s \in D$, with simple pole at $s = 1$ having residue

$$\frac{\log q^h}{q^h\xi - 1}\left(\frac{1-q^{hF}\xi^F}{1-q^h\xi} - \frac{1-q^{hpF}}{1-q^h\xi}\right)$$

when $\chi = 1$. In addition, for each $n \in \mathbb{Z}^+$, we have

$$L^{(h)}_{\xi,p,q}(1-n,t,\chi) = -\frac{B^{(h)}_{n,\chi_n,\xi}(p^*t,q) - \chi_n(p)p^{n-1}B^{(h)}_{n,\chi_n,1}(p^{-1}p^*t,q^p)}{n}.$$

Remark 12. Observe that
$\lim_{\xi\to 1} L^{(h)}_{\xi,p,q}(s,t,\chi) = L^{(h)}_{p,q}(s,t,\chi)$ cf. [17].
$\lim_{h\to 1} L^{(h)}_{p,q}(s,0,\chi) = L_{p,q}(s,\chi)$ cf. ([7], [8]).
$\lim_{q\to 1} L_{p,q}(s,\chi) = L_p(s,\chi)$, cf. ([1], [3], [5], [20], [21], [23], [33]).

In [29], we defined twisted (h,q) partial zeta function. By (10), we now define q-analogue of the partial p-adic twisted zeta function as follows:

$$(14)\qquad H^{(h)}_{\xi,p,q}(s,a:f) = \frac{1}{(s-1)F}\langle a\rangle^{1-s}\sum_{k=0}^{\infty}\binom{1-s}{k}\left(\frac{F}{a}\right)^k B^{(h)}_{k,\xi^F}(q^F),$$

where $s\in D$, $s\neq 1$, $k\in\mathbb{Z}$ with $(a,p)=1$, and F is a multiple of p^*, $f=f_\chi$ cf ([7], [8], [17]). This function is a meromorphic for $s\in D$ with a simple pole at $s=1$. We now calculate residue of this functions at $s=1$ as follows:

$$\lim_{s\to 1}(s-1)H^{(h)}_{\xi,p,q}(s,a:f) = \frac{\log_p q^h}{q^{hf}\xi^f - 1}.$$

Substituting $s=1-n$, $n\in\mathbb{Z}^+$ into (14), we have

$$\begin{aligned} H^{(h)}_{\xi,p,q}(1-n,a:F) &= -\frac{1}{n}\langle a\rangle^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{F}{a}\right)^k B^{(h)}_{k,\xi^F}(q^F) \\ &= -\frac{1}{n}w^{-n}(a)a^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{F}{a}\right)^k B^{(h)}_{k,\xi^F}(q^F). \end{aligned}$$

By using the following formula

$$B^{(h)}_{n,\xi}(z:q) = \sum_{k=0}^{n}\binom{n}{k}z^{n-k}B^{(h)}_k(q),\ n\geq 0,\ \text{(cf. Theorem 7 in [28])},$$

we obtain

$$\begin{aligned} H^{(h)}_{\xi,p,q}(1-n,a:F) &= -\frac{w^{-n}(a)F^{n-1}}{n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{a}{F}\right)^{n-k} B^{(h)}_{k,\xi^F}(q^F) \\ &= -\frac{w^{-n}(a)F^{n-1}}{n}B^{(h)}_{k,\xi^F}\left(\frac{a}{F},q^F\right). \end{aligned}$$

Thus we arrive at the following theorem:

Theorem 13. *Let $n\in\mathbb{Z}^+$.*

$$H^{(h)}_{\xi,p,q}(1-n,a:F) = -\frac{w^{-n}(a)F^{n-1}}{n}B^{(h)}_{k,\xi^F}\left(\frac{a}{F},q^F\right).$$

By (12) and (14) we construct the following twisted p-adic (h,q)-L-function as follows:

Theorem 14. *Let F be a multiple of p^*, $f=f_\chi$. We have*

$$L^{(h)}_{\xi,p,q}(s,\chi)=\sum_{\substack{a=1\\(a,p)=1}}^{F}\chi(a)q^{ha}\xi^{a}H^{(h)}_{\xi,p,q}(s,a:F).$$

In [29], we defined twisted $H^{(h)}_{\xi,q}(s,x+a:F)$ partial zeta function. By (10) and (13), the partial p-adic twisted zeta function is defined. Substituting $a=b+tp^*$ into (14), we obtain

$$\begin{aligned}&H^{(h)}_{\xi,p,q}(s,b+tp^*:f)\\=&\frac{\langle b+tp^*\rangle^{1-s}}{(s-1)F}\sum_{k=0}^{\infty}\binom{1-s}{k}\left(\frac{F}{b+tp^*}\right)^{k}B^{(h)}_{k,\xi^f}(q^f).\end{aligned}$$

Then by the similar method in the above, we define two variable twisted p-adic (h,q)-L-functions as follows:

$$L^{(h)}_{\xi,p,q}(s,t,\chi)=\sum_{\substack{b=1\\(b,p)=1}}^{F}\chi(b)q^{hb}\xi^{b}H^{(h)}_{\xi,p,q}(s,b+tp^*:F).$$

Observe that $H^{(h)}_{\xi,p,q}(s,b+tp^*:F)$ is analytic for $t\in\mathbb{C}_p$ with $|t|_p\leq 1$, where $s\in D$, except $s=1$, and meromorphic for $s\in D$, with a simple pole at $s=1$, when $t\in\mathbb{C}_p$ with $|t|_p\leq 1$.

We now find $\frac{\partial}{\partial s}L^{(h)}_{\xi,p,q}(0,t,\chi)$ below. The value of $\frac{\partial}{\partial s}L^{(h)}_{\xi,p,q}(0,t,\chi)$ is the coefficient of s in the expansion of $L^{(h)}_{\xi,p,q}(s,t,\chi)$ at $s=0$. By using Taylor expansion at $s=0$, we also need the following relations cf. [17]

$$\begin{aligned}\frac{1}{1-s}&=1+s+s^2+\cdots,\\\binom{1-s}{k}&=\frac{(-1)^{m+1}}{m(m-1)}s+\cdots,\end{aligned}$$

$$\langle b+tp^*\rangle^{1-s}=\langle b+tp^*\rangle\left(1-s\log_p\langle b+tp^*\rangle+\cdots\right),\text{ cf. [17]}.$$

By the following definition

$$\begin{aligned}&L^{(h)}_{\xi,p,q}(s,t,\chi)\\=&\frac{1}{(s-1)F}\sum_{\substack{a=1\\(a,p)=1}}^{F}\chi(a)\langle a+p^*t\rangle^{1-s}q^{ha}\xi^{a}\\&\times\sum_{k=0}^{\infty}\binom{1-s}{k}\left(\frac{F}{a+p^*t}\right)^{k}B^{(h)}_{k,\xi^F}(q^F),\end{aligned}$$

and $w(a)$ is a root of unity for $(a,p)=1$,

$$\log_p\langle b+tp^*\rangle = \log_p(b+tp^*)+\log_p w^{-1}(a) = \log_p(b+tp^*) \text{ cf. ([33], [17]).} \tag{15}$$

We obtain

$$\begin{aligned}
&\frac{\partial}{\partial s}L^{(h)}_{\xi,p,q}(0,t,\chi) \\
=& -\frac{1}{F}\sum_{\substack{a=1\\(a,p)=1}}^{F}\chi(a)\langle a+p^*t\rangle^{1-s}q^{ha}\xi^{a} \\
&\times\left(\frac{F}{a+p^*t}B^{(h)}_{1,\xi^F}(q^F)+B^{(h)}_{0,\xi^F}(q^F)\right) \\
&+\frac{1}{F}\sum_{\substack{a=1\\(a,p)=1}}^{F}\chi(a)\langle a+p^*t\rangle\left(\log_p\frac{a+p^*t}{F}+\log_p F\right) \\
&\times q^{ha}\xi^{a}\left(\frac{F}{a+p^*t}B^{(h)}_{1,\xi^F}(q^F)+B^{(h)}_{0,\xi^F}(q^F)\right) \\
&-\frac{1}{F}\sum_{\substack{a=1\\(a,p)=1}}^{F}\chi(a)\langle a+p^*t\rangle q^{ha}\xi^{a} \\
&\left(\frac{F}{a+p^*t}B^{(h)}_{1,\xi^F}(q^F)+\sum_{m=2}^{\infty}\frac{(-1)^m}{m(m-1)}\left(\frac{a+p^*t}{F}\right)^{-m}B^{(h)}_{m,\xi^F}(q^F)\right).
\end{aligned}\tag{16}$$

For calculations the above equation, we need the following relations and definitions:

The Diamond gamma function defined by

$$G_p(x)=(x-\frac{1}{2})\log_p x - x+\sum_{m=2}^{\infty}\frac{x^{1-m}B_m}{m(m-1)},$$

for $|x|_p>1$, cf. ([1], [3], [17], [20], [21]).

We now define a twisted locally analytic function $G^{(h)}_{p,q,\xi}(x)$, which is the (h,q)-extension of the Diamond gamma function, as follows:

$$G^{(h)}_{p,q,\xi}(x)=\int_{\mathbb{Z}_p}\left((x+z)\log_p(x+z)-(x+z)\right)d\mu_1(x),\quad |x|_p>1,$$

where $G^{(h)}_{p,q,\xi}(x)$ is locally analytic function on $\mathbb{Z}_p\backslash\mathbb{C}_p$. By (2), and Theorem 4 in [28], we easily obtain

$$G^{(h)}_{p,q,\xi}(x) = (xB^{(h)}_{0,\xi}(q)+B^{(h)}_{0,\xi}(q))\log_p x - B^{(h)}_{0,\xi}(q) + \sum_{m=1}^{\infty}\frac{(-1)^{1+m}B^{(h)}_{m,\xi}(q)}{m(m+1)}x^{-n}$$

for $|x|_p>1$. By substituting the above equation and (16) into (16), after some calculations, we obtain

$$\frac{\partial}{\partial s}L^{(h)}_{\xi,p,q}(0,t,\chi) = \sum_{\substack{a=1\\(a,p)=1}}^{F} \chi_1(a)q^{ha}\xi^a G^{(h)}_{p,q^F,\xi^F}\left(\frac{a+p^*t}{F}\right) - L^{(h)}_{\xi,p,q}(0,t,\chi)\log_p F - \sum_{\substack{a=1\\(a,p)=1}}^{F}\chi_1(a)q^{ha}\xi^a B^{(h)}_{1,\xi^F}(q^F).$$

Consequently, we complete the proof of Theorem 2.

Observe that if $\xi\to 1$, then $\frac{\partial}{\partial s}L^{(h)}_{\xi,p,q}(0,t,\chi)$ is reduced to $\frac{\partial}{\partial s}L^{(h)}_{p,q}(0,t,\chi)$ cf. [17]. If $\xi\to 1$, $q\to 1$, $h=1$, then Theorem 2 is reduced to Proposition 1 in [3].

Acknowledgement. The author expresses his sincere gratitude to referee for his/her valuable comments and suggestions.

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