J. Korean Math. Soc. **44** (2007), No. 4, pp. 903–913

WEAKLY STABLE CONDITIONS FOR EXCHANGE RINGS

HUANYIN CHEN

ABSTRACT. A ring R has weakly stable range one provided that $aR + bR = R$ implies that there exists a $y \in R$ such that $a + by \in R$ is right or left invertible. We prove, in this paper, that every regular element in an exchange ring having weakly stable range one is the sum of an idempotent and a weak unit. This generalize the corresponding result of one-sided unit-regular ring. Extensions of power comparability and power cancellation are also studied.

1. Introduction

A ring R is said to be an exchange ring if for every right R-module A and any two decompositions $A = M \oplus N = \bigoplus_{i\in I} A_i$, where $M_R \cong R_R$ and I is a finite index set, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus (\bigoplus_{i\in I} A_i')$. It is well known that regular rings, π-regular rings, unit C^*-algebras of real rank zero, semiperfect rings, left or right continuous rings and clean rings are all exchange rings. For general theory of exchange rings, we refer the readers to [10]. Following Wei and Tong (cf. [11]), a ring R is said to have weakly stable range one provided that $aR + bR = R$ implies that there exists a $y \in R$ such that $a + by \in R$ is right or left invertible. Weakly stable range one is a natural generalization of stable range one (cf. [10]). The class of rings satisfying weakly stable range one is very large. It includes rings having stable range one, one-sided unit-regular rings (cf. [3]), exchange rings satisfying comparability axiom, etc. By [12, Theorem 3.4], an exchange ring R has weakly stable range one if and only if every regular element in R is one-sided unit-regular. Clearly, a regular ring has weakly stable range one if and only if it is one-sided unit-regular. Many authors have studied exchange rings of weakly stable range one, for example [8] and [11-13].

An element $a \in R$ is regular if there exists a $x \in R$ such that $a = axa$. In this paper, we investigate regular elements in exchange rings having weakly stable range one. For such exchange rings, we proved that every regular element is the sum of an idempotent and a weak unit. This gives a nontrivial generalization

Received February 11, 2006.

2000 *Mathematics Subject Classification.* 16E50, 16U99.

Key words and phrases. exchange rings, weakly stable range one, power comparability, power cancellation.

of [1, Theorem 1]. A ring R is said to satisfy power comparability provided that $aR + bR = R$ with $a, b \in R$ implies that there exist a positive integer n and a $Y \in M_n(R)$ such that $aI_n + bY$ is right or left invertible (cf. [9]). Let R be an exchange ring satisfying power comparability. Furthermore, we show that for any regular $a \in R$, there exist a positive integer n, an idempotent $(e_{ij}) \in M_n(R)$ and a weak unit $(u_{ij}) \in M_n(R)$ such that

$$e_{ij} + u_{ij} = \begin{cases} a, & i = j; \\ 0, & i \neq j. \end{cases}$$

A ring R is said to satisfy power cancellation provided that $aR + bR = R$ with $a, b \in R$ implies that there exist a positive integer n and a $Y \in M_n(R)$ such that $aI_n + bY$ is invertible. Many authors studied such exchange rings(cf. [6] and [14]). We also extend the previous result to exchange rings satisfying power cancellation.

Throughout, all rings are associative with identity. We use $M_n(R)$ to denote the ring of all $n \times n$ matrices over the ring R and $\mathbb{N}$ to denote the set of all natural numbers. The notation $A \lesssim^{\oplus} B$ means that A is isomorphic to a direct summand of B. We say that $u \in R$ is a weak unit in case $l.ann(u) = 0$ or $r.ann(u) = 0$.

2. Weakly stable range one

A ring R is a unit-regular ring provided that for any $a \in R$, there exists an invertible $u \in R$ such that $a = aua$ (see [7]). In [1, Theorem 1], Camillo and Khurana proved that every element in a unit-regular ring is the sum of an idempotent and an invertible element. We now extend this result to exchange rings having weakly stable range one by a new route.

Theorem 2.1. *Let R be an exchange ring having weakly stable range one. Then for any regular $a \in R$, there exist an idempotent $e \in R$ and a weak unit $u \in R$ such that $x = e + u$.*

Proof. Since $a \in R$ is regular, there exists a $x \in R$ such that $a = axa$. Hence, $R = \operatorname{Im} a \oplus (1 - ax)R = xaR \oplus \operatorname{Ker} a$. Since R is an exchange ring, so is $End_R(\operatorname{Im} a)$ by [10, Theorem 29.2]. Thus, we have right R-modules X_1, Y_1 such that $R = \operatorname{Im} a \oplus X_1 \oplus Y_1$ with $X_1 \subseteq \operatorname{Ker} a$ and $Y_1 \subseteq xaM$. Clearly, $\operatorname{Ker} a = \operatorname{Ker} a \cap (X_1 \oplus \operatorname{Im} a \oplus Y_1) = X_1 \oplus X_2$, where $X_2 = \operatorname{Ker} a \cap (\operatorname{Im} a \oplus Y_1)$. Similarly, there exists a right R-module Y_2 such that $xaM = Y_1 \oplus Y_2$. It is easy to see that $R = \operatorname{Im} a \oplus X_1 \oplus Y_1 = xaR \oplus X_1 \oplus X_2$. Clearly, $\varphi : \operatorname{Im} a \cong xaR$ given by $\pi(ar) = xar$ for any $r \in R$. As a result, $\operatorname{Im} a \oplus X_1 \cong xaR \oplus X_1$. Thus, we see that $X_2 \lesssim^{\oplus} Y_1$ or $Y_1 \lesssim^{\oplus} X_2$.

Assume that $X_2 \lesssim^{\oplus} Y_1$. Then there exist $\phi : X_2 \to Y_1$ and $\psi : Y_1 \to X_2$ such that $\psi\phi = 1_{X_2}$. Let $k : X_1 \oplus X_2 \to X_1 \oplus Y_1$ given by $k(x_1 + x_2) = x_1 + \phi(x_2)$ for any $x_1 \in X_1, x_2 \in X_2$. Let $l : X_1 \oplus Y_1 \to X_1 \oplus X_2$ given by $l(x_1 + y_1) = x_1 + \psi(y_1)$ for any $x_1 \in X_1, y_1 \in Y_1$. Let $h : M = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 \to X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 = M$ given by $h(x_1 + x_2 + y_1 + y_2) = k(x_1 + x_2) + y_1$ for any $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$. Let $v : M = X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 \to X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 = M$ given by $v(x_1+y_1+x_2+y_2) = l(x_1+y_1)+\phi(x_2)$ for any $x_1 \in X_1, y_1 \in Y_1, x_2 \in X_2, y_2 \in Y_2$. For any $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$, we have

$$\begin{aligned} hvhv(x_1 + x_2 + y_1 + y_2) &= hvh\big(l(x_1 + y_1) + \phi(x_2)\big) \\ &= hv\big(kl(x_1 + y_1) + \phi(x_2)\big) \\ &= h\big(lkl(x_1 + y_1) + l\phi(x_2)\big) \\ &= klkl(x_1 + y_1) + \phi\psi\phi(x_2) \\ &= kl(x_1 + y_1) + \phi(x_2) \\ &= hv(x_1 + x_2 + y_1 + y_2). \end{aligned}$$

Thus, we see that $hv = hvhv$. Set $e = hv$. Then $e \in R$ is an idempotent.

Assume that $(a - hv)(x_1 + y_1 + x_2 + y_2) = 0$ for any $x_1 \in X_1, y_1 \in Y_1, x_2 \in X_2, y_2 \in Y_2$. Then $a(y_1 + y_2) = kl(x_1 + y_1) + \phi(x_2) = x_1 + \phi\psi(y_1) + \phi(x_2) \in \operatorname{Im} a \cap (X_1 \oplus Y_1) = 0$, and then $x_1 = \phi\psi(y_1) - \phi(x_2) \in X_1 \cap Y_1 = 0$. It follows from $a(y_1+y_2) = 0$ that $y_1+y_2 \in (X_1 \oplus X_2) \cap (Y_1 \oplus Y_2) = 0$; hence $y_1+y_2 = 0$. This infers that $y_1 = -y_2 \in Y_1 \cap Y_2 = 0$, and then $y_1 = y_2 = 0$. Furthermore, we have that $\phi(x_2) = 0$. As $\psi\phi = 1$, we get $x_2 = 0$. Thus $x_1+y_1+x_2+y_2 = 0$. Let $u = a - e$. Then $a = e + u$ with $r.ann(u) = 0$.

Assume that $Y_1 \lesssim^{\oplus} X_2$. Then there exist $\phi : X_2 \to Y_1$ and $\psi : Y_1 \to X_2$ such that $\phi\psi = 1_{X_1}$. Let $k : X_1 \oplus X_2 \to X_1 \oplus Y_1$ given by $k(x_1 + x_2) = x_1 + \phi(x_2)$ for any $x_1 \in X_1, x_2 \in X_2$. Let $l : X_1 \oplus Y_1 \to X_1 \oplus X_2$ given by $l(x_1 + y_1) = x_1 + \psi(y_1)$ for any $x_1 \in X_1, y_1 \in Y_1$. Let $h : M = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 \to X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 = M$ given by $h(x_1 + x_2 + y_1 + y_2) = k(x_1 + x_2) + y_1$ for any $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$. Let $v : M = X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 \to X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 = M$ given by $v(x_1+y_1+x_2+y_2) = l(x_1+y_1)+\phi(x_2)$ for any $x_1 \in X_1, y_1 \in Y_1, x_2 \in X_2, y_2 \in Y_2$. For any $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$, we have

$$\begin{aligned} hvhv(x_1 + x_2 + y_1 + y_2) &= hvh\big(l(x_1 + y_1) + \phi(x_2)\big) \\ &= hv\big(x_1 + y_1 + \phi(x_2)\big) \\ &= h\big(l(x_1 + y_1) + l\phi(x_2)\big) \\ &= x_1 + y_1 + \phi(x_2) \\ &= hv(x_1 + x_2 + y_1 + y_2). \end{aligned}$$

This implies that $hv = hvhv$. Set $e = hv$. Then $e \in R$ is an idempotent.

Given any $w + x_1 + y_1 \in R$ with $w \in \operatorname{Im} a, x_1 \in X_1, y_1 \in Y_1$, we can find some $y_1' \in Y_1, y_2' \in Y_2$ such that $w = a(y_1' + y_2')$. Choose $x_1' = -x_1$. Since ϕ is a R-epimorphism, there exists a $x_2' \in X_2$ such that $\phi(x_2') = -y_1 - y_1'$. It is easy to verify that

$$\begin{aligned} (a - hv)(x_1' + y_1' + x_2' + y_2') &= a(y_1' + y_2') - x_1' - y_1' - \phi(x_2') \\ &= w + x_1 + y_1. \end{aligned}$$

Thus $a - hv : R \to R$ is a R-epimorphism. Since R is a projective right R-module, $a - hv$ splits. Let $u = a - hv$. Then $a = e + u$ with $l.ann(u) = 0$. In any case, we can find an idempotent $e \in R$ and a weak unit $u \in R$ such that $a = e + u$, as asserted. □

Corollary 2.2. *Let R be a one-sided unit-regular ring. Then for any $a \in R$, there exist an idempotent $e \in R$ and a right or left invertible $u \in R$ such that $a = e + u$.*

Proof. Since R is a one-sided unit-regular ring, it follows by [12, Theorem 3.4], R is an exchange ring having weakly stable range one. Let $a \in R$. In view of Theorem 2.1, we can find an idempotent $e \in R$ and a weak unit $u \in R$ such that $a = e + u$. Since R is regular, there exists a $v \in R$ such that $u = uvu$; hence, either $u(1-vu) = 0 = (1-uv)u$. As $r(u) = 0$ or $l(u) = 0$, we get $vu = 1$ or $uv = 1$. Thus, $u \in R$ is right or left invertible, and therefore we complete the proof. □

Theorem 2.3. *Let R be an exchange ring having weakly stable range one. Then for any regular $a \in R$, there exist weak units $u, v \in R$ such that $2a = u + v$.*

Proof. Let $a \in R$ be regular. Then there exists a $x \in R$ such that $a = axa$. As in the proof of Theorem 2.1, we have right R-modules X_1, Y_1, X_2 and Y_2 such that $R = \operatorname{Im} a \oplus X_1 \oplus Y_1$ with $X_1 \subseteq \operatorname{Ker} a$ and $Y_1 \subseteq xaM$. Furthermore, we have $R = \operatorname{Im} a \oplus X_1 \oplus Y_1 = xaR \oplus X_1 \oplus X_2$, and so $X_2 \lesssim^{\oplus} Y_1$ or $Y_1 \lesssim^{\oplus} X_2$.

Assume that $X_2 \lesssim^{\oplus} Y_1$. Then there exist $\phi : X_2 \to Y_1$ and $\psi : Y_1 \to X_2$ such that $\psi\phi = 1_{X_2}$. Let $k : X_1 \oplus X_2 \to X_1 \oplus Y_1$ given by $k(x_1 + x_2) = x_1 + \phi(x_2)$ for any $x_1 \in X_1, x_2 \in X_2$. Let $l : X_1 \oplus Y_1 \to X_1 \oplus X_2$ given by $l(x_1 + y_1) = x_1 + \psi(y_1)$ for any $x_1 \in X_1, y_1 \in Y_1$. Let $h : M = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 \to X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 = M$ given by $h(x_1 + x_2 + y_1 + y_2) = k(x_1 + x_2) + y_1$ for any $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$. Let $v : M = X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 \to X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 = M$ given by $v(x_1 + y_1 + x_2 + y_2) = l(x_1 + y_1) + \phi(x_2)$ for any $x_1 \in X_1, y_1 \in Y_1, x_2 \in X_2, y_2 \in Y_2$. For any $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$, we see that $hv \in R$ is an idempotent. As in the proof of Theorem 2.1, we show that $a + hv \in R$ is a weak unit. Analogously to the previous discussion, it is easy to verify that $a - hv \in R$ is a weak unit. Therefore $2a = (a - hv) + (a + hv)$ is the sum of two weak units, as asserted. □

Corollary 2.4. *Let R be a one-sided unit-regular ring. If $\frac{1}{2} \in R$, then for any $a \in R$, there exist right or left invertible $u, v \in R$ such that $a = u + v$.*

Proof. Let R be a one-sided unit-regular ring with $\frac{1}{2} \in R$. By Theorem 2.3, we can find two weak units $u, v \in R$ such that $2a = u + v$, and therefore $a = \frac{u}{2} + \frac{v}{2}$, as desired. □

Lemma 2.5. *If R is an exchange ring having weakly stable range one, then so is $M_n(R)$ for all $n \geq 1$.*

Proof. Given $M = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong nR \cong A_2$, we have $M = A_{11} \oplus \cdots \oplus A_{1n} \oplus B = A_{21} \oplus \cdots \oplus A_{2n} \oplus C$ with $A_{1i} \cong R \cong A_{2i}$ for all i. As $End_R(R) \cong R$, $End_R(R)$ has weakly stable range one. Hence, we can find some $D_1, E_1 \subseteq M$

such that $M = D_1 \oplus E_1 \oplus (A_{12} \oplus \cdots \oplus A_{1n} \oplus B) = D_1 \oplus (A_{22} \oplus \cdots \oplus A_{2n} \oplus C)$ or $M = D_1 \oplus (A_{12} \oplus \cdots \oplus A_{1n} \oplus B) = D_1 \oplus E_1 \oplus (A_{22} \oplus \cdots \oplus A_{2n} \oplus C)$. Thus we get $M = (E_1 \oplus A_{12}) \oplus (A_{13} \oplus \cdots \oplus A_{1n} \oplus B \oplus D_1) = A_{22} \oplus (A_{23} \oplus \cdots \oplus A_{2n} \oplus C \oplus D_1)$ or $M = A_{12} \oplus (A_{13} \oplus \cdots \oplus A_{1n} \oplus B \oplus D_1) = (E_1 \oplus A_{22}) \oplus (A_{23} \oplus \cdots \oplus A_{2n} \oplus C \oplus D_1)$. As a result, $M = A'_{12} \oplus (A_{13} \oplus \cdots \oplus A_{1n} \oplus B \oplus D_1) = A'_{22} \oplus (A_{23} \oplus \cdots \oplus A_{2n} \oplus C \oplus D_1)$, where $A'_{12} = E_1 \oplus A_{12}$ or $A'_{12} = A_{12}$ and $A'_{22} = A_{22}$ or $A'_{22} = E_1 \oplus A_{22}$. Clearly, $A'_{12} \cong A \cong A'_{22}$. By [3, Proposition 2] again, we can find $D_2 \subseteq M$ such that $M = A'_{13} \oplus (A_{14} \oplus \cdots \oplus A_{1n} \oplus B \oplus D_1 \oplus D_2) = A'_{23} \oplus (A_{24} \oplus \cdots \oplus A_{2n} \oplus C \oplus D_1 \oplus D_2)$ with $A'_{13} \cong A \cong A'_{23}$. Continuing in this way, we get $D_3, \dots, D_{n-1} \subseteq M$ such that $M = A'_{1n} \oplus (D_1 \oplus D_2 \oplus \cdots \oplus D_{n-1} \oplus B) = A'_{2n} \oplus (D_1 \oplus D_2 \oplus \cdots \oplus D_{n-1} \oplus C)$ with $A'_{1n} \cong A \cong A'_{2n}$. Thus we can find $D_n, E \subseteq M$ such that $M = (D_1 \oplus D_2 \oplus \cdots \oplus D_n) \oplus E \oplus B = (D_1 \oplus D_2 \oplus \cdots \oplus D_n) \oplus C$ or $M = (D_1 \oplus D_2 \oplus \cdots \oplus D_n) \oplus B = (D_1 \oplus D_2 \oplus \cdots \oplus D_n) \oplus E \oplus C$. Therefore $M_n(R) \cong End_R(nR)$ has weakly stable range one. □

Theorem 2.6. *Let R be an exchange ring having weakly stable range one. Then for any regular $A \in M_n(R)$, the following hold:*

(1) *There exist an idempotent matrix $E \in M_n(R)$ and a weak unit $U \in M_n(R)$ such that $A = E + U$.*
(2) *There exist weak unit $U, V \in M_n(R)$ such that $2A = U + V$.*

Proof. In view of Lemma 2.5, $M_n(R)$ is an exchange ring having weakly stable range one. Therefore we complete the proof by Theorem 2.1 and Theorem 2.3. □

Corollary 2.7. *Let R be a one-sided unit-regular ring. If $\frac{1}{2} \in R$, then for any regular $A \in M_n(R)$, there exist right and left invertible $U, V \in M_n(R)$ such that $A = U + V$.*

Proof. Since R is a one-sided unit-regular, it is an exchange ring having weakly stable range one. By virtue of Theorem 2.6, there exist weak unit $U', V' \in M_n(R)$ such that $2A = U' + V'$; hence, $A = \frac{1}{2}U' + \frac{1}{2}V'$. Therefore we complete the proof. □

3. Power stable ranges

It is well known that an exchange ring R satisfies power comparability if and only if $R = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$ implies that $B_1^n \lesssim^{\oplus} B_2^n$ or $B_2^n \lesssim^{\oplus} B_1^n$ for some $n \in \mathbb{N}$. In this section, we extend Theorem 2.1 to exchange rings satisfying power comparability.

Theorem 3.1. *Let R be an exchange ring satisfying power comparability. Then for any regular $a \in R$, there exist a positive integer n, an idempotent $(e_{ij}) \in M_n(R)$ and a weak unit $(u_{ij}) \in M_n(R)$ such that*

$$e_{ij} + u_{ij} = \begin{cases} a, & i = j; \\ 0, & i \neq j. \end{cases}$$

Proof. Since $a \in R$ is regular, there exists a $x \in R$ such that $a = axa$. Hence, $R = \operatorname{Im} a \oplus (1 - ax)R = xaR \oplus \operatorname{Ker} a$. Since R is an exchange ring, we have right R-modules X_1, Y_1 such that $R = \operatorname{Im} a \oplus X_1 \oplus Y_1$ with $X_1 \subseteq \operatorname{Ker} a$ and $Y_1 \subseteq xaM$. Obviously, $\operatorname{Ker} a = \operatorname{Ker} a \cap (X_1 \oplus \operatorname{Im} a \oplus Y_1) = X_1 \oplus X_2$, where $X_2 = \operatorname{Ker} a \cap (\operatorname{Im} a \oplus Y_1)$. Similarly, there exists a right R-module Y_2 such that $xaM = Y_1 \oplus Y_2$. Thus, $R = \operatorname{Im} a \oplus X_1 \oplus Y_1 = xaR \oplus X_1 \oplus X_2$ with $\varphi : \operatorname{Im} a \cong xaR$, where $\pi(ar) = xar$ for any $r \in R$. Thus, $\operatorname{Im} a \oplus X_1 \cong xaR \oplus X_1$. Clearly, $End_R(\operatorname{Im} a \oplus X_1)$ satisfies power comparability. Hence, there exists some $n \in \mathbb{N}$ such that $X_2^n \lesssim^{\oplus} Y_1^n$ or $Y_1^n \lesssim^{\oplus} X_2^n$.

Assume that $X_2^n \lesssim^{\oplus} Y_1^n$. Then there exist $\phi : X_2^n \to Y_1^n$ and $\psi : Y_1^n \to X_2^n$ such that $\psi\phi = 1_{X_2^n}$. Let $k : X_1^n \oplus X_2^n \to X_1^n \oplus Y_1^n$ given by $k(x_1 + x_2) = x_1 + \phi(x_2)$ for any $x_1 \in X_1^n, x_2 \in X_2^n$. Let $l : X_1^n \oplus Y_1^n \to X_1^n \oplus X_2^n$ given by $l(x_1 + y_1) = x_1 + \psi(y_1)$ for any $x_1 \in X_1^n, y_1 \in Y_1^n$. Let $h : M^n = X_1^n \oplus X_2^n \oplus Y_1^n \oplus Y_2^n \to X_1^n \oplus Y_1^n \oplus X_2^n \oplus Y_2^n = M^n$ given by $h(x_1 + x_2 + y_1 + y_2) = k(x_1 + x_2) + y_1$ for any $x_1 \in X_1^n, x_2 \in X_2^n, y_1 \in Y_1^n, y_2 \in Y_2^n$. Let $v : M = X_1^n \oplus Y_1^n \oplus X_2^n \oplus Y_2^n \to X_1^n \oplus X_2^n \oplus Y_1^n \oplus Y_2^n = M^n$ given by $v(x_1 + y_1 + x_2 + y_2) = l(x_1 + y_1) + \phi(x_2)$ for any $x_1 \in X_1^n, y_1 \in Y_1^n, x_2 \in X_2^n, y_2 \in Y_2^n$. As in the proof of Theorem 2.1, we see that $e : R^n \to R^n$ is an idempotent. Assume that $(aI_n - hv)(x_1 + y_1 + x_2 + y_2) = 0$ for any $x_1 \in X_1^n, y_1 \in Y_1^n, x_2 \in X_2^n, y_2 \in Y_2^n$. Also we have $x_1 + y_1 + x_2 + y_2 = 0$. Let $u = aI_n - e$. Then $aI_n = e + u$ with $r.ann(u) = 0$.

Assume that $Y_1 \lesssim^{\oplus} X_2$. Then there exist $\phi : X_2 \to Y_1$ and $\psi : Y_1 \to X_2$ such that $\phi\psi = 1_{X_1}$. Let $k : X_1^n \oplus X_2^n \to X_1^n \oplus Y_1^n$ given by $k(x_1 + x_2) = x_1 + \phi(x_2)$ for any $x_1 \in X_1^n, x_2 \in X_2^n$. Let $l : X_1^n \oplus Y_1^n \to X_1^n \oplus X_2^n$ given by $l(x_1 + y_1) = x_1 + \psi(y_1)$ for any $x_1 \in X_1^n, y_1 \in Y_1^n$. Let $h : M^n = X_1^n \oplus X_2^n \oplus Y_1^n \oplus Y_2^n \to X_1^n \oplus Y_1^n \oplus X_2^n \oplus Y_2^n = M^n$ given by $h(x_1 + x_2 + y_1 + y_2) = k(x_1 + x_2) + y_1$ for any $x_1 \in X_1^n, x_2 \in X_2^n, y_1 \in Y_1^n, y_2 \in Y_2^n$. Let $v : M^n = X_1^n \oplus Y_1^n \oplus X_2^n \oplus Y_2^n \to X_1^n \oplus X_2^n \oplus Y_1^n \oplus Y_2^n = M^n$ given by $v(x_1 + y_1 + x_2 + y_2) = l(x_1 + y_1) + \phi(x_2)$ for any $x_1 \in X_1^n, y_1 \in Y_1^n, x_2 \in X_2^n, y_2 \in Y_2^n$. Similarly to the previous consideration, $e \in R^n \to R^n$ is an idempotent. Furthermore, $aI_n - hv : R^n \to R^n$ is a R-epimorphism. Since R^n is a projective right R-module, $aI_n - hv$ splits. Let $u = aI_n - hv$. Then $aI_n = e + u$ with $l.ann(u) = 0$. Therefore we complete the proof. □

Corollary 3.2. *Let R be a regular ring satisfying power comparability. Then for any $a \in R$, there exist a positive integer n, an idempotent $(e_{ij}) \in M_n(R)$ and a right or left invertible $(u_{ij}) \in M_n(R)$ such that*

$$e_{ij} + u_{ij} = \begin{cases} a, & i = j; \\ 0, & i \neq j. \end{cases}$$

Proof. Let $a \in R$. It follows by Theorem 3.1 that we can find a positive integer n, an idempotent $(e_{ij}) \in M_n(R)$ and a weak unit $(u_{ij}) \in M_n(R)$ such that

$$e_{ij} + u_{ij} = \begin{cases} a, & i = j; \\ 0, & i \neq j. \end{cases}$$

Since R is regular, we can find a $V \in M_n(R)$ such that $(u_{ij}) = (u_{ij})V(u_{ij})$. As $(u_{ij}) \in M_n(R)$ is a weak unit, we deduce that $(u_{ij})V = I_n$ or $V(u_{ij}) = I_n$, and therefore we complete the proof. □

The following lemma gives a generalization of [5, Lemma 3.1].

Lemma 3.3. *Let R be an exchange ring. If for any regular $x, y \in R$, there exist $n \in \mathbb{N}$ and $A \in M_n(R)$ such that $xI_n - A$ is an idempotent and a one-sided unit, and that $I_n - yA \in GL_n(R)$, then R satisfies power comparability.*

Proof. Given $ax + b = 1$ with $a, x, b \in R$, then we can find an idempotent $e \in R$ such that $e = bs$ and $1 - e = (1 - b)t$ by [10, Theorem 1], where $s, t \in R$. Since $axt + e = (1 - b)t + e = 1$, we have $(1 - e)axt(1 - e) + e = 1$. In addition, we see that $(1 - e)a \in R$ and $xt(1 - e) \in R$ are regular. Thus, we can find $n \in \mathbb{N}$ and $A \in M_n(R)$ such that $(1 - e)aI_n - A = EU$ and $I_n - xt(1-e)A \in GL_n(R)$, where $E \in M_n(R)$ is an idempotent and $U \in M_n(R)$ is a one-sided unit. This implies that $I_n - xt(1 - e)A \in GL_n(R)$, and so $I_n - Axt(1 - e) = V \in GL_n(R)$, i.e., $EUxt(1 - e) + eI_n = V$. Furthermore, $V^{-1}EUxt(1 - e) + V^{-1}eI_n = I_n$. Obviously, $V^{-1}EU \in M_n(R)$ is a one-sided unit. In view of [2, Lemma 1], there exists a $Y \in M_n(R)$ such that $xt(1 - e)I_n + ZV^{-1}e \in M_n(R)$ is one-sided unit. By using [2, Lemma 1] again, we can find a $Z \in M_n(R)$ such that $(1 - e)aI_n + eZ \in M_n(R)$ is a one-sided unit. Consequently, $aI_n + e(Z - aI_n) = aI_n + bs(Z - aI_n) \in M_n(R)$ is one-sided unit, as required. □

Following Li and Tong, a ring R is a one-sided unit π-regular ring provided that for any $a \in R$ there exist a positive integer n and a right or left invertible $u \in R$ such that $a^n = a^n u a^n$ (cf. [8]).

Theorem 3.4. *Let R be a one-sided unit π-regular ring. Then for any regular $a \in R$, there exist a positive integer n, an idempotent $(e_{ij}) \in M_n(R)$ and a weak unit $(u_{ij}) \in M_n(R)$ such that*

$$e_{ij} + u_{ij} = \begin{cases} a, & i = j; \\ 0, & i \neq j. \end{cases}$$

Proof. Let $x, y \in R$ be regular. Then we have some $m \in \mathbb{N}$ such that $x^m = x^m u x^m$, where $u \in R$ is a one-sided unit. In view of [11, Lemma 3.3], x^m is the product of an idempotent $e \in R$ and a one-sided unit $u \in R$. Let

$$A = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & x & \cdots & x^{m-1} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} y^{m-1} & \cdots & y & 1 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \in M_m(R).$$

Clearly,

$$B(xI_m - A) = \begin{pmatrix} 0 & \cdots & 0 & x^m \\ -1 & \cdots & 0 & * \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & x & * \\ 0 & \cdots & -1 & * \end{pmatrix},$$

$$C(I_m - yA) = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \cdots & 0 & * \\ -y & \cdots & 0 & * \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & * \end{pmatrix}.$$

Obviously, $B, C \in GL_m(R)$; hence, $I_m - yA \in GL_m(R)$. Furthermore,

$$xI_m - A = B^{-1} \begin{pmatrix} 0 & \cdots & 0 & x^m \\ -1 & \cdots & 0 & * \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & x & * \\ 0 & \cdots & -1 & * \end{pmatrix}.$$

Since $x^m = eu$, we get

$$xI_m - A = B^{-1} \begin{pmatrix} e & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & u \\ -1 & \cdots & 0 & * \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & x & * \\ 0 & \cdots & -1 & * \end{pmatrix}.$$

Let $E = B^{-1}diag(e, 1, \cdots, 1)B$. Then $B^{-1}diag(e, 1, \cdots, 1) = EB^{-1}$ with $E = E^2$. As $u \in R$ is a one-sided unit, we see that $\begin{pmatrix} 0 & \cdots & 0 & u \\ -1 & \cdots & 0 & * \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & x & * \\ 0 & \cdots & -1 & * \end{pmatrix}$ is a one-sided unit, and so

$$B^{-1}\begin{pmatrix} 0 & \cdots & 0 & u \\ -1 & \cdots & 0 & * \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & x & * \\ 0 & \cdots & -1 & * \end{pmatrix} \in M_m(R)$$

is a one-sided unit. In view of [11, Lemma 3.3], $xI_{mn} - A$ is the product of an idempotent and a one-sided unit. Hence, R is an exchange ring satisfying power comparability by Lemma 3.4. Therefore we complete the proof from Corollary 3.2. □

Corollary 3.5. *Let R be a one-sided unit π-regular ring. Then for any $a \in R$, there exist a positive integer n, an idempotent $(e_{ij}) \in M_n(R)$ and a right or left invertible $(u_{ij}) \in M_n(R)$ such that*

$$e_{ij} + u_{ij} = \begin{cases} a, & i = j; \\ 0, & i \neq j. \end{cases}$$

Proof. Since R is a regular ring, so is $M_n(R)$ by [1, Theorem 1]. In view of Theorem 3.4, there exist a positive integer n, an idempotent $(e_{ij}) \in M_n(R)$ and a weak unit $(u_{ij}) \in M_n(R)$ such that

$$e_{ij} + u_{ij} = \begin{cases} a, & i = j; \\ 0, & i \neq j. \end{cases}$$

Clearly, we have a $(v_{ij}) \in M_n(R)$ such that $(u_{ij}) = (u_{ij})(v_{ij})(u_{ij})$. It follows that $(u_{ij}) \in R$ is right or left invertible, as required. □

Theorem 3.6. *Let R be an exchange ring satisfying power cancellation. Then for any regular $a \in R$, there exist a positive integer n, an idempotent $(e_{ij}) \in M_n(R)$ and an invertible $(u_{ij}) \in M_n(R)$ such that*

$$e_{ij} + u_{ij} = \begin{cases} a, & i = j; \\ 0, & i \neq j. \end{cases}$$

Proof. Since $a \in R$ is regular, there exists a $x \in R$ such that $a = axa$. As in the proof on Theorem 2.1, we see that $R = \operatorname{Im} a \oplus X_1 \oplus Y_1 = xaR \oplus X_1 \oplus X_2$ with $\operatorname{Im} a \oplus X_1 \cong xaR \oplus X_1$. Since R satisfies power-cancellation, so does $End_R(\operatorname{Im} a \oplus X_1)$. Thus, we get $\phi : X_2^n \cong Y_1^n$ for some $n \in \mathbb{N}$. Thus, we have $\psi : Y_1^n \to X_2^n$ such that $\psi\phi = 1_{X_2^n}$ and $\phi\psi = 1_{Y_1^n}$. Let $k : X_1^n \oplus X_2^n \to X_1^n \oplus Y_1^n$ given by $k(x_1 + x_2) = x_1 + \phi(x_2)$ for any $x_1 \in X_1^n, x_2 \in X_2^n$. Let $l : X_1^n \oplus Y_1^n \to X_1^n \oplus X_2^n$ given by $l(x_1 + y_1) = x_1 + \psi(y_1)$ for any $x_1 \in X_1^n, y_1 \in Y_1^n$. Let $h : R^n = X_1^n \oplus X_2^n \oplus Y_1^n \oplus Y_2^n \to X_1^n \oplus Y_1^n \oplus X_2^n \oplus Y_2^n = R^n$ given by $h(x_1 + x_2 + y_1 + y_2) = k(x_1 + x_2) + y_1$ for any $x_1 \in X_1^n, x_2 \in X_2^n, y_1 \in Y_1^n, y_2 \in Y_2^n$. Let $v : R^n = X_1^n \oplus Y_1^n \oplus X_2^n \oplus Y_2^n \to X_1^n \oplus X_2^n \oplus Y_1^n \oplus Y_2^n = R^n$ given by $v(x_1 + y_1 + x_2 + y_2) = l(x_1 + y_1) + \phi(x_2)$ for any $x_1 \in X_1^n, y_1 \in Y_1^n, x_2 \in X_2^n, y_2 \in Y_2^n$. As in the proof of Theorem 2.1, we see that $e : R^n \to R^n$

is an idempotent. Assume that $(aI_n - hv)(x_1 + y_1 + x_2 + y_2) = 0$ for any $x_1 \in X_1^n, y_1 \in Y_1^n, x_2 \in X_2^n, y_2 \in Y_2^n$. Also we have $x_1 + y_1 + x_2 + y_2 = 0$. Furthermore, $aI_n - hv : R^n \to R^n$ is a R-epimorphism. Since R^n is a projective right R-module, $aI_n - hv$ splits. Let $u = aI_n - hv$. Then $u \in M_n(R)$ is invertible and $aI_n = e + u$, as asserted. □

Corollary 3.7. *Let R be an exchange ring satisfying power cancellation. Then for any regular $a \in R$, there exist a positive integer n and invertible $(u_{ij}), (v_{ij}) \in M_n(R)$ such that*

$$u_{ij} + v_{ij} = \begin{cases} 2a, & i = j; \\ 0, & i \neq j. \end{cases}$$

Proof. Let $a \in R$ be regular. By virtue of Theorem 3.6, there exist a positive integer n, an idempotent $(e_{ij}) \in M_n(R)$ and an invertible $(u_{ij}) \in M_n(R)$ such that

$$e_{ij} + u_{ij} = \begin{cases} a, & i = j; \\ 0, & i \neq j. \end{cases}$$

Furthermore, we have $(v_{ij}) \in M_n(R)$ such that

$$-e_{ij} + v_{ij} = \begin{cases} a, & i = j; \\ 0, & i \neq j. \end{cases}$$

analogously to the consideration in Theorem 3.6. Thus we complete the proof. □

Recall that a ring R is a unit π-regular ring provided that for any $a \in R$ there exist a positive integer n and an invertible $u \in R$ such that $a^n = a^n u a^n$.

Corollary 3.8. *Let R be a unit π-regular ring. If $\frac{1}{2} \in R$, then for any regular $a \in R$, there exist a positive integer n and invertible $(u_{ij}), (v_{ij}) \in M_n(R)$ such that*

$$u_{ij} + v_{ij} = \begin{cases} a, & i = j; \\ 0, & i \neq j. \end{cases}$$

Proof. Since R is a unit π-regular ring, it is an exchange ring. In view of [5, Theorem 3.2], it satisfies power cancellation. Let $a \in R$ be regular. According to Corollary 3.6, there exist a positive integer n and invertible $(u'_{ij}), (v'_{ij}) \in M_n(R)$ such that

$$e_{ij} + u_{ij} = \begin{cases} a, & i = j; \\ 0, & i \neq j. \end{cases}$$

Let

$$e_{ij} + u_{ij} = \begin{cases} a, & i = j; \\ 0, & i \neq j. \end{cases}$$

Then the result follows. □

References

[1] V. P. Camillo and D. Khurana, *A characterization of unit regular rings*, Comm. Algebra **29** (2001), no. 5, 2293–2295.

[2] H. Chen, *Elements in one-sided unit regular rings*, Comm. Algebra **25** (1997), no. 8, 2517–2529.

[3] ———, *Comparability of modules over regular rings*, Comm. Algebra **25** (1997), no. 11, 3531–3543.

[4] ———, *Exchange rings with artinian primitive factors*, Algebr. Represent Theory **2** (1999), no. 2, 201–207.

[5] ———, *Power-substitution, exchange rings and unit π-regularity*, Comm. Algebra **28** (2000), no. 11, 5223–5233.

[6] ———, *Exchange rings satisfying power ideal-cancellation*, Sci. China Ser. A **46** (2003), no. 6, 804–814.

[7] K. R. Goodearl, *von Neumann Regular Rings*, Pitman, London, San Francisco, Melbourne, 1979; second ed., Krieger, Malabar, Fl., 1991.

[8] Q. Li and W. Tong, *Weak cancellation of modules and weakly stable range conditions for exchange rings*, Acta Math. Sinica (Chinese) **45** (2002), no. 6, 1121–1126.

[9] Q. Li, J. Zhu, and W. Tong, *On related power comparability of modules*, Comm. Algebra **31** (2003), no. 10, 4925–4938.

[10] A. Tuganbaev, *Rings Close to Regular*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[11] J. Wei, *Exchange rings with weakly stable range one*, Vietnam J. Math. **32** (2004), no. 4, 441–449.

[12] ———, *Unit-regularity and stable range conditions*, Comm. Algebra **33** (2005), no. 6, 1937–1946.

[13] T. Wu, *The problem of weak cancellation of modules and qu-regular rings*, Acta Math. Sinica (Chinese), **38** (1995), no. 6, 746–751.

[14] ———, *The power-substitution condition of endomorphism rings of quasi-projective modules*, Comm. Algebra **28** (2000), no. 1, 407–418.

Department of Mathematics
Hunan Normal University
Changsha 410006, P. R. China
E-mail address: chyzxl@hunnu.edu.cn