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PARAMETRIC GENERALIZED MIXED IMPLICIT QUASI-VARIATIONAL INCLUSIONS

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Abstract. An existence theorem for a new class of parametric generalized mixed implicit quasi-variational inclusion problems is established in Hilbert spaces. Further, we study the behavior and sensitivity analysis of the solution set in this class of parametric variational inclusion problems.

1. Introduction

Sensitivity analysis of solutions for variational inequalities has been studied extensively by many authors via quite different techniques.

By using the projection method, Dafermos [3], Yen [13], Mukherjee and Verma [6], Noor [8] dealt with the sensitivity analysis of solutions for some variational inequalities with single-valued mappings in finite dimensional spaces and Hilbert spaces.

By using the resolvent operator technique, Adly [1], Noor and Noor [9], and Agarwal, Cho and Huang [2] studied sensitivity analysis of the solution set with single-valued mappings in Hilbert spaces.

Recently, by using the resolvent operator technique, Park and Jeong [10] and Salahuddin [12] studied the behavior and sensitivity analysis of the solution set for parametric generalized mixed variational inequalities and parametric generalized variational inclusion problems with set-valued mappings in Hilbert spaces, respectively.

The purpose of this paper is to introduce and study the behavior and sensitivity analysis of the solution set for a class of parametric generalized mixed implicit quasi-variational inclusion problems in Hilbert spaces by using the concept of the resolvent operator and the property of fixed point sets of set-valued contractive mappings. In particular, the classes of problems studied by Park and Jeong [10], Ding and Luo [4], and Dafermos [3] will be special cases of our problems.

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2. Preliminaries

Let H be a real Hilbert space with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Let $C(H)$ denote the family of all nonempty compact subsets of H and $\tilde{H}(\cdot,\cdot)$ denote the Hausdorff metric on $C(H)$ defined by

$$\tilde{H}(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}, \quad \forall A,B \in C(H),$$

where $d(A,b) = \inf_{a\in A}\|a-b\|$ and $d(a,B) = \inf_{b\in B}\|a-b\|$.

Let Ω be a nonempty open subset of H in which the parameter λ takes values. Let $M : H\times H\times\Omega \to H$, $N : H\times H\times H\times\Omega \to H$, $g : H\times\Omega \to H$ be single-valued mappings. Let $A,B,C,D,E,F : H\times\Omega \to C(H)$ be set-valued mappings. Let $z \in H$ be given and let $\varphi : H\times H\times\Omega \to \mathbb{R}\cup\{+\infty\}$ be such that for each fixed $(z,\lambda)\in H\times\Omega$, $\varphi(\cdot,z,\lambda)$ is a proper convex lower semicontinuous functional satisfying $g(H,\lambda)\cap\mathrm{dom}(\partial\varphi(\cdot,z,\lambda)) \neq \phi$, where $\partial\varphi(\cdot,z,\lambda)$ is the subdifferential of $\varphi(\cdot,z,\lambda)$. By [11], $\partial\varphi(\cdot,z,\lambda) : H \to 2^H$ is a maximal monotone mapping. Let $b : H\times H\times\Omega \to \mathbb{R}$ be a real-valued function satisfying

(i) $b(x,y,\lambda)$ is linear in first argument,

(ii) $b(x,y,\lambda)$ is bounded, i.e., there exists a constant $\nu > 0$ such that

$$b(x,y,\lambda) \le \nu\|x\|\|y\|,$$

(iii) for all $(x,y,z,\lambda)\in H\times H\times H\times\Omega$,

$$b(x,y,\lambda) - b(x,z,\lambda) \le b(x,y-z,\lambda).$$

In this paper, we shall consider the following parametric generalized mixed implicit quasi-variational inclusion problem(PGMIQVIP): for each fixed $\lambda\in\Omega$, find $x\in H$, $a\in A(x,\lambda)$, $b\in B(x,\lambda)$, $c\in C(x,\lambda)$, $d\in D(x,\lambda)$, $e\in E(x,\lambda)$, $f\in F(x,\lambda)$ such that

$$\begin{aligned} &\langle M(g(x,\lambda),a,\lambda) - N(b,c,d,\lambda) + z, y - g(x,\lambda)\rangle \\ &\quad + b(y,e,\lambda) - b(g(x,\lambda),e,\lambda) \\ &\ge \varphi(g(x,\lambda),f,\lambda) - \varphi(y,f,\lambda), \quad \forall y\in H. \end{aligned} \tag{2.1}$$

Special Cases

(I) If $b(x,y,\lambda) = 0$ for all $(x,y,\lambda)\in H\times H\times\Omega$, then problem (2.1) reduces to the following parametric quasi-variational inclusion problem: for each fixed $\lambda\in\Omega$, find $x\in H$, $a\in A(x,\lambda)$, $b\in B(x,\lambda)$, $c\in C(x,\lambda)$, $d\in D(x,\lambda)$, $f\in F(x,\lambda)$ such that

$$\begin{aligned} &\langle M(g(x,\lambda),a,\lambda) - N(b,c,d,\lambda) + z, y - g(x,\lambda)\rangle \\ &\ge \varphi(g(x,\lambda),f,\lambda) - \varphi(y,f,\lambda), \quad \forall y\in H. \end{aligned} \tag{2.2}$$

(II) If $M(x,a,\lambda) = 0$ for all $(x,a,\lambda)\in H\times H\times\Omega$, $N(b,c,d,\lambda) = -N_1(b,c,d,\lambda)$ for all $(b,c,d,\lambda)\in H\times H\times H\times\Omega$, and $z = 0$, then problem (2.2) reduces to the following parametric problem: for each fixed $\lambda \in \Omega$, find $x \in H$, $b \in B(x,\lambda)$, $c \in C(x,\lambda)$, $d \in D(x,\lambda)$, $f \in F(x,\lambda)$ such that

$$\langle N_1(b,c,d,\lambda), y - g(x,\lambda)\rangle \geq \varphi(g(x,\lambda), f, \lambda) - \varphi(y, f, \lambda), \tag{2.3}$$

for all $y \in H$.

(III) If $K : H \times \Omega \to 2^H$ is a set-valued mapping such that for each $(y,\lambda) \in H \times \Omega$, $K(y,\lambda)$ is a closed convex subset of H and for each $(y,\lambda) \in H \times \Omega$, $\varphi(\cdot, y, \lambda) = I_{K(y,\lambda)}(\cdot)$ is the indicator function of $K(y,\lambda)$, then problem (2.3) reduces to the following parametric problem: for each fixed $\lambda \in \Omega$, find $x \in H$, $b \in B(x,\lambda)$, $c \in C(x,\lambda)$, $d \in D(x,\lambda)$ such that

$$\langle N_1(b,c,d,\lambda), y - g(x,\lambda)\rangle \geq 0, \quad \forall y \in H. \tag{2.4}$$

In this paper, our main aim is to study the behavior of the solution set $S(\lambda)$ of (PGMIQVIP) (2.1), and the conditions on these mappings $A, B, C, D, E, F, M, N, g, b, \varphi$ under which the function $S(\lambda)$ is Lipschitz continuous (or continuous) with respect to the parameter $\lambda \in \Omega$.

Let H be a Hilbert space and let $G : H \to 2^H$ be a maximal monotone mapping. For any fixed $\rho > 0$, the mapping $J_\rho^G : H \to H$ defined by

$$J_\rho^G(x) = (I + \rho G)^{-1}(x), \quad \forall x \in H,$$

is said to be the resolvent operator of G, where I is the identity mapping on H.

It is well known that J_ρ^G is a nonexpansive mapping (see [11]).

Definition 2.1. A mapping $g : H \times \Omega \to H$ is called

(1) α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle g(x,\lambda) - g(y,\lambda), x - y\rangle \geq \alpha\|x-y\|^2, \quad \forall (x,y,\lambda) \in H \times H \times \Omega.$$

(2) β-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|g(x,\lambda) - g(y,\lambda)\| \leq \beta\|x-y\|, \quad \forall (x,y,\lambda) \in H \times H \times \Omega.$$

Definition 2.2. Let $A : H \times \Omega \to C(H)$ be a set-valued mapping and $M : H \times H \times \Omega \to H$ be a single-valued mapping. Then

(1) A is said to be η-relaxed Lipschitz continuous with respect to the second argument of M if there exists a constant $\eta > 0$ such that

$$\langle M(z,u,\lambda) - M(z,v,\lambda), x - y\rangle \leq -\eta\|x-y\|^2,$$
$$\forall (x,y,z,\lambda) \in H \times H \times H \times \Omega, u \in A(x,\lambda), v \in A(y,\lambda).$$

(2) A is said to be λ_A-Lipschitz continuous if there exists a constant $\lambda_A > 0$ such that

$$\tilde{H}(A(x,\lambda), A(y,\lambda)) \leq \lambda_A\|x-y\|, \quad \forall (x,y,\lambda) \in H \times H \times \Omega.$$

Definition 2.3. A mapping $M : H \times H \times \Omega \to H$ is said to be M_1-Lipschitz continuous in the first argument if there exists a constant $M_1 > 0$ such that

$$\|M(x,a,\lambda) - M(y,a,\lambda)\| \leq M_1\|x-y\|, \quad \forall (x,y,a,\lambda) \in H \times H \times H \times \Omega.$$

In similar way, one can define the Lipschitz continuity of M in the second argument.

Definition 2.4. Let $B, C : H \times \Omega \to C(H)$ be set-valued mappings and $N : H \times H \times H \times \Omega \to H$ be a single-valued mapping. Then

(1) B is said to be γ-relaxed Lipschitz continuous with respect to the first argument of N if there exists a constant $\gamma > 0$ such that

$$\langle N(u,c,d,\lambda) - N(v,c,d,\lambda), x-y\rangle \le -\gamma\|x-y\|^2,$$

$$\forall (x,y,c,d,\lambda) \in H \times H \times H \times H \times \Omega, u \in B(x,\lambda), v \in B(y,\lambda).$$

(2) C is said to be σ-generalized pseudo-contractive with respect to the second argument of N if there exists a constant $\sigma > 0$ such that

$$\langle N(b,u,d,\lambda) - N(b,v,d,\lambda), x-y\rangle \le \sigma\|x-y\|^2,$$

$$\forall (x,y,b,d,\lambda) \in H \times H \times H \times H \times \Omega, u \in C(x,\lambda), v \in C(y,\lambda).$$

(3) N is said to be N_1-Lipschitz continuous in the first argument if there exists a constant $N_1 > 0$ such that

$$\|N(x,c,d,\lambda) - N(y,c,d,\lambda)\| \le N_1\|x-y\|,$$

$$\forall (x,y,c,d,\lambda) \in H \times H \times H \times H \times \Omega.$$

In similar way, one can define the Lipschitz continuity of N in the second and third argument, respectively.

Lemma 2.1 ([5]). *Let (X,d) be a complete metric space and $T_1, T_2 : X \to C(X)$ be two set-valued contractive mappings with same contractive constant $\theta \in (0,1)$, i.e.,*

$$\tilde{H}(T_i(x), T_i(y)) \le \theta d(x,y), \quad \forall x,y \in X, i = 1,2.$$

Then

$$\tilde{H}(F(T_1), F(T_2)) \le \frac{1}{1-\theta} \sup_{x \in X} \tilde{H}(T_1(x), T_2(x)),$$

where $F(T_i)$ is the fixed point set of T_i, $i = 1,2$.

Lemma 2.2. *For each fixed $\lambda \in \Omega$, let $b : H \times H \times \Omega \to \mathbb{R}$ be a real valued function satisfying the conditions* (i)-(iii) *as mentioned before. Then for each $y \in H$, there exists a unique mapping $h(y,\lambda) \in H$ such that*

$$b(x,y,\lambda) = \langle h(y,\lambda), x\rangle, \quad \forall x \in H,$$

and

$$\|h(y,\lambda) - h(z,\lambda)\| \le \nu\|y-z\|, \quad \forall y,z \in H,$$

where ν is a positive constant, i.e., the mapping $h : H \times \Omega \to H$ is Lipschitz continuous.

Proof. By the condition (ii) on $b(\cdot,\cdot,\lambda)$, we have

$$|b(x,y,\lambda)| \le \nu\|x\|\|y\|, \quad \forall x,y \in H,$$

and hence

$$b(x,0,\lambda) = b(0,y,\lambda) = 0,$$

and for each $y \in H$, $x \mapsto b(x,y,\lambda)$ is continuous. By the conditions (ii) and (iii), we have

$$\begin{aligned} |b(x,y,\lambda) - b(x,z,\lambda)| &\le |b(x,y-z,\lambda)| \\ &\le \nu\|x\|\|y-z\|, \quad \forall x,y,z \in H, \end{aligned}$$

and so for each $x \in H$, $y \mapsto b(x,y,\lambda)$ is also continuous. Hence for each given $y \in H$, $x \mapsto b(x,y,\lambda)$ is a continuous linear functional on H. By the Riesz representation theorem, there exists a unique point $h(y,\lambda) \in H$ such that

$$b(x,y,\lambda) = \langle h(y,\lambda), x\rangle, \quad \forall x \in H,$$

and

$$\begin{aligned} \|h(y,\lambda) - h(z,\lambda)\| &= \sup_{\|x\|\le 1} |\langle h(y,\lambda) - h(z,\lambda), x\rangle| \\ &= \sup_{\|x\|\le 1} |b(x,y,\lambda) - b(x,z,\lambda)| \\ &\le \sup_{\|x\|\le 1} |b(x,y-z,\lambda)| \\ &\le \sup_{\|x\|\le 1} \nu\|x\|\|y-z\| \\ &\le \nu\|y-z\|, \quad \forall y,z \in H. \end{aligned}$$

□

3. Sensitivity analysis of solution set

We first transfer the PGMIQVIP (2.1) into a parametric fixed point problem.

Theorem 3.1. *For each fixed $\lambda \in \Omega$, (x,a,b,c,d,e,f) is a solution of the PGMIQVIP (2.1) if and only if there exist $x \in H$, $a \in A(x,\lambda)$, $b \in B(x,\lambda)$, $c \in C(x,\lambda)$, $d \in D(x,\lambda)$, $e \in E(x,\lambda)$, and $f \in F(x,\lambda)$ such that the following relation holds:*

$$\begin{aligned} g(x,\lambda) = J_\rho^{\partial\varphi(\cdot,f,\lambda)}[g(x,\lambda) - \rho\{M(g(x,\lambda),a,\lambda) - N(b,c,d,\lambda) \\ + z + h(e,\lambda)\}], \end{aligned} \tag{3.1}$$

Proof. For each $\lambda \in \Omega$, suppose that (x,a,b,c,d,e,f) is a solution of the PGMIQVIP (2.1). Then there exist $x \in H$, $a \in A(x,\lambda)$, $b \in B(x,\lambda)$, $c \in C(x,\lambda)$, $d \in D(x,\lambda)$, $e \in E(x,\lambda)$, $f \in F(x,\lambda)$ such that

$$\begin{aligned} &\langle M(g(x,\lambda),a,\lambda) - N(b,c,d,\lambda) + z, y - g(x,\lambda)\rangle \\ &\quad + b(y,e,\lambda) - b(g(x,\lambda),e,\lambda) \\ &\ge \varphi(g(x,\lambda),f,\lambda) - \varphi(y,f,\lambda), \quad \forall y \in H. \end{aligned} \tag{3.2}$$

By Lemma 2.2, we have

$$\begin{aligned} b(y, e, \lambda) - b(g(x, \lambda), e, \lambda) &= b(y - g(x, \lambda), e, \lambda) \\ &= \langle h(e, \lambda), y - g(x, \lambda) \rangle, \quad \forall y \in H. \end{aligned}$$

Hence the relation (3.2) holds if and only if

$$\begin{aligned} \varphi(y, f, \lambda) - \varphi(g(x, \lambda), f, \lambda) \geq \langle &N(b, c, d, \lambda) - z - M(g(x, \lambda), a, \lambda) \\ &- h(e, \lambda), y - g(x, \lambda) \rangle. \end{aligned} \tag{3.3}$$

The relation (3.3) holds if and only if

$$N(b, c, d, \lambda) - z - M(g(x, \lambda), a, \lambda) - h(e, \lambda) \in \partial\varphi(\cdot, f, \lambda)(g(x, \lambda)). \tag{3.4}$$

By the definition of $J_\rho^{\partial\varphi(\cdot, f, \lambda)}$, the relation (3.4) holds if and only if

$$\begin{aligned} g(x, \lambda) = J_\rho^{\partial\varphi(\cdot, f, \lambda)}[&g(x, \lambda) - \rho\{M(g(x, \lambda), a, \lambda) - N(b, c, d, \lambda) \\ &+ z + h(e, \lambda)\}], \end{aligned}$$

where $\rho > 0$ is a constant. Hence we obtain that (x, a, b, c, d, e, f) is a solution of the PGMIQVIP (2.1) if only if there exist $x \in H$, $a \in A(x, \lambda)$, $b \in B(x, \lambda)$, $c \in C(x, \lambda)$, $d \in D(x, \lambda)$, $e \in E(x, \lambda)$, and $f \in F(x, \lambda)$ such that (3.1) holds. □

Theorem 3.2. *Let $A, B, C, D, E, F : H \to C(H)$ be $\tilde{H}$-Lipschitz continuous with constants $\lambda_A, \lambda_B, \lambda_C, \lambda_D, \lambda_E$, and λ_F, respectively. Let $M : H \times H \times \Omega \to H$ be M_1, M_2-Lipschitz continuous in first and second arguments, respectively. Let $N : H \times H \times H \times \Omega \to H$ be N_1, N_2, and N_3-Lipschitz continuous in first, second, and third arguments, respectively. Let $B : H \times \Omega \to H$ be γ-relaxed Lipschitz continuous with respect to the first argument of N and C be σ-generalized pseudo-contractive with respect to the second argument of N. Let $g : H \times \Omega \to H$ be α-strongly monotone and β-Lipschitz continuous. Let $b : H \times H \times \Omega \to \mathbb{R}$ be a function satisfying the conditions* (i)-(iii) *mentioned before. Let $\varphi : H \times H \times \Omega \to \mathbb{R} \cup \{+\infty\}$ be proper convex lower semicontinuous in the first argument with $g(H) \cap dom\partial\varphi(\cdot, z, \lambda) \neq \phi$ for all $(z, \lambda) \in H \times \Omega$ such that*

$$\|J_\rho^{\partial\varphi(\cdot, x, \lambda)}(z) - J_\rho^{\partial\varphi(\cdot, y, \lambda)}(z)\| \leq \mu\|x - y\|, \quad \forall (x, y, z, \lambda) \in H \times H \times H \times \Omega.$$

Suppose that there exists a constant $\rho > 0$ satisfying

$$k = 2\sqrt{1 - 2\alpha + \beta^2} + \mu\lambda_F,$$

$$p = N_1\lambda_B + N_2\lambda_C > M_1\beta + M_2\lambda_A + N_3\lambda_D + \nu\lambda_E = q,$$

$$\gamma - \sigma > (1 - k)q + \sqrt{(p^2 - q^2)k(2 - k)},$$

$$|\rho - \frac{\gamma - \sigma - (1 - k)q}{p^2 - q^2}| < \frac{\sqrt{[(\gamma - \sigma) - (1 - k)q]^2 - (p^2 - q^2)k(2 - k)}}{p^2 - q^2}. \tag{3.5}$$

Then, for any fixed $\lambda \in \Omega$, we have the following:

(1) *the solution set $S(\lambda)$ of the PGMIQVIP* (2.1) *is nonempty,*

(2) *$S(\lambda)$ is a closed subset in H.*

Proof. (1) Define a set-valued mapping $\mathcal{F} : H \times \Omega \to 2^H$ by

$$\mathcal{F}(x,\lambda) = \bigcup_{\substack{a\in A(x,\lambda), b\in B(x,\lambda), c\in C(x,\lambda)\\ d\in D(x,\lambda), e\in E(x,\lambda), f\in F(x,\lambda)}} \big(x - g(x,\lambda) + J_\rho^{\partial\varphi(\cdot,f,\lambda)}[g(x,\lambda) - \rho\{M(g(x,\lambda),a,\lambda) - N(b,c,d,\lambda) + z + h(e,\lambda)\}]\big)$$

for all $(x,\lambda) \in H \times \Omega$. For any $(x,\lambda) \in H \times \Omega$, we have $\mathcal{F}(x,\lambda) \in C(H)$. Now for each fixed $\lambda \in \Omega$, we prove that $\mathcal{F}(x,\lambda)$ is a set-valued contractive mapping.

For any $(x,\lambda),(y,\lambda) \in H \times \Omega$ and $u \in F(x,\lambda)$, there exist $a_1 \in A(x,\lambda)$, $b_1 \in B(x,\lambda)$, $c_1 \in C(x,\lambda)$, $d_1 \in D(x,\lambda)$, $e_1 \in E(x,\lambda)$, $f_1 \in F(x,\lambda)$ such that

$$u = x - g(x,\lambda) + J_\rho^{\partial\varphi(\cdot,f_1,\lambda)}[g(x,\lambda) - \rho\{M(g(x,\lambda),a_1,\lambda) - N(b_1,c_1,d_1,\lambda) + z + h(e_1,\lambda)\}].$$

Since $A(y,\lambda)$, $B(y,\lambda)$, $C(y,\lambda)$, $D(y,\lambda)$, $E(y,\lambda)$, $F(y,\lambda) \in C(H)$, there exist $a_2 \in A(y,\lambda)$, $b_2 \in B(y,\lambda)$, $c_2 \in C(y,\lambda)$, $d_2 \in D(y,\lambda)$, $e_2 \in E(y,\lambda)$, and $f_2 \in F(y,\lambda)$ such that

$$\begin{aligned}
&\|a_1 - a_2\| \le \tilde{H}(A(x,\lambda),A(y,\lambda)),\\
&\|b_1 - b_2\| \le \tilde{H}(B(x,\lambda),B(y,\lambda)),\\
&\|c_1 - c_2\| \le \tilde{H}(C(x,\lambda),C(y,\lambda)),\\
&\|d_1 - d_2\| \le \tilde{H}(D(x,\lambda),D(y,\lambda)),\\
&\|e_1 - e_2\| \le \tilde{H}(E(x,\lambda),E(y,\lambda)),\\
&\|f_1 - f_2\| \le \tilde{H}(F(x,\lambda),F(y,\lambda)).
\end{aligned} \tag{3.6}$$

Let

$$v = y - g(y,\lambda) + J_\rho^{\partial\varphi(\cdot,f_2,\lambda)}[g(y,\lambda) - \rho\{M(g(y,\lambda),a_2,\lambda) - N(b_2,c_2,d_2,\lambda) + z + h(e_2,\lambda)\}].$$

Then we have $v \in \mathcal{F}(y,\lambda)$. It follows that

$$\begin{aligned}
\|u - v\| &\le \|x - y - (g(x,\lambda) - g(y,\lambda))\| + \|J_\rho^{\partial\varphi(\cdot,f_1,\lambda)}[g(x,\lambda)\\
&\quad - \rho\{M(g(x,\lambda),a_1,\lambda) - N(b_1,c_1,d_1,\lambda) + z + h(e_1,\lambda)\}]\\
&\quad - J_\rho^{\partial\varphi(\cdot,f_1,\lambda)}[g(y,\lambda) - \rho\{M(g(y,\lambda),a_2,\lambda) - N(b_2,c_2,d_2,\lambda)\\
&\quad + z + h(e_2,\lambda)\}]\| + \|J_\rho^{\partial\varphi(\cdot,f_1,\lambda)}[g(y,\lambda) - \rho\{M(g(y,\lambda),a_2,\lambda)\\
&\quad - N(b_2,c_2,d_2,\lambda) + z + h(e_2,\lambda)\}] - J_\rho^{\partial\varphi(\cdot,f_2,\lambda)}[g(y,\lambda)\\
&\quad - \rho\{M(g(y,\lambda),a_2,\lambda) - N(b_2,c_2,d_2,\lambda) + z + h(e_2,\lambda)\}]\|\\
&\le 2\|x - y - (g(x,\lambda) - g(y,\lambda))\|\\
&\quad + \rho[\|M(g(x,\lambda),a_1,\lambda) - M(g(y,\lambda),a_1,\lambda)\|
\end{aligned}$$

$$
\begin{aligned}
&+ \|M(g(y,\lambda),a_1,\lambda) - M(g(y,\lambda),a_2,\lambda)\|] \\
&+ \|x - y + \rho(N(b_1,c_1,d_1,\lambda) - N(b_2,c_2,d_1,\lambda))\| \\
&+ \rho[\|N(b_2,c_2,d_1,\lambda) - N(b_2,c_2,d_2,\lambda)\| \\
&+ \|h(e_1,\lambda) - h(e_2,\lambda)\|] + \mu\|f_1 - f_2\|.
\end{aligned} \tag{3.7}
$$

By the strong monotonicity and the Lipschitz continuity of g, we have

$$
\begin{aligned}
&\|x - y - (g(x,\lambda) - g(y,\lambda))\|^2 \\
&\le \|x-y\|^2 - 2\langle g(x,\lambda) - g(y,\lambda), x - y\rangle + \|g(x,\lambda) - g(y,\lambda)\|^2 \\
&\le (1 - 2\alpha + \beta^2)\|x-y\|^2.
\end{aligned} \tag{3.8}
$$

By the Lipschitz continuity of M in first and second argument, the Lipschitz continuity of g, A and (3.6), we obtain

$$
\begin{aligned}
&\|M(g(x,\lambda),a_1,\lambda) - M(g(y,\lambda),a_1,\lambda)\| \\
&\quad + \|M(g(y,\lambda),a_1,\lambda) - M(g(y,\lambda),a_2,\lambda)\| \\
&\le M_1\|g(x,\lambda) - g(y,\lambda)\| + M_2\|a_1 - a_2\| \\
&\le M_1\beta\|x-y\| + M_2\tilde{H}(A(x,\lambda),A(y,\lambda)) \\
&\le (M_1\beta + M_2\lambda_A)\|x-y\|.
\end{aligned} \tag{3.9}
$$

By the relaxed Lipschitz continuity with respect to the first argument of N, the generalized pseudo-contractive of C with respect to the second argument of N, the Lipschitz continuity of N in the first and second arguments and (3.6), we have

$$
\begin{aligned}
&\|x - y + \rho(N(b_1,c_1,d_1,\lambda) - N(b_2,c_2,d_1,\lambda))\|^2 \\
&\le \|x-y\|^2 + 2\rho\{\langle x-y, N(b_1,c_1,d_1,\lambda) - N(b_2,c_1,d_1,\lambda)\rangle \\
&\quad + \langle x-y, N(b_2,c_1,d_1,\lambda) - N(b_2,c_2,d_1,\lambda)\rangle\} \\
&\quad + \rho^2[\|N(b_1,c_1,d_1,\lambda) - N(b_2,c_1,d_1,\lambda)\| \\
&\quad + \|N(b_2,c_1,d_1,\lambda) - N(b_2,c_2,d_1,\lambda)\|]^2 \\
&\le \|x-y\|^2 - 2\rho(\gamma - \sigma)\|x-y\|^2 + \rho^2[N_1\|b_1 - b_2\| + N_2\|c_1 - c_2\|]^2 \\
&\le \|x-y\|^2 - 2\rho(\gamma - \sigma)\|x-y\|^2 \\
&\quad + \rho^2[N_1\tilde{H}(B(x,\lambda),B(y,\lambda)) + N_2\tilde{H}(C(x,\lambda),C(y,\lambda))]^2 \\
&\le \|x-y\|^2 - 2\rho(\gamma - \sigma)\|x-y\|^2 + \rho^2[N_1\lambda_B\|x-y\| + N_2\lambda_C\|x-y\|]^2 \\
&= [1 - 2\rho(\gamma - \sigma) + \rho^2(N_1\lambda_B + N_2\lambda_C)^2]\|x-y\|^2,
\end{aligned} \tag{3.10}
$$

and

$$
\begin{aligned}
\|N(b_2,c_2,d_1,\lambda) - N(b_2,c_2,d_2,\lambda)\| &\le N_3\|d_1 - d_2\| \\
&\le N_3\tilde{H}(D(x,\lambda),D(y,\lambda)) \\
&\le N_3\lambda_D\|x-y\|.
\end{aligned} \tag{3.11}
$$

By Lemma 2.2, (3.6) and the Lipschitz continuity of E and F, we have

$$\begin{aligned}\|h(e_1,\lambda)-h(e_2,\lambda)\| &\le \nu\|e_1-e_2\| \\ &\le \nu\tilde{H}(E(x,\lambda),E(y,\lambda)) \\ &\le \nu\lambda_E\|x-y\|,\end{aligned} \tag{3.12}$$

$$\begin{aligned}\|f_1-f_2\| &\le \tilde{H}(F(x,\lambda),F(y,\lambda)) \\ &\le \lambda_F\|x-y\|.\end{aligned} \tag{3.13}$$

From (3.7)-(3.13) it follows that

$$\begin{aligned}\|u-v\| &\le [2\sqrt{1-2\alpha+\beta^2}+\rho(M_1\beta+M_2\lambda_A)4 \\ &\quad+\sqrt{1-2\rho(\gamma-\sigma)+\rho^2(N_1\lambda_B+N_2\lambda_C)^2} \\ &\quad+\rho(N_3\lambda_D+\nu\lambda_E)+\mu\lambda_F]\|x-y\| \\ &=(k+t(\rho))\|x-y\| \\ &=\theta\|x-y\|,\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}k &= 2\sqrt{1-2\alpha+\beta^2}+\mu\lambda_F, \\ t(\rho) &= \sqrt{1-2\rho(\gamma-\sigma)+\rho^2(N_1\lambda_B+N_2\lambda_C)^2} \\ &\quad+\rho(M_1\beta+M_2\lambda_A+N_3\lambda_D+\nu\lambda_E),\end{aligned}$$

and

$$\theta = k+t(\rho).$$

It follows from condition (3.5) that $\theta<1$. Hence we have

$$\begin{aligned}d(u,\mathcal{F}(y,\lambda)) &= \inf_{v\in\mathcal{F}(y,\lambda)}\|u-v\| \\ &\le \theta\|x-y\|.\end{aligned}$$

Since $u\in\mathcal{F}(x,\lambda)$ is arbitrary, we obtain

$$\sup_{u\in\mathcal{F}(x,\lambda)} d(u,\mathcal{F}(y,\lambda)) \le \theta\|x-y\|.$$

By using same argument, we can prove

$$\sup_{v\in\mathcal{F}(y,\lambda)} d(\mathcal{F}(x,\lambda),v) \le \theta\|x-y\|.$$

By the definition of the Hausdorff metric $\tilde{H}$ on $C(H)$, we obtain that for all $(x,y,\lambda)\in H\times H\times\Omega$,

$$\tilde{H}(\mathcal{F}(x,\lambda),\mathcal{F}(y,\lambda)) \le \theta\|x-y\|,$$

i.e., $\mathcal{F}(x,\lambda)$ is a set-valued contractive mapping which is uniform with respect to $\lambda\in\Omega$. By a fixed point theorem of Nadler [7], for each $\lambda\in\Omega$, $\mathcal{F}(x,\lambda)$

has a fixed point $x \in \mathcal{F}(x,\lambda)$. By the definition of $\mathcal{F}$, there exist $a \in A(x,\lambda)$, $b \in B(x,\lambda)$, $c \in C(x,\lambda)$, $d \in D(x,\lambda)$, $e \in E(x,\lambda)$, and $f \in F(x,\lambda)$ such that

$$\begin{aligned} x = x - g(x,\lambda) + J_\rho^{\partial\varphi(\cdot,f,\lambda)}[g(x,\lambda) - \rho\{M(g(x,\lambda),a,\lambda) \\ - N(b,c,d,\lambda) + z + h(e,\lambda)\}], \end{aligned}$$

and so

$$\begin{aligned} g(x,\lambda) = J_\rho^{\partial\varphi(\cdot,f,\lambda)}[g(x,\lambda) - \rho\{M(g(x,\lambda),a,\lambda) - N(b,c,d,\lambda) \\ + z + h(e,\lambda)\}]. \end{aligned}$$

By Theorem 3.1, (x,a,b,c,d,e,f) is a solution of the PGMIQVIP(2.1). Thus we have $S(\lambda) \neq \phi$.

(2) For each $\lambda \in \Omega$, let $\{x_n\} \subset S(\lambda)$ and $x_n \to x_0$ as $n \to \infty$. Then we have $x_n \in \mathcal{F}(x_n,\lambda)$ for all $n = 1,2,\ldots$. By the proof of (1), we have

$$\tilde{H}(\mathcal{F}(x_n,\lambda),\mathcal{F}(x_0,\lambda)) \leq \theta\|x_n - x_0\|.$$

It follows that

$$\begin{aligned} d(x_0,\mathcal{F}(x_0,\lambda)) &\leq \|x_0 - x_n\| + d(x_n,\mathcal{F}(x_n,\lambda)) \\ &\quad + \tilde{H}(\mathcal{F}(x_n,\lambda),\mathcal{F}(x_0,\lambda)) \\ &\leq (1+\theta)\|x_n - x_0\| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence, we have $x_0 \in \mathcal{F}(x_0,\lambda)$ and $x_0 \in S(\lambda)$. Therefore, $S(\lambda)$ is a nonempty closed subset of H. □

Theorem 3.3. *Under the hypotheses of Theorem 3.2, further assume that*

(i) *for any $x \in H$, $\lambda \to A(x,\lambda)$, $\lambda \to B(x,\lambda)$, $\lambda \to C(x,\lambda)$, $\lambda \to D(x,\lambda)$, $\lambda \to E(x,\lambda)$, $\lambda \to F(x,\lambda)$ are Lipschitz continuous with Lipschitz constants $l_A, l_B, l_C, l_D, l_E, l_F$, respectively,*

(ii) *for any $p,q,u,v,w,x,y,z \in H$, $\lambda \mapsto M(p,q,\lambda)$, $\lambda \mapsto N(u,v,w,\lambda)$, $\lambda \mapsto g(x,\lambda)$, $\lambda \mapsto h(x,\lambda)$, and $\lambda \mapsto J_\rho^{\partial\varphi(\cdot,y,\lambda)}(z)$ are Lipschitz continuous with Lipschitz constants l_M, l_N, l_g, l_h, l_J, respectively.*

Then the solution set $S(\lambda)$ of the PGIQVIP (2.1) *is a Lipschitz continuous mapping from Ω to H.*

Proof. For each $\lambda, \bar{\lambda} \in \Omega$, by Theorem 3.2, $S(\lambda)$ and $S(\bar{\lambda})$ are both nonempty closed subsets. By the proof of Theorem 3.2, $\mathcal{F}(x,\lambda)$ and $\mathcal{F}(x,\bar{\lambda})$ are both set-valued contractive mappings with same contraction constant $\theta \in (0,1)$. By Lemma 2.1, we obtain

$$\tilde{H}(S(\lambda),S(\bar{\lambda})) \leq \frac{1}{1-\theta}\sup_{x\in H}\tilde{H}(\mathcal{F}(x,\lambda),\mathcal{F}(x,\bar{\lambda})).$$

Taking any $u \in \mathcal{F}(x,\lambda)$, there exist $a \in A(x,\lambda)$, $b \in B(x,\lambda)$, $c \in C(x,\lambda)$, $d \in D(x,\lambda)$, $e \in E(x,\lambda)$, $f \in F(x,\lambda)$ such that

$$u = x - g(x,\lambda) + J_\rho^{\partial\varphi(\cdot,f,\lambda)}[g(x,\lambda) - \rho\{M(g(x,\lambda),a,\lambda) - N(b,c,d,\lambda) + z + h(e,\lambda)\}].$$

Since $A(x,\lambda) \in C(H)$ and $A(x,\bar\lambda) \in C(H)$, there exists $\bar a \in A(x,\bar\lambda)$ such that

$$\|a - \bar a\| \le \tilde H(A(x,\lambda), A(x,\bar\lambda)).$$

Similarly, there exist $\bar b \in B(x,\bar\lambda)$, $\bar c \in C(x,\bar\lambda)$, $\bar d \in D(x,\bar\lambda)$, $\bar e \in E(x,\bar\lambda)$, $\bar f \in F(x,\bar\lambda)$ such that

$$\begin{aligned}
\|b - \bar b\| &\le \tilde H(B(x,\lambda), B(x,\bar\lambda),\\
\|c - \bar c\| &\le \tilde H(C(x,\lambda), C(x,\bar\lambda),\\
\|d - \bar d\| &\le \tilde H(D(x,\lambda), D(x,\bar\lambda)),\\
\|e - \bar e\| &\le \tilde H(E(x,\lambda), E(x,\bar\lambda)),\\
\|f - \bar f\| &\le \tilde H(F(x,\lambda), F(x,\bar\lambda)).
\end{aligned}$$

Let

$$v = x - g(x,\bar\lambda) + J_\rho^{\partial\varphi(\cdot,\bar f,\bar\lambda)}[g(x,\bar\lambda) - \rho\{M(g(x,\bar\lambda),\bar a,\bar\lambda) - N(\bar b,\bar c,\bar d,\bar\lambda) + z + h(\bar e,\bar\lambda)\}].$$

Then $v \in \mathcal{F}(x,\bar\lambda)$. It follows that

(3.15)

$$\begin{aligned}
&\|u - v\|\\
&\le \|g(x,\lambda) - g(x,\bar\lambda)\|\\
&\quad + \|J_\rho^{\partial\varphi(\cdot,f,\lambda)}[g(x,\lambda) - \rho\{M(g(x,\lambda),a,\lambda) - N(b,c,d,\lambda) + z + h(e,\lambda)\}]\\
&\quad - J_\rho^{\partial\varphi(\cdot,\bar f,\lambda)}[g(x,\lambda) - \rho\{M(g(x,\lambda),a,\lambda) - N(b,c,d,\lambda) + z + h(e,\lambda)\}]\|\\
&\quad + \|J_\rho^{\partial\varphi(\cdot,\bar f,\lambda)}[g(x,\lambda) - \rho\{M(g(x,\lambda),a,\lambda) - N(b,c,d,\lambda) + z + h(e,\lambda)\}]\\
&\quad - J_\rho^{\partial\varphi(\cdot,\bar f,\lambda)}[g(x,\bar\lambda) - \rho\{M(g(x,\bar\lambda),\bar a,\bar\lambda) - N(\bar b,\bar c,\bar d,\bar\lambda) + z + h(\bar e,\bar\lambda)\}]\|\\
&\quad + \|J_\rho^{\partial\varphi(\cdot,\bar f,\lambda)}[g(x,\bar\lambda) - \rho\{M(g(x,\bar\lambda),\bar a,\bar\lambda) - N(\bar b,\bar c,\bar d,\bar\lambda) + z + h(\bar e,\bar\lambda)\}]\\
&\quad - J_\rho^{\partial\varphi(\cdot,\bar f,\bar\lambda)}[g(x,\bar\lambda) - \rho\{M(g(x,\bar\lambda),\bar a,\bar\lambda) - N(\bar b,\bar c,\bar d,\bar\lambda) + z + h(\bar e,\bar\lambda)\}]\|\\
&\le \|g(x,\lambda) - g(x,\bar\lambda)\| + \mu\|f - \bar f\|\\
&\quad + \|g(x,\lambda) - \rho\{M(g(x,\lambda),a,\lambda) - N(b,c,d,\lambda) + z + h(e,\lambda)\}\\
&\quad - (g(x,\bar\lambda) - \rho\{M(g(x,\bar\lambda),\bar a,\bar\lambda) - N(\bar b,\bar c,\bar d,\bar\lambda) + z + h(\bar e,\bar\lambda)\})\|\\
&\quad + l_J\|\lambda - \bar\lambda\|\\
&\le 2\|g(x,\lambda) - g(x,\bar\lambda)\| + \mu\|f - \bar f\|\\
&\quad + \rho\|M(g(x,\lambda),a,\lambda) - M(g(x,\bar\lambda),\bar a,\bar\lambda)\|\\
&\quad + \rho\|N(b,c,d,\lambda) - N(\bar b,\bar c,\bar d,\bar\lambda)\| + \rho\|h(e,\lambda) - h(\bar e,\bar\lambda)\|.
\end{aligned}$$

By the Lipschitz continuity of g, F, M and A in λ, we have

$$\|g(x,\lambda)-g(x,\bar{\lambda})\| \leq l_g\|\lambda-\bar{\lambda}\|, \tag{3.16}$$

$$\begin{aligned}\|f-\bar{f}\| &\leq \tilde{H}(F(x,\lambda),F(x,\bar{\lambda}))\\ &\leq l_F\|\lambda-\bar{\lambda}\|,\end{aligned} \tag{3.17}$$

$$\begin{aligned}&\|M(g(x,\lambda),a,\lambda)-M(g(x,\bar{\lambda}),\bar{a},\bar{\lambda})\|\\ &\leq \|M(g(x,\lambda),a,\lambda)-M(g(x,\bar{\lambda}),a,\lambda)\|+\|M(g(x,\bar{\lambda}),a,\lambda)\\ &\qquad -M(g(x,\bar{\lambda}),\bar{a},\lambda)\|+\|M(g(x,\bar{\lambda}),\bar{a},\lambda)-M(g(x,\bar{\lambda}),\bar{a},\bar{\lambda})\|\\ &\leq M_1\|g(x,\lambda)-g(x,\bar{\lambda})\|+M_2\|a-\bar{a}\|+l_M\|\lambda-\bar{\lambda}\|\\ &\leq M_1l_g\|\lambda-\bar{\lambda}\|+M_2\tilde{H}(A(x,\lambda),A(x,\bar{\lambda}))+l_M\|\lambda-\bar{\lambda}\|\\ &\leq (M_1l_g+M_2l_A+l_M)\|\lambda-\bar{\lambda}\|.\end{aligned} \tag{3.18}$$

By the Lipschitz continuity of E, h in $\lambda \in \Omega$, and Lemma 2.2, we have

$$\begin{aligned}\|h(e,\lambda)-h(\bar{e},\bar{\lambda})\| &\leq \|h(e,\lambda)-h(\bar{e},\lambda)\|+\|h(\bar{e},\lambda)-h(\bar{e},\bar{\lambda})\|\\ &\leq \nu\|e-\bar{e}\|+l_h\|\lambda-\bar{\lambda}\|\\ &\leq \nu\tilde{H}(E(x,\lambda),E(x,\bar{\lambda}))+l_h\|\lambda-\bar{\lambda}\|\\ &\leq (\nu l_E+l_h)\|\lambda-\bar{\lambda}\|.\end{aligned} \tag{3.19}$$

By the Lipschitz continuity of N and B, C, D in $\lambda \in \Omega$, we have

$$\begin{aligned}&\|N(b,c,d,\lambda)-N(\bar{b},\bar{c},\bar{d},\bar{\lambda})\|\\ &\leq \|N(b,c,d,\lambda)-N(\bar{b},c,d,\lambda)\|+\|N(\bar{b},c,d,\lambda)-N(\bar{b},\bar{c},d,\lambda)\|\\ &\quad +\|N(\bar{b},\bar{c},d,\lambda)-N(\bar{b},\bar{c},\bar{d},\lambda)\|+\|N(\bar{b},\bar{c},\bar{d},\lambda)-N(\bar{b},\bar{c},\bar{d},\bar{\lambda})\|\\ &\leq N_1\|b-\bar{b}\|+N_2\|c-\bar{c}\|+N_3\|d-\bar{d}\|+l_N\|\lambda-\bar{\lambda}\|\\ &\leq N_1\tilde{H}(B(x,\lambda),B(x,\bar{\lambda}))+N_2\tilde{H}(C(x,\lambda),C(x,\bar{\lambda}))\\ &\quad +N_3\tilde{H}(D(x,\lambda),D(x,\bar{\lambda}))+l_N\|\lambda-\bar{\lambda}\|\\ &\leq (N_1l_B+N_2l_C+N_3l_D+l_N)\|\lambda-\bar{\lambda}\|.\end{aligned} \tag{3.20}$$

It follows from (3.15)-(3.20) that

$$\begin{aligned}\|u-v\| &\leq [2l_g+\mu l_F+\rho(M_1l_g+M_2l_A+l_M+N_1l_B+N_2l_C\\ &\qquad +N_3l_D+l_N+\nu l_E+l_h)]\|\lambda-\bar{\lambda}\|\\ &\leq L\|\lambda-\bar{\lambda}\|,\end{aligned}$$

where

$$\begin{aligned}L = 2lg+\mu l_F+\rho(M_1l_g+M_2l_A+l_M+N_1l_B+N_2l_C\\ +N_3l_D+l_N+\nu l_E+l_h).\end{aligned}$$

Hence, we obtain

$$\sup_{u\in\mathcal{F}(x,\lambda)} d(u,\mathcal{F}(x,\bar{\lambda})) \leq L\|\lambda-\bar{\lambda}\|.$$

By using a similar argument as above, we can obtain

$$\sup_{v \in \mathcal{F}(x,\bar{\lambda})} d(\mathcal{F}(x,\lambda), v) \leq L\|\lambda - \bar{\lambda}\|.$$

It follows that

$$\tilde{H}(\mathcal{F}(x,\lambda), \mathcal{F}(x,\bar{\lambda})) \leq L\|\lambda - \bar{\lambda}\|.$$

By Lemma 2.1, we obtain

$$\tilde{H}(S(\lambda), S(\bar{\lambda})) \leq \frac{L}{1-L}\|\lambda - \bar{\lambda}\|.$$

This proves that $S(\lambda)$ is Lipschitz continuous in $\lambda \in \Omega$.

If each mapping in Conditions (i) and (ii) is assumed to be continuous in $\lambda \in \Omega$, then by similar argument as above, we can show that $S(\lambda)$ is also continuous in $\lambda \in \Omega$. □

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