J. Korean Math. Soc. **44** (2007), No. 4, pp. 863–872

SPECTRAL LOCALIZING SYSTEMS THAT ARE t-SPLITTING MULTIPLICATIVE SETS OF IDEALS

GYU WHAN CHANG

ABSTRACT. Let D be an integral domain with quotient field K, Λ a nonempty set of height-one maximal t-ideals of D, $\mathcal{F}(\Lambda) = \{I \subseteq D \mid I$ is an ideal of D such that $I \nsubseteq P$ for all $P \in \Lambda\}$, and $D_{\mathcal{F}(\Lambda)} = \{x \in K \mid xA \subseteq D$ for some $A \in \mathcal{F}(\Lambda)\}$. In this paper, we prove that if each $P \in \Lambda$ is the radical of a finite type v-ideal (resp., a principal ideal), then $D_{\mathcal{F}(\Lambda)}$ is a weakly Krull domain (resp., generalized weakly factorial domain) if and only if the intersection $D_{\mathcal{F}(\Lambda)} = \cap_{P \in \Lambda} D_P$ has finite character, if and only if $\mathcal{F}(\Lambda)$ is a t-splitting set of ideals, if and only if $\mathcal{F}(\Lambda)$ is v-finite.

1. Introduction

Throughout this paper D will be an integral domain with quotient field K and an ideal means an integral ideal. A nonempty set $\mathcal{S}$ of ideals of D is said to be *multiplicative* if $\mathcal{S}$ is multiplicatively closed, i.e., if $A, B \in \mathcal{S}$ implies $AB \in \mathcal{S}$. Let $\mathcal{S}$ be a multiplicative set of ideals of D. The following overring of D

$$D_{\mathcal{S}} = \{x \in K \mid xA \subseteq D \text{ for some } A \in \mathcal{S}\}$$

is called the *$\mathcal{S}$-transform* of D or the *generalized ring of fractions of D with respect to $\mathcal{S}$* (cf. [5]). Let $\mathrm{Sat}(\mathcal{S})$ be the set of ideals C of D such that $A \subseteq C$ for some $A \in \mathcal{S}$ and $\mathcal{S}^{\perp} = \{B \subseteq D \mid B$ is an ideal of D such that $(B+J)_t = D$ for all $J \in \mathcal{S}\}$. If $\mathcal{S} = \mathrm{Sat}(\mathcal{S})$, then $\mathcal{S}$ is called *saturated.* We say that $\mathcal{S}$ is *finitely generated* if every ideal $I \in \mathcal{S}$ contains a finitely generated ideal which is still in $\mathcal{S}$, while $\mathcal{S}$ is *v-finite* if each t-ideal $A \in \mathrm{Sat}(\mathcal{S})$ contains a finitely generated ideal J such that $J_v \in \mathrm{Sat}(\mathcal{S})$. Clearly, each finitely generated multiplicative set of ideals is v-finite, but the converse does not hold (see [11, p.124]). If Λ is a nonempty set of nonzero prime ideals of D, we define

$$\mathcal{F}(\Lambda) = \{A \subseteq D \mid A \text{ is an ideal of } D \text{ such that } A \nsubseteq P \text{ for all } P \in \Lambda\}.$$

Received February 3, 2006.

2000 *Mathematics Subject Classification.* 13A15,13F05, 13G05.

Key words and phrases. spectral localizing system, t-splitting set of ideals, weakly Krull domain, generalized weakly factorial domain.

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2005-202-C00010).

Then $\mathcal{F}(\Lambda)$, called a *spectral localizing system*, is a saturated multiplicative set of ideals of D and $D_{\mathcal{F}(\Lambda)} = \cap_{P\in\Lambda} D_P$ [10, Proposition 5.1.4]. If P is a prime ideal of D, we denote $\mathcal{F}(\{P\})$ by $\mathcal{F}(P)$. It is obvious that $\mathcal{F}(\Lambda) = \cap_{P\in\Lambda}\mathcal{F}(P)$.

A multiplicative subset N of D is called a *t-splitting set* if for each $0 \neq d \in D$, we have $dD = (AB)_t$ for some integral ideals A and B of D, where $A_t \cap sD = sA_t$ for all $s \in N$ and $B_t \cap N \neq \emptyset$ (see [1, 7]). Anderson-Anderson-Zafrullah introduced the concept of t-splitting sets and proved that the ring $D+XD_N[X]$ is a PVMD if and only if D is a PVMD and N is a t-splitting set [1, Theorem 2.5]. (Recall that D is a *Prüfer v-multiplication domain* (PVMD) if each nonzero finitely generated ideal of D is t-invertible.) Chang-Dumitrescu-Zafrullah further studied t-splitting sets [7] and extended the notion of t-splitting sets to multiplicative sets of ideals as follows [8]; $\mathcal{S}$ is a *t-splitting set of ideals* if every nonzero principal ideal dD of D can be written as $dD = (AB)_t$ with $A \in \mathrm{Sat}(\mathcal{S})$ and $B \in \mathcal{S}^{\perp}$. Clearly, if $\mathcal{S}$ is a t-splitting set of ideals, then $\mathcal{S}^{\perp}$ is also a t-splitting set of ideals [8, Proposition 2]. It is proved that $\mathcal{S}$ is a t-splitting set of ideals if and only if $\mathcal{S}$ is v-finite and $dD_{\mathcal{S}} \cap D$ is t-invertible for each $0 \neq d \in D$ [8, Proposition 5]. Also, a multiplicative subset N of D is a t-splitting set if and only if $\mathcal{N} = \{sD | s \in N\}$ is a t-splitting set of ideals (cf. [1, Corollary 2.3]).

Let Λ be a nonempty set of height-one maximal t-ideals of D. The purpose of this paper is to study when $\mathcal{F}(\Lambda)$ is a t-splitting set of ideals. In particular, we show that if each $P \in \Lambda$ is the radical of a finite type v-ideal (resp., principal ideal), then $D_{\mathcal{F}(\Lambda)}$ is a weakly Krull domain (resp., generalized weakly factorial domain) if and only if the intersection $D_{\mathcal{F}(\Lambda)} = \cap_{P\in\Lambda} D_P$ has finite character, if and only if $\cap_n P_1 \cdots P_n = (0)$ for each infinite sequence (P_n) of distinct elements of Λ, if and only if $\mathcal{F}(\Lambda)$ is a t-splitting set of ideals, if and only if $t\text{-Max}(D_{\mathcal{F}(\Lambda)}) = \{P_{\mathcal{F}(\Lambda)} | P \in \Lambda\}$, if and only if $\mathcal{F}(\Lambda)$ is finitely generated, if and only if $\mathcal{F}(\Lambda)$ is v-finite.

We first review some notation and definitions. Let $\mathcal{F}(D)$ be the set of nonzero fractional ideals of D. For each $I \in \mathcal{F}(D)$, let $I^{-1} = \{x \in K | xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, and $I_t = \cup\{J_v | J \subseteq I$ is a nonzero finitely generated fractional ideal of $D\}$. Obviously, if $I \in \mathcal{F}(D)$ is finitely generated, then $I_v = I_t$. An $I \in \mathcal{F}(D)$ is called a *divisorial ideal* (resp., *t-ideal*) if $I_v = I$ (resp., $I_t = I$). A t-ideal I is called a *finite type v-ideal* if $I = (x_1, \ldots, x_n)_v$ for some $(0) \neq (x_1, \ldots, x_n) \subseteq I$. An $I \in \mathcal{F}(D)$ is said to be *t-invertible* if $(II^{-1})_t = D$. It is known that if I is t-invertible, then I_t is a finite type v-ideal. Let $t\text{-Max}(D)$ be the set of ideals maximal among proper integral t-ideals of D. It is well known that each ideal $P \in t\text{-Max}(D)$ is a prime ideal, $t\text{-Max}(D) \neq \emptyset$ if D is not a field, and $D = \cap_{P\in t\text{-Max}(D)} D_P$. We say that an ideal $P \in t\text{-Max}(D)$ is a *maximal t-ideal* and that D has a *t-dimension one*, denoted by $t\text{-dim}(D) = 1$, if each maximal t-ideal of D has height-one. Let $X^1(D)$ be the set of height-one prime ideals of D; so $t\text{-dim}(D) = 1 \Leftrightarrow t\text{-Max}(D) = X^1(D)$. Examples of integral domains of t-dimension one include (weakly) Krull domains and one-dimensional integral domains. For more on the v- and the t-operation, the reader may consult [12, Sections 32 and 34].

Let $\mathcal{S}$ be a multiplicative set of ideals of D. If I is a fractional ideal of D, then $I_{\mathcal{S}} = \{x \in K | xA \subseteq I$ for some $A \in \mathcal{S}\}$ is a fractional ideal of $D_{\mathcal{S}}$. In particular, if I is a prime ideal of D, then $I_{\mathcal{S}}$ is a prime ideal of $D_{\mathcal{S}}$. We call $\mathcal{S}^{\perp}$ the *t-complement* of $\mathcal{S}$. Let A, B_1, B_2, C be ideals of D such that $A \in \mathcal{S}$, $B_i \in \mathcal{S}^{\perp}$, and $B_1 \subseteq C$. Then $D = (A + B_1)_t \subseteq (A + C)_t \subseteq D$, and hence $C \in \mathcal{S}^{\perp}$. Also, $D = (A + B_1)_t(A + B_2)_t \subseteq ((A + B_1)_t(A + B_2)_t)_t \subseteq (A + B_1B_2)_t \subseteq D$; so $(A + B_1B_2)_t = D$. Thus $\mathcal{S}^{\perp}$ is a saturated multiplicative set of ideals. Also, $\mathrm{Sat}(\mathcal{S})$ is a saturated multiplicative set of ideals. It is known that $D_{\mathcal{S}} = D_{\mathrm{Sat}(\mathcal{S})}$ and $D = D_{\mathcal{S}} \cap D_{\mathcal{S}^{\perp}}$ [8, Lemma 7].

A nonempty family $\mathcal{F}$ of ideals of D is called a *localizing system* if

(*i*) $I \in \mathcal{F}, J$ an ideal of $D, I \subseteq J \Rightarrow J \in \mathcal{F}$;

(*ii*) $I \in \mathcal{F}, J$ an ideal of $D, (J :_D iD) \in \mathcal{F}$ for all $i \in I \Rightarrow J \in \mathcal{F}$.

It can be easily shown that a localizing system is a saturated multiplicative set of ideals [10, Proposition 5.1.1] and that if Λ is a nonempty set of prime ideals of D, then $\mathcal{F}(\Lambda)$ is a localizing system [10, Proposition 5.1.4]. A localizing system $\mathcal{F}$ is said to be *spectral* if $\mathcal{F} = \mathcal{F}(\Lambda)$ for some nonempty set Λ of prime ideals of D. The reader is referred to the papers [1, 7, 8] for t-splitting sets. For more on multiplicative sets of ideals, generalized ring of fractions of D, and localizing systems, see, for example, [5], [10, Section 5.1], or [11].

2. Weakly Krull domains

Let R be a commutative ring with identity, and let I be an ideal of R. Then there exist only a finite number of prime ideals of R minimal over I under one of the following conditions;

(1) ([16, Theorem 88]) R satisfies the ascending chain condition on radical ideals.

(2) ([13, Theorem 1.6] or [6, Theorem 2.1]) Every prime ideal of R minimal over I is the radical of a finitely generated ideal.

As the t-operation analog, El Baghdadi showed that if D satisfies the ascending chain conditions on radical t-ideals, then each t-ideal of D has a finite number of minimal prime ideals [9, Lemma 3.8]. The following lemma is a generalization of El Baghdadi's result. The proof is similar to the proofs of [6, Theorem 2.1] and [9, Lemma 3.8], and hence omitted.

Lemma 2.1. *Let I be a proper integral t-ideal of D. If every prime ideal of D minimal over I is the radical of a finite type v-ideal, then I has only a finite number of minimal prime ideals.*

Lemma 2.2. *Let Λ be a nonempty subset of t-$Max(D)$ and $\Sigma = t$-$Max(D) \setminus \Lambda$.*

(1) $\mathcal{F}(\Lambda)^{\perp} = \mathcal{F}(\Sigma)$.

(2) *If $\Lambda \subseteq X^1(D)$ and $\mathcal{F}(\Lambda)$ is v-finite, then t-$Max(D_{\mathcal{F}(\Lambda)}) = \{P_{\mathcal{F}(\Lambda)} | P \in \Lambda\}$.*

Proof. (1) ($\subseteq$) Let $A \in \mathcal{F}(\Lambda)^{\perp}$. If $Q \in \Sigma$, then $Q \not\subseteq P$ for all $P \in \Lambda$, and hence $Q \in \mathcal{F}(\Lambda)$. So $(A+Q)_t = D$, and since $Q \in t\text{-Max}(D)$, we have $A \not\subseteq Q$. Thus $A \in \mathcal{F}(\Sigma)$. ($\supseteq$) Conversely, assume that B is an ideal of D such that $B \notin \mathcal{F}(\Lambda)^{\perp}$. Then $(B+C')_t \subsetneq D$ for some $C' \in \mathcal{F}(\Lambda)$, and since $C' \not\subseteq P$ for all $P \in \Lambda$, there exists a maximal t-ideal $Q \in \Sigma$ such that $B \subseteq (B+C')_t \subseteq Q$; hence $B \notin \mathcal{F}(\Sigma)$. Thus $\mathcal{F}(\Sigma) \subseteq \mathcal{F}(\Lambda)^{\perp}$.

(2) ($\subseteq$) Let Q be a maximal t-ideal of $D_{\mathcal{F}(\Lambda)}$ and $P = Q \cap D$. Then P is a prime t-ideal of D [11, Proposition 1.3]. If $P \notin \Lambda$, then $P \in \mathcal{F}(\Lambda)$ (note that each prime ideal in Λ has height-one), and since $\mathcal{F}(\Lambda)$ is v-finite, there exists a finite type v-ideal I of D such that $I \in \mathcal{F}(\Lambda)$ and $I \subseteq P$; so $Q \supseteq (ID_{\mathcal{F}(\Lambda)})_v = (I_{\mathcal{F}(\Lambda)})_v = (D_{\mathcal{F}(\Lambda)})_v = D_{\mathcal{F}(\Lambda)}$ [11, Propositions 1.1(a) and 1.2(b)]. This contradiction shows that $P \in \Lambda$, and thus $Q = P_{\mathcal{F}(\Lambda)}$ [5, Theorem 1.1(2)] since $P = Q \cap D$ implies that $AD_{\mathcal{F}(\Lambda)} \not\subseteq Q$ for all $A \in \mathcal{F}(\Lambda)$. ($\supseteq$) Let $P \in \Lambda$. Since $(D_{\mathcal{F}(\Lambda)})_{P_{\mathcal{F}(\Lambda)}} = D_P$ [5, Theorem 1.1], we have $\text{ht}(P_{\mathcal{F}(\Lambda)}) = \text{ht}P = 1$, and hence $P_{\mathcal{F}(\Lambda)}$ is a prime t-ideal of $D_{\mathcal{F}(\Lambda)}$ (cf. [11, Proposition 1.6(a)]). Thus $P_{\mathcal{F}(\Lambda)}$ is a maximal t-ideal of $D_{\mathcal{F}(\Lambda)}$ (see the proof of the " $\subseteq$ " case). □

An integral domain D is called a *weakly Krull domain* if $D = \cap_{P \in X^1(D)} D_P$ and this intersection has finite character. One can easily show that D is a weakly Krull domain if and only if $t\text{-dim}(D) = 1$ and for each $P \in X^1(D)$, $P = \sqrt{(a,b)}$ for some $a, b \in D$ (cf. [4, Theorem 2.6]). Let D be a weakly Krull domain, and let Λ be a nonempty set of prime t-ideals of D. Then $\mathcal{F}(\Lambda)$ is finitely generated [11, Lemma 1.16], and hence $t\text{-Max}(D_{\mathcal{F}(\Lambda)}) = \{P_{\mathcal{F}(\Lambda)} | P \in \Lambda\}$ by Lemma 2.2(2) (cf. [11, Proposition 1.17]). We next give the main result of this paper.

Theorem 2.3. *Let Λ be a nonempty set of height-one maximal t-ideals of D such that each $P \in \Lambda$ is the radical of a finite type v-ideal. Then the following statements are equivalent.*

(1) *$D_{\mathcal{F}(\Lambda)}$ is a weakly Krull domain.*
(2) *The intersection $D_{\mathcal{F}(\Lambda)} = \cap_{P \in \Lambda} D_P$ has finite character.*
(3) *$\cap_n P_1 \cdots P_n = (0)$ for each infinite sequence (P_n) of distinct elements of Λ.*
(4) *$\mathcal{F}(\Lambda)$ is a t-splitting set of ideals.*
(5) *$t\text{-Max}(D_{\mathcal{F}(\Lambda)}) = \{P_{\mathcal{F}(\Lambda)} | P \in \Lambda\}$.*
(6) *$\mathcal{F}(\Lambda)$ is finitely generated.*
(7) *$\mathcal{F}(\Lambda)$ is v-finite.*

Proof. (1) $\Rightarrow$ (3) This follows directly from the fact that $D_P = (D_{\mathcal{F}(\Lambda)})_{P_{\mathcal{F}(\Lambda)}}$ for all $P \in \Lambda$ [5, Theorem 1.1(4)]. (2) $\Rightarrow$ (1) This appears in [11, Lemma 2.5]. For (2) $\Rightarrow$ (6), see [11, Lemma 1.16].

(3) $\Rightarrow$ (4) Suppose that $\cap_n P_1 \cdots P_n = (0)$ for each infinite sequence (P_n) of distinct elements of Λ. Let $0 \neq d \in D$. By assumption, the number of prime ideals in Λ containing d is finite, say $P_1, \ldots, P_n$. Let $A_i = dD_{P_i} \cap D$ and $A = (A_1 \cdots A_n)_t$.

We first show that each A_i, and hence A, is t-invertible. Note that since each P_i is of height-one, dD_{P_i} is $P_iD_{P_i}$-primary, and hence A_i is P_i-primary. Also, note that A_i is t-locally principal since P_i is a maximal t-ideal. Hence it suffices to show that each A_i is of finite type [15, Corollary 2.7]. Let I_i be a finitely generated ideal of D such that $\sqrt{(I_i)_t} = P_i$. Since I_i is finitely generated, there is a positive integer m such that $I_i^m D_{P_i} = (I_iD_{P_i})^m \subseteq dD_{P_i}$; hence $(I_i^m)_tD_{P_i} \subseteq ((I_i^m)_tD_{P_i})_t = (I_i^mD_{P_i})_t \subseteq dD_{P_i}$ (cf. [11, Proposition 1.3] for the second equality). Replacing I_i with I_i^m, we may assume that $I_iD_{P_i} \subseteq dD_{P_i}$. Let $J_i = (d, I_i)_t$. Then J_i is a finite type v-ideal and a P_i-primary ideal [4, Lemma 2.1]. Hence $(A_i)_Q = D_Q = (J_i)_Q$ for any $Q \in t\text{-Max}(D) \setminus \{P_i\}$ and $(A_i)_{P_i} = dD_{P_i} = (d, I_i)D_{P_i} = ((d, I_i)D_{P_i})_t = ((d, I_i)_tD_{P_i})_t = (J_iD_{P_i})_t \supseteq J_iD_{P_i} \supseteq dD_{P_i}$. Thus $A_i = J_i$ [15, Proposition 2.8(3)].

Now, let $B = dA^{-1}$; then $dD = (AB)_t$. We next show that $A \in \mathcal{F}(\Lambda)^\perp$ and $B \in \mathcal{F}(\Lambda)$, which means that $\mathcal{F}(\Lambda)$ is a t-splitting set of ideals. Note that each A_i is P_i-primary, $d \in A_i$, and P_i is a maximal t-ideal of D. So $A = (A_1 \cdots A_n)_t = A_1 \cap \cdots \cap A_n$, and thus $d \in A$, $A \subseteq D$, and $B \subseteq D$. If $C \in \mathcal{F}(\Lambda)$, then $(A + C)_t = D$ since $C \nsubseteq P$ for all $P \in \Lambda$ and $A \nsubseteq Q$ for all $Q \in t\text{-Max}(D) \backslash \Lambda$ (for $A \subseteq Q \Rightarrow A_i \subseteq Q$ for some $i \Rightarrow P_i = \sqrt{A_i} \subseteq Q \Rightarrow Q = P_i \in \Lambda$). Hence $A \in \mathcal{F}(\Lambda)^\perp$. Next, assume that $B \notin \mathcal{F}(\Lambda)$. Then $B \subseteq P$ for some $P \in \Lambda$, and since $d \in B$, we have $d \in P$; hence $P = P_i$ for some i. Hence $B = dA^{-1} \subseteq P_i \Rightarrow dAA^{-1} \subseteq P_iA \Rightarrow d(AA^{-1})_t = dD \subseteq (P_iA)_t$ since A is t-invertible by the above paragraph $\Rightarrow A_iD_{P_i} = dD_{P_i} \subseteq (P_iA)_tD_{P_i} \subseteq ((P_iA)_tD_{P_i})_t = ((P_iA)D_{P_i})_t = (P_iD_{P_i}AD_{P_i})_t = (P_iD_{P_i}A_iD_{P_i})_t = (P_iD_{P_i}dD_{P_i})_t \Rightarrow D_{P_i} \subseteq (P_iD_{P_i})_t \Rightarrow D_{P_i} = (P_iD_{P_i})_t$. But since $\text{ht}(P_iD_{P_i}) = \text{ht}P_i = 1$, we have $D_{P_i} \subseteq (P_iD_{P_i})_t = P_iD_{P_i} \subsetneq D_{P_i}$, a contradiction. Thus $B \in \mathcal{F}(\Lambda)$.

(4) $\Rightarrow$ (5) Let $\Sigma = t\text{-Max}(D) \setminus \Lambda$. Then $\mathcal{F}(\Lambda)^\perp = \mathcal{F}(\Sigma)$ by Lemma 2.2(1), and hence $t\text{-Max}(D) \cap \mathcal{F}(\Lambda)^\perp = \Lambda$. Therefore, $t\text{-Max}(D_{\mathcal{F}(\Lambda)}) = \{P_{\mathcal{F}(\Lambda)} | P \in \Lambda\}$ by the remark before [8, Corollary 15].

(5) $\Rightarrow$ (2) For any $P \in \Lambda$, let I be a finite type v-ideal such that $\sqrt{I} = P$. Since $\text{ht}P = 1$, we have $(P_{\mathcal{F}(\Lambda)})_t = P_{\mathcal{F}(\Lambda)}$ [11, Proposition 1.6(a)]; so $(ID_{\mathcal{F}(\Lambda)})_t \subseteq (PD_{\mathcal{F}(\Lambda)})_t \subseteq (P_{\mathcal{F}(\Lambda)})_t \subsetneq D_{\mathcal{F}(\Lambda)}$. Let Q be a prime ideal of $D_{\mathcal{F}(\Lambda)}$ minimal over $(ID_{\mathcal{F}(\Lambda)})_t$. Since $I \subseteq ID_{\mathcal{F}(\Lambda)} \cap D \subseteq Q \cap D$ and $\sqrt{I} = P$, we have $P \subseteq Q \cap D$, and hence $P = Q \cap D$ since P is a maximal t-ideal and $Q \cap D$ is a t-ideal [11, Proposition 1.3]. In particular, $P = Q \cap D$ implies that $AD_{\mathcal{F}(\Lambda)} \nsubseteq Q$ for all $A \in \mathcal{F}(\Lambda)$, and so $Q = (Q \cap D)_{\mathcal{F}(\Lambda)}$ [5, Theorem 1.1(2)]. Therefore, $P_{\mathcal{F}(\Lambda)} = \sqrt{(ID_{\mathcal{F}(\Lambda)})_t}$, and since $(J_tD_{\mathcal{F}(\Lambda)})_t = (JD_{\mathcal{F}(\Lambda)})_t$ for any nonzero finitely generated ideal J of D [11, Proposition 1.2(b)], $P_{\mathcal{F}(\Lambda)}$ is the radical of a finite type v-ideal. Note that $(D_{\mathcal{F}(\Lambda)})_{P_{\mathcal{F}(\Lambda)}} = D_P$ for all $P \in \Lambda$ [5, Theorem 1.1]. Thus the intersection $D_{\mathcal{F}(\Lambda)} = \cap_{P \in \Lambda} D_P$ has finite character by Lemma 2.1.

(6) $\Rightarrow$ (7) Clear. (7) $\Rightarrow$ (5) See Lemma 2.2(2). $\square$

Corollary 2.4. *Let Λ be a nonempty set of t-invertible height-one prime ideals of D. Then the following statements are equivalent.*

(1) *$D_{\mathcal{F}(\Lambda)}$ is a Krull domain.*
(2) *$D_{\mathcal{F}(\Lambda)}$ is a weakly Krull domain.*
(3) *The intersection $D_{\mathcal{F}(\Lambda)} = \cap_{P\in\Lambda} D_P$ has finite character.*
(4) *$\cap_n P_1 \cdots P_n = (0)$ for each infinite sequence (P_n) of distinct elements of Λ.*
(5) *$\mathcal{F}(\Lambda)$ is a t-splitting set of ideals.*
(6) *t-Max$(D_{\mathcal{F}(\Lambda)}) = \{P_{\mathcal{F}(\Lambda)} | P \in \Lambda\}$.*
(7) *$\mathcal{F}(\Lambda)$ is finitely generated.*
(8) *$\mathcal{F}(\Lambda)$ is v-finite.*

Proof. (1) $\Rightarrow$ (2) is clear and (3) $\Rightarrow$ (1) appears in [11, Theorem 2.9]. The other implications are immediate consequences of Theorem 2.3 since t-invertible prime t-ideals are maximal t-ideals [14, Proposition 1.3] and of finite type. □

An integral domain D is said to be of *t-finite character* if each nonzero nonunit of D is contained in only a finite number of maximal t-ideals of D, i.e., if the intersection $D = \cap_{P\in t\text{-Max}(D)} D_P$ has finite character. It is clear that a weakly Krull domain is of t-finite character.

Corollary 2.5. *Let Λ be a nonempty set of height-one maximal t-ideals of D. If D is of t-finite character, then $\mathcal{F}(\Lambda)$ is a t-splitting set of ideals.*

Proof. First, note that $\mathcal{F}(\Lambda)$ is finitely generated [11, Proposition 1.17]. Next, let $P \in \Lambda$, and choose a nonzero element $x \in P$. Since D is of t-finite character, there are only finitely many maximal t-ideals of D containing x. So we can choose an $y \in P$ such that $P = \sqrt{(x,y)}$ since ht$P = 1$ (cf. [16, Theorem 83]). Hence $P = \sqrt{(x,y)_v}$. Thus $\mathcal{F}(\Lambda)$ is a t-splitting set of ideals by Theorem 2.3. □

Note that the integral domain $\mathbb{Z} + X\mathbb{Q}[X]$ does not have t-finite character, even though $\mathcal{F}(\Lambda)$ is finitely generated for each nonempty subset Λ of prime t-ideals (see [10, Example 8.4.7] or [11, p.129]). Our next result shows that if t-dim$(D) = 1$, then D has t-finite character if and only if $\mathcal{F}(\Lambda)$ is finitely generated for all nonempty subsets Λ of maximal t-ideals of D.

Corollary 2.6. *The following statements are equivalent.*

(1) *D is a weakly Krull domain.*
(2) *$\mathcal{F}(\Lambda)$ is t-splitting for every nonempty subset Λ of prime t-ideals of D.*
(3) *t-dim$(D) = 1$ and $\mathcal{F}(\Lambda)$ is finitely generated for every nonempty subset Λ of prime t-ideals of D.*

Proof. (1) $\Rightarrow$ (2) and (3) Suppose that D is a weakly Krull domain, and let Λ be a nonempty set of prime t-ideals of D. Then t-dim$(D) = 1$, and hence each prime t-ideal of D is a height-one maximal t-ideal. Thus $\mathcal{F}(\Lambda)$ is finitely generated [11, Proposition 1.17]. Also, since a weakly Krull domain is of t-finite character, $\mathcal{F}(\Lambda)$ is a t-splitting set of ideals by Corollary 2.5.

(2) $\Rightarrow$ (1) Let P be a prime ideal of D minimal over a nonzero principal ideal. Note that $D_{\mathcal{F}(P)} = D_P$; so $dD_P \cap D$ is t-invertible for all $0 \neq d \in D$ [8, Proposition 5]. Hence $D \setminus P$ is a t-splitting set [1, Corollary 2.3]). Thus D is a weakly Krull domain [1, p.8].

(3) $\Rightarrow$ (1) Let $P \in t\text{-Max}(D)$ and $\Lambda = t\text{-Max}(D) \setminus \{P\}$. Then $P \nsubseteq Q$ for all $Q \in \Lambda$, and hence $P \in \mathcal{F}(\Lambda)$; so there is a finitely generated ideal I of D such that $I \subseteq P$ and $I \nsubseteq Q$ for all $Q \in \Lambda$. So $P = \sqrt{I_t}$ since $t\text{-dim}(D) = 1$, and thus the intersection $D = \cap_{P \in X^1(D)} D_P$ has finite character by Lemma 2.1. □

An integral domain D is called a *Mori domain* if D satisfies the ascending chain condition on integral divisorial ideals of D; equivalently, if each t-ideal of D is a finite type v-ideal. It is well known, and easily verified, that a Mori domains (and hence Noetherian domain) has t-finite character. So if D is a Mori domain with $t\text{-dim}(D) = 1$, then every spectral localizing system of D is finitely generated by Corollary 2.6. Our next result is a restatement of Corollary 2.6 for a Mori domain.

Corollary 2.7. *The following statements are equivalent for a Mori domain D.*
(1) *D is a weakly Krull domain.*
(2) *$t\text{-dim}(D) = 1$.*
(3) *$\mathcal{F}(\Lambda)$ is t-splitting for every nonempty subset Λ of prime t-ideals of D.*

3. Generalized weakly factorial domains

A nonzero element $x \in D$ is said to be *primary* if xD is a primary ideal, while D is called a *generalized weakly factorial domain* (GWFD) if each nonzero prime ideal of D contains a primary element (see [4]). This concept is a generalization of the well-known property of a UFD; D is a UFD if and only if each nonzero prime ideal of D contains a principal prime [16, Theorem 5]. It is known that D is a GWFD if and only if $t\text{-dim}(D) = 1$ and for each $P \in X^1(D)$, $P = \sqrt{aD}$ for some $a \in D$ [4, Theorem 2.2]; so a GWFD is a weakly Krull domain. We next give the GWFD analog of Theorem 2.3. To do this, we need a lemma.

Lemma 3.1. *Let Λ be a nonempty set of maximal t-ideals of D, and let $P \in \Lambda$. If $P = \sqrt{aD}$, then $aD_{\mathcal{F}(\Lambda)}$ is $P_{\mathcal{F}(\Lambda)}$-primary and $P_{\mathcal{F}(\Lambda)}$ is a maximal t-ideal of $D_{\mathcal{F}(\Lambda)}$.*

Proof. First, recall that aD is P-primary [4, Lemma 2.1] and $(aD)_{\mathcal{F}(\Lambda)} = \cap_{P \in \Lambda} aD_P = a(\cap_{P \in \Lambda} D_P) = aD_{\mathcal{F}(\Lambda)}$ (see [11, p.120] for the first equality). Let $b \in D_{\mathcal{F}(\Lambda)}$ such that $ab \in D$. Then there is an $I \in \mathcal{F}(\Lambda)$ such that $bI \subseteq D$; so $abI \subseteq aD$. Since $I \in \mathcal{F}(\Lambda)$ and $P \in \Lambda$, we have $I \nsubseteq P$, and since aD is P-primary, $ab \in aD$ and $b \in D$. Hence $aD_{\mathcal{F}(\Lambda)} \cap D \subseteq aD$, and thus $aD_{\mathcal{F}(\Lambda)} \cap D = aD$.

Let $xy \in aD_{\mathcal{F}(\Lambda)}$, where $x, y \in D_{\mathcal{F}(\Lambda)}$ with $y \notin P_{\mathcal{F}(\Lambda)}$. Then there are $I, J \in \mathcal{F}(\Lambda)$ such that $xI \subseteq D$ and $yJ \subseteq D$; hence $(xI)(yJ) \subseteq aD_{\mathcal{F}(\Lambda)} \cap D = aD$. Since $y \notin P_{\mathcal{F}(\Lambda)}$ and $J \nsubseteq P = P_{\mathcal{F}(\Lambda)} \cap D$, we have $yJ \nsubseteq P$, and thus $xI \subseteq aD$; so

$x \in (aD)_{\mathcal{F}(\Lambda)} = aD_{\mathcal{F}(\Lambda)}$. Thus if we show that $\sqrt{aD_{\mathcal{F}(\Lambda)}} = P_{\mathcal{F}(\Lambda)}$, then $aD_{\mathcal{F}(\Lambda)}$ is a $P_{\mathcal{F}(\Lambda)}$-primary ideal, and hence $P_{\mathcal{F}(\Lambda)}$ is a maximal t-ideal [4, Lemma 2.1]. Let Q be a prime ideal of $D_{\mathcal{F}(\Lambda)}$ minimal over $aD_{\mathcal{F}(\Lambda)}$. Then Q, and hence $Q \cap D$, is a prime t-ideal [11, Proposition 1.3]. Also, since $aD \subseteq Q \cap D$, we have $P = \sqrt{aD} \subseteq Q \cap D$. Hence the maximality of P implies that $P = Q \cap D$, and thus $Q = P_{\mathcal{F}(\Lambda)}$ [5, Theorem 1.1(2)]. This implies that $\sqrt{aD_{\mathcal{F}(\Lambda)}} = P_{\mathcal{F}(\Lambda)}$. □

The following theorem is the GWFD analog of Theorem 2.3.

Theorem 3.2. *Let Λ be a nonempty set of height-one maximal t-ideals of D such that each $P \in \Lambda$ is the radical of a principal ideal. Then the following statements are equivalent.*

(1) *$D_{\mathcal{F}(\Lambda)}$ is a GWFD.*
(2) *$D_{\mathcal{F}(\Lambda)}$ is a weakly Krull domain.*
(3) *The intersection $D_{\mathcal{F}(\Lambda)} = \cap_{P\in\Lambda} D_P$ has finite character.*
(4) *$\cap_n P_1 \cdots P_n = (0)$ for each infinite sequence (P_n) of elements of Λ.*
(5) *$\mathcal{F}(\Lambda)$ is a t-splitting set of ideals.*
(6) *t-$Max(D_{\mathcal{F}(\Lambda)}) = \{P_{\mathcal{F}(\Lambda)} | P \in \Lambda\}$.*
(7) *$\mathcal{F}(\Lambda)$ is finitely generated.*
(8) *$\mathcal{F}(\Lambda)$ is v-finite.*

Proof. (1) ⇒ (2) [4, Corollary 2.3]. For (2) ⇔ (3) ⇔ (4) ⇔ (5) ⇔ (6) ⇔ (7) ⇔ (8), see Theorem 2.3. (3) ⇒ (1) Note that $D_{\mathcal{F}(\Lambda)}$ is a weakly Krull domain and $X^1(D_{\mathcal{F}(\Lambda)}) = t\text{-Max}(D_{\mathcal{F}(\Lambda)}) = \{P_{\mathcal{F}(\Lambda)} | P \in \Lambda\}$ by Theorem 2.3. Also, note that for $P \in \Lambda$, if $P = \sqrt{aD}$, then $P_{\mathcal{F}(\Lambda)} = \sqrt{aD_{\mathcal{F}(\Lambda)}}$ by Lemma 3.1. Thus $D_{\mathcal{F}(\Lambda)}$ is a GWFD [4, Theorem 2.2]. □

Let $T(D)$ be the group of t-invertible fractional t-ideals of D under the t-multiplication $I * J = (IJ)_t$, and let $Prin(D)$ be its subgroup of nonzero principal fractional ideals of D. Then $Cl(D) = T(D)/Prin(D)$, called the *class group of D*, is an abelian group. Recall that D is a *weakly factorial domain* (WFD) if each nonzero element of D can be written as a product of primary elements and that D is an *almost weakly factorial domain* (AWFD) if for each nonzero $d \in D$, there exists a natural number $n = n(d)$ such that d^n can be written as a product of primary elements. It is well known that D is a WFD if and only if D is a weakly Krull domain and $Cl(D) = 0$ [3, Theorem] and that D is an AWFD if and only if D is a weakly Krull domain and $Cl(D)$ is torsion [2, Theorem 3.4].

Let $\mathcal{S}$ be a t-splitting set of ideals of D and $\mathcal{S}^{\perp}$ the t-complement of $\mathcal{S}$. Then the map $\alpha : Cl(D) \to Cl(D_{\mathcal{S}}) \oplus Cl(D_{\mathcal{S}^{\perp}})$ defined by $\alpha([I]) = ([(ID_{\mathcal{S}})_t], [(ID_{\mathcal{S}^{\perp}})_t])$ is a group epimorphism [8, Remark 13], and thus the homomorphism $\beta : Cl(D) \to Cl(D_{\mathcal{S}})$ defined by $\beta([I]) = [(ID_{\mathcal{S}})_t]$ is surjective. Let Λ be a nonempty set of prime t-ideals of D. Then $\cap_{P\in\Lambda} D_P$ is called a *subintersection* of D.

Corollary 3.3. *Any subintersection of a GWFD (resp., AWFD, WFD) is a GWFD (resp., AWFD, WFD).*

Proof. Recall that a GWFD, an AWFD, and a WFD are weakly Krull domains. Let D be a weakly Krull domain, and let R be a subintersection of D. Then $R = \cap_{P \in \Lambda} D_P$ for some $\emptyset \neq \Lambda \subseteq t\text{-Max}(D)$, and hence $R = D_{\mathcal{F}(\Lambda)}$ [10, Proposition 5.1.4].

If D is a GWFD, then $t\text{-dim}(D) = 1$, each prime ideal $P \in \Lambda$ is the radical of a principal ideal [4, Theorem 2.2], and $\mathcal{F}(\Lambda)$ is a t-splitting set of ideals (Corollary 2.6). Thus $R = D_{\mathcal{F}(\Lambda)}$ is a GWFD by Theorem 3.2. Next, assume that D is a WFD (resp., AWFD). Since WFDs and AWFDs are both GWFDs, $R = D_{\mathcal{F}(\Lambda)}$ is a GWFD. Also, since the homomorphism $\beta : Cl(D) \to Cl(D_{\mathcal{F}(\Lambda)})$ defined by $\beta([I]) = [(ID_{\mathcal{F}(\Lambda)})_t]$ is surjective (see the remark before Corollary 3.3), $Cl(R) = 0$ if $Cl(D) = 0$ and $Cl(R)$ is torsion if $Cl(D)$ is torsion. Therefore, if D is a WFD (resp., AWFD), then R is a WFD (resp., AWFD). □

We end this paper with an example which shows that $\mathcal{F}(\Lambda)$ need not be a t-splitting set of ideals for a nonempty set Λ of height-one principal prime ideals (and hence maximal t-ideals).

Example 3.4. Let D be the ring of entire functions, $\mathbb{C}$ the field of complex numbers, and $\Lambda = \{M_z = (X - z)D | z \in \mathbb{C}\}$. Then $\Lambda \subseteq t\text{-Max}(D) \cap X^1(D)$, $D = \cap_{M_z \in \Lambda} D_{M_z}$ [17, p.267], and D is a Bezout domain with $\dim(D) = \infty$ (and hence $t\text{-dim}(D) = \infty$) [10, Proposition 8.1.1]. Hence D is not a GWFD, and thus $\mathcal{F}(\Lambda)$ is not a t-splitting set of ideals by Theorem 3.2. The ring of entire functions also serves as a counterexample of the following generalization of [13, Theorem 1.6] that if each minimal prime ideal of the ideal I is the radical of a finitely generated ideal, then I has only finitely many minimal prime ideals.

References

[1] D. D. Anderson, D. F. Anderson, and M. Zafrullah, *The ring $D+XD_S[X]$ and t-splitting sets*, Arab. J. Sci. Eng. Sect. C Theme Issues **26** (2001), no. 1, 3–16.

[2] D. D. Anderson, J. L. Mott, and M. Zafrullah, *Finite character representations for integral domains*, Boll. Un. Mat. Ital. B (7) **6** (1992), no. 3, 613–630.

[3] D. D. Anderson and M. Zafrullah, *Weakly factorial domains and groups of divisibility*, Proc. Amer. Math. Soc. **109** (1990), no. 4, 907–913.

[4] D. F. Anderson, G. W. Chang, and J. Park, *Generalized weakly factorial domains*, Houston Math. J. **29** (2003), no. 1, 1–13.

[5] J. T. Arnold and J. W. Brewer, *On flat overrings, ideal transforms and generalized transforms of a commutative ring*, J. Algebra **18** (1971), 254–263.

[6] G. W. Chang, *Weakly factorial rings with zero divisors*, Lecture Notes in Pure and Appl. Math., 220, Dekker, New York, 2001.

[7] G. W. Chang, T. Dumitrescu, and M. Zafrullah, *t-splitting sets in integral domains*, J. Pure Appl. Algebra **187** (2004), no. 1-3, 71–86.

[8] ———, *t-splitting multiplicative sets of ideals in integral domains*, J. Pure Appl. Algebra **197** (2005), no. 1-3, 239–248.

[9] S. El Baghdadi, *On a class of Prüfer v-multiplication domains*, Comm. Algebra **30** (2002), no. 8, 3723–3742.

[10] M. Fontana, J. A. Huckaba, and I. J. Papick, *Prüfer domains*, Marcel Dekker, Inc., New York, 1997.

[11] S. Gabelli, *On Nagata's theorem for the class group*, II, Lecture Notes in Pure and Appl. Math., Vol 206, Marcel Dekker, New York, 1999.

[12] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.

[13] R. Gilmer and W. Heinzer, *Primary ideals with finitely generated radical in a commutative ring*, Manuscripta Math. **78** (1993), no. 2, 201–221.

[14] E. Houston and M. Zafrullah, *On t-invertibility,* II, Comm. Algebra **17** (1989), no. 8, 1955–1969.

[15] B. G. Kang, *Prüfer v-multiplication domains and the ring* $R[X]_{N_v}$, J. Algebra **123** (1989), no. 1, 151–170.

[16] I. Kaplansky, *Commutative Rings, Revised Ed.*, The University of Chicago Press, Chicago, Ill.-London, 1974.

[17] K. A. Loper, *A class of Prüfer domains that are similar to the ring of entire functions*, Rocky Mountain J. Math. **28** (1998), no. 1, 267–285.

Department of Mathematics
University of Incheon
Incheon 402-749, Korea
E-mail address: whan@incheon.ac.kr