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CONVERGENCE PROPERTIES OF HYPERSPACES

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Abstract. In this paper we investigate relationships between closure-type and convergence-type properties of hyperspaces over a space X and covering properties of X.

Introduction

Let X be a Hausdorff space. By 2^X we denote the family of all closed subsets of X. If A is a subset of X and $\mathcal{A}$ a family of subsets of X, then we write

$$A^c = X \setminus A \text{ and } \mathcal{A}^c = \{A^c : A \in \mathcal{A}\},$$
$$A^- = \{F \in 2^X : F \cap A \neq \emptyset\},$$
$$A^+ = \{F \in 2^X : F \subset A\}.$$

Let Δ be a subset of 2^X closed for finite unions and containing all singletons. We mention three important special cases:

1. Δ is the collection $\mathsf{CL}(X) = 2^X \setminus \{\emptyset\}$;
2. Δ is the family $\mathbb{K}(X)$ of all non-empty compact subsets of X;
3. Δ is the family $\mathbb{F}(X)$ of all non-empty finite subsets of X.

Given $\Delta \subset 2^X$, the associated *upper Δ-topology*, denoted by Δ^+, is the topology whose base is the collection

$$\{(D^c)^+ : D \in \Delta\} \cup \{2^X\}.$$

For $\Delta = \mathsf{CL}(X)$ we have the well-known *upper Vietoris topology* V^+, for $\Delta = \mathbb{K}(X)$ we have the extensively studied *upper Fell topology* (or *co-compact topology*) F^+, and for $\Delta = \mathbb{F}(X)$ we have the topology that we call Z^+-topology.

The *lower Vietoris topology* V^- is generated by all the sets U^-, $U \subset X$ open.

The *Δ-topology*, denoted τ_Δ, is defined by $\tau_\Delta = \Delta^+ \vee \mathsf{V}^-$. Recall that τ_Δ-basic sets are of the form

$$(D^c)^+ \cap (\cap_{i \leq m} V_i^-),\ D \in \Delta,\ V_1, \ldots, V_m \text{ open in } X.$$

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The most popular among Δ-topologies are the *Vietoris topology* $\mathsf{V} = \mathsf{V}^+ \vee \mathsf{V}^-$ and the *Fell topology* $\mathsf{F} = \mathsf{F}^+ \vee \mathsf{V}^-$ [7]. We also consider the topology $\mathsf{Z} = \mathsf{Z}^+ \vee \mathsf{V}^-$.

τ_Δ-topologies are a large and significant class of hyperspace topologies and were intensively studied in the last decades (see [19], [5]).

Let us fix some terminology and notation.

Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose elements are families of subsets of an infinite set X. Then (see [21], [10]):

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle:

> For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n $b_n \in A_n$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

> For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n $B_n \subset A_n$ and $\bigcup_{n\in\mathbb{N}} B_n$ is an element of $\mathcal{B}$.

For a space X, a collection $\Delta \subset 2^X$ and a point $x \in X$ we consider the following sets $\mathcal{A}$ and $\mathcal{B}$:

- $\mathcal{O}$: the collection of open covers of X;
- Ω: the collection of ω-covers of X;
- $\mathcal{K}$: the collection of k-covers of X;
- Γ: the collection of γ-covers;
- Γ_k: the collection of γ_k-covers;
- Γ_Δ: the collection of γ_Δ-covers;
- Ω_x: the set $\{A \subset X \setminus \{x\} : x \in \overline{A}\}$;
- Σ_x: the set of all nontrivial sequences in X that converge to x.

Let us recall that if $\Delta \subset 2^X$, then an open cover $\mathcal{U}$ of X is called a Δ*-cover* if each $D \in \Delta$ is contained in an element of $\mathcal{U}$ and X does not belong to $\mathcal{U}$ (i.e. the cover is not trivial). $\mathbb{F}(X)$-covers (resp. $\mathbb{K}(X)$-covers) are called ω*-covers* (resp. k*-covers*). An open cover $\mathcal{U}$ of X is said to be a γ_Δ*-cover* if it is infinite and for each $D \in \Delta$ the set $\{U \in \mathcal{U} : D \not\subseteq U\}$ is finite. $\gamma_{\mathbb{F}(X)}$-covers (resp. $\gamma_{\mathbb{K}(X)}$-covers) are called γ*-covers* [8] (resp. γ_k*-covers* [12]). Observe that each infinite subset of a γ_Δ-cover is still a γ_Δ-cover. So, we may suppose that such covers are countable.

We also suppose that all spaces are infinite and Hausdorff.

A number of results in the literature show that there is a nice duality between closure properties of hyperspaces and covering properties of the basic space, i.e. the closure properties of hyperspaces 2^X can be characterized by covering properties of X (see, for instance, [3] [9], [11], [12], [4]).

In this paper we show that when the space of closed subsets of a space X is endowed with a Δ^+-topology or τ_Δ-topology some of its convergence properties can be expressed in a transparent way by covering properties of X.

In Section 1 we consider Arhangel'skii's α_i, $i = 1, 2, 3, 4$, properties and show that in hyperspaces the properties α_2, α_3 and α_4 coincide. Section 2 contains some results related to sequential-type properties of hyperspaces.

1. α_i-properties in hyperspaces

In this section we investigate α_i properties of Δ^+ hyperspace topologies. These properties were introduced by Arhangel'skiĭ in [1] (in a bit different formulations) as follows.

A space X has *property* α_i, $i = 1, 2, 3, 4$, if for each $x \in X$ and each sequence $(\sigma_n : n \in \mathbb{N})$ of elements of Σ_x there is a $\sigma \in \Sigma_x$ such that:

α_1: for each $n \in \mathbb{N}$ the set $\sigma_n \setminus \sigma$ is finite;
α_2: for each $n \in \mathbb{N}$ the set $\sigma_n \cap \sigma$ is infinite;
α_3: for infinitely many $n \in \mathbb{N}$ the set $\sigma_n \cap \sigma$ is infinite;
α_4: for infinitely many $n \in \mathbb{N}$ the set $\sigma_n \cap \sigma$ is nonempty.

Evidently,

$$\alpha_1 \Rightarrow \alpha_2 \Rightarrow \alpha_3 \Rightarrow \alpha_4.$$

A specific behavior of these properties in topological groups was investigated in [23], [18], and in function spaces $\mathsf{C}_p(X)$ in [22] and [20]. We prove that in some hyperspaces the last three properties coincide.

Theorem 1. *For a space X and a collection $\Delta \subset 2^X$ the following statements are equivalent:*

(1) *$(2^X, \Delta^+)$ is an α_2-space;*
(2) *$(2^X, \Delta^+)$ is an α_3-space;*
(3) *$(2^X, \Delta^+)$ is an α_4-space;*
(4) *For each $E \in 2^X$, $(2^X, \Delta^+)$ satisfies $\mathsf{S}_1(\Sigma_E, \Sigma_E)$;*
(5) *Each open set $Y \subset X$ satisfies $\mathsf{S}_1(\Gamma_\Delta, \Gamma_\Delta)$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial. So, we prove the remaining implications.

(3) $\Rightarrow$ (4): Let $(\sigma_n : n \in \mathbb{N})$ be a sequence of elements of Σ_E for some $E \in (2^X, \Delta^+)$. Assume that for each $n \in \mathbb{N}$, $\sigma_n = (F_{n,m} : m \in \mathbb{N})$. For all $n, m \in \mathbb{N}$ define $S_{n,m} = F_{1,m} \cup F_{2,m} \cup \cdots \cup F_{n,m}$. Then each $S_{n,m}$ is a closed subset of X and for each n the sequence $s_n = (S_{n,m} : m \in \mathbb{N})$ belongs to Σ_E as can be easily verified. Apply now (3) to the sequence $(s_n : n \in \mathbb{N})$ of elements of Σ_E. There is an increasing sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ and a sequence $s = (S_{n_i,m_i} : i \in \mathbb{N}) \in \Sigma_E$ such that for each $i \in \mathbb{N}$, $S_{n_i,m_i} \in s_{n_i}$. Then:

(*i*) If $S_{n_1,m_1} = \bigcup_{i=1}^{n_1} F_{i,m_1}$, then for each $i \leq n_1$ put $T_i = F_{i,m_1}$;
(*ii*) If $j \geq 1$ and $S_{n_{j+1},m_{j+1}} = \bigcup_{i=1}^{n_{j+1}} F_{i,m_{j+1}}$, then for each i with $n_j < i \leq n_{j+1}$ put $T_i = F_{i,m_{j+1}}$.

Note that for each $i \in \mathbb{N}$, $T_i \in \sigma_i$. The sequence $t = (T_n : n \in \mathbb{N})$ is an element of Σ_E and is a selector showing that $(2^X, \Delta^+)$ satisfies (4).

(4) $\Rightarrow$ (5): Let $(\mathcal{U}_n : n \in \mathbb{N})$, $\mathcal{U} = \{U_{n,m} : m \in \mathbb{N}\}$, be a sequence of countable γ_Δ-covers of an open set Y in X. If we put for each $n \in \mathbb{N}$, $\mathcal{S}_n = \mathcal{U}_n^c$ we get a sequence $(\mathcal{S}_n : n \in \mathbb{N})$ of sequences in 2^X such that each $\mathcal{S}_n$ Δ^+-converges to Y^c, i.e. $\mathcal{S}_n \in \Sigma_{Y^c}$. Indeed, fix n and let $W = (D^c)^+$, be a basic Δ^+-neighborhood of Y^c. Since $\mathcal{U}_n$ is a γ_Δ-cover of Y, $D \subset Y$ and $D \in \Delta$, there is $m_0 \in \mathbb{N}$ such that $D \subset U_{n,m}$ for each $m \geq m_0$. It follows that for each $m \geq m_0$ we have $U_{n,m}^c \in W$, i.e. $\mathcal{S}_n$ Δ^+-converges to Y^c. As $(2^X, \Delta^+)$ is an $\mathsf{S}_1(\Sigma_{Y^c}, \Sigma_{Y^c})$-set there is a sequence $\mathcal{S} = (S_n : n \in \mathbb{N})$ in 2^X Δ^+-converging to Y^c and such that for each n, $S_n \in \mathcal{S}_n$. Then $(S_n^c \equiv U_{n,m_n} : n \in \mathbb{N})$ is a sequence such that for each n, U_{n,m_n} is an element of $\mathcal{U}_n$. We claim that $\{U_{n,m_n} : n \in \mathbb{N}\}$ is a γ_Δ-cover of Y. Indeed, let $D \in \Delta$, $D \subset Y$. Since $\mathcal{S}$ Δ^+-converges to Y^c, it follows that the Δ^+-neighborhood $(D^c)^+$ of Y^c contains all but finitely many elements S_n. It implies that D is a subset of all but finitely many elements U_{n,m_n}.

(5) $\Rightarrow$ (1): Let $E \in (2^X, \Delta^+)$ and let $(\sigma_n : n \in \mathbb{N})$ be a sequence of elements of Σ_E. Suppose that for each $n \in \mathbb{N}$, $\sigma_n = (F_{n,m} : m \in \mathbb{N})$. For all $n, m \in \mathbb{N}$ define $S_{n,m} = F_{1,m} \cup F_{2,m} \cup \cdots \cup F_{n,m}$. Then each $S_{n,m}$ is a closed subset of X and for each n the sequence $s_n = (S_{n,m} : m \in \mathbb{N})$ belongs to Σ_E. For each $n \in \mathbb{N}$, $\mathcal{U}_n := \{(S_{n,m} \cup E)^c : m \in \mathbb{N}\}$ is a γ_Δ-cover of the open set $E^c \subset X$; it is easily verified. By (5), there is a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\}$ is a γ_Δ-cover of E^c. Let for each n, $U_n = (S_{n,m_n} \cup E)^c$. By taking subsequences if necessary we may suppose that for each n, $m_n < m_{n+1}$. Note that since $\{U_n : n \in \mathbb{N}\}$ is a γ_Δ-cover of E^c, we have $(S_{n,m_n} : n \in \mathbb{N}) \in \Sigma_E$. As a consequence, $\sigma := (F_{n,m} : m \geq m_i, i \geq n; i, n \in \mathbb{N})$ is an element of Σ_E which has infinite intersection with each σ_n, hence witnesses for the sequence $(\sigma_n : n \in \mathbb{N})$ that $(2^X, \Delta^+)$ satisfies α_2. □

Let us observe that one can prove the following result:

Remark 2. For a space X and a collection $\Delta \subset 2^X$, the statement (1) implies (2):

(1) For each $E \in 2^X$, $(2^X, \tau_\Delta)$ satisfies $\mathsf{S}_1(\Sigma_E, \Sigma_E)$;
(2) Each open set $Y \subset X$ satisfies $\mathsf{S}_1(\Gamma_\Delta, \Gamma_\Delta)$ (equivalently, $\mathsf{S}_{fin}(\Gamma_\Delta, \Gamma_\Delta)$).

Proof. The proof of (1) $\Rightarrow$ (2) is similar to the proof of (4) $\Rightarrow$ (5) in the previous theorem. A change is that we have to prove that each $\mathcal{S}_n$ τ_Δ-converges to Y^c. Indeed, fix n and let $W = (D^c)^+ \cap (\cap_{i \leq k} V_i^-)$ be a basic τ_Δ-neighborhood of Y^c. Since $\mathcal{U}_n$ is a γ_Δ-cover of Y, $D \subset Y$ and $D \in \Delta$, there is $m_0 \in \mathbb{N}$ such that $D \subset U_{n,m}$ for each $m \geq m_0$. On the other hand, from $Y^c \in V_i^-$ for each $i \leq k$ and $U_{n,m}^c \supset Y^c$ for each $m \in \mathbb{N}$, it follows that for each $m \geq m_0$ we have $U_{n,m}^c \in W$, i.e. $\mathcal{S}_n$ τ_Δ-converges to Y^c. Then use the fact that $(2^X, \tau_\Delta)$ is an $\mathsf{S}_1(\Sigma_{Y^c}, \Sigma_{Y^c})$-set, find a sequence $\mathcal{S} = (S_n : n \in \mathbb{N})$ in 2^X τ_Δ-converging to Y^c and such that for each n, $S_n \in \mathcal{S}_n$, and show, in the same way, that $\{S_n^c : n \in \mathbb{N}\}$ is a γ_Δ-cover of Y.

The fact that Y satisfies $\mathsf{S}_1(\Gamma_\Delta, \Gamma_\Delta)$ if and only if it satisfies $\mathsf{S}_{fin}(\Gamma_\Delta, \Gamma_\Delta)$ is proved by a small modification of the proof of Theorem 5 in [12], or of the proof of Theorem 1.1 in [10]. □

Two important special cases are consequences of Theorem 1.

Corollary 3. *For a space X the following statements are equivalent:*

(1) $(2^X, \mathsf{F}^+)$ *is an* α_4*-space;*
(2) *Each open set* $Y \subset X$ *is an* $\mathsf{S}_1(\Gamma_k, \Gamma_k)$*-set.*

Corollary 4. *For a space X the following statements are equivalent:*

(1) $(2^X, \mathsf{Z}^+)$ *is an* α_4*-space;*
(2) *Each open set* $Y \subset X$ *is an* $\mathsf{S}_1(\Gamma, \Gamma)$*-set.*

Recall that a space X is said to be *Fréchet-Urysohn* (briefly, FU) if for each $x \in X$ and each $A \in \Omega_x$ there is a sequence $(x_n : n \in \mathbb{N})$ in A belonging to Σ_x. X is *strictly Fréchet-Urysohn* (SFU) if for each $x \in X$ it satisfies the selection principle $\mathsf{S}_1(\Omega_x, \Sigma_x)$.

According to [8] a space is said to be a γ*-set* if it satisfies the selection principle $\mathsf{S}_1(\Omega, \Gamma)$. In [12] spaces satisfying the selection principle $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$ were called γ_k'*-sets*. It was also shown in [12] that for a space X, $(2^X, \mathsf{Z}^+)$ is a Fréchet-Urysohn space if and only if each open set $Y \subset X$ is a γ-set, and that $(2^X, \mathsf{F}^+)$ is a strictly Fréchet-Urysohn space if and only if each open set $Y \subset X$ is a γ_k'-set. Because $\Gamma \subset \Omega$ and $\Gamma_k \subset \mathcal{K}$, it follows that $\mathsf{S}_1(\Omega, \Gamma) \subset \mathsf{S}_1(\Gamma, \Gamma)$ and $\mathsf{S}_1(\mathcal{K}, \Gamma_k) \subset \mathsf{S}_1(\Gamma_k, \Gamma_k)$. So from Corollary 4 and Corollary 3 we have the following proposition.

Proposition 5. *For a space X the following statements hold:*

(1) *If* $(2^X, \mathsf{Z}^+)$ *is a Fréchet-Urysohn space, then it is an* α_2*-space;*
(2) *If* $(2^X, \mathsf{F}^+)$ *is strictly Fréchet-Urysohn, then it is an* α_2*-space.*

We consider now the α_1 property in hyperspaces.

Theorem 6. *For a space X and a collection $\Delta \subset 2^X$ the following are equivalent:*

(1) $(2^X, \Delta^+)$ *is an* α_1*-space;*
(2) *For each open set* $Y \subset X$ *and each sequence* $(\mathcal{U}_n : n \in \mathbb{N})$ *of* γ_Δ*-covers of* Y *there is a* γ_Δ*-cover* $\mathcal{U}$ *of* Y *intersecting each* $\mathcal{U}_n$ *in all but finitely many elements.*

Proof. (1) ⇒ (2): Let Y be an open subset of X and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of γ_Δ-covers of Y. Put for each $n \in \mathbb{N}$, $\mathcal{S}_n = \mathcal{U}_n^c$. We get a sequence $(\mathcal{S}_n : n \in \mathbb{N})$ of elements of Σ_{Y^c} in $(2^X, \Delta^+)$. As $(2^X, \Delta^+)$ is an α_1-space there is a sequence $\mathcal{S}$ in 2^X which Δ^+-converges to Y^c and such that for each n, $\mathcal{S}_n \setminus \mathcal{S}$ is a finite set. Set $\mathcal{U} = \mathcal{S}^c$. Then $\mathcal{U}$ is a γ_Δ-cover of Y which shows that (2) holds.

(2) ⇒ (1): Let $(\mathcal{S}_n : n \in \mathbb{N})$ be a sequence of sequences in 2^X which all Δ^+-converge to $E \in 2^X$. It is easy to see that for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{(S \cup E)^c :$

$S \in \mathcal{S}_n\}$ is a γ_Δ-cover of the open set $E^c \subset X$. By (2), there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subset \mathcal{U}_n$, $\mathcal{U}_n \setminus \mathcal{V}_n$ is a finite set, and $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a γ_Δ-cover of E^c. The sequence $\mathcal{S} = \{A : (A \cup E)^c \in \mathcal{V}\}$ intersects each $\mathcal{S}_n$ in all but finitely many elements and Δ^+-converges to E. □

2. FU-type properties

In this section we consider hyperspaces endowed with the Fell topology and investigate several FU-type properties.

Recall that a space X is *sequential* if for each non-closed set $A \subset X$ there are a point $x \in X \setminus A$ and a sequence $(x_n : n \in \mathbb{N})$ in A that belongs to Σ_x. X has *countable tightness* if for each $x \in X$ and each $A \in \Omega_x$ there is a countable element $B \in \Omega_x$ such that $B \subset A$.

E. Reznichenko introduced a property close to the Fréchet-Urysohn property and called the *Reznichenko property* in [13] and [14] (somewhere called also the *weak Fréchet-Urysohn property*). A space X has the Reznichenko property if for each $x \in X$ and each $A \in \Omega_x$ there is a sequence $(B_n : n \in \mathbb{N})$ of pairwise disjoint, finite subsets of A such that each neighborhood U of x intersects all but finitely many sets B_n.

In 1983, E. Pytkeev considered the following property called now the *Pytkeev property* (see, for example, [11]). A space X has the Pytkeev property if for each $x \in X$ and each $A \in \Omega_x$ there is a sequence $(B_n : n \in \mathbb{N})$ of infinite, countable subsets of A such that each neighborhood U of x contains some B_n.

Recall that a *filter-base* on a space X is a non-empty family $\mathcal{F}$ of subsets of X satisfying: (i) $\emptyset \notin \mathcal{F}$, (ii) if $A, B \in \mathcal{F}$, then there is $C \in \mathcal{F}$ with $C \subset A \cap B$ (see [6]).

The following four generalizations of the Fréchet-Urysohn property were introduced in [2].

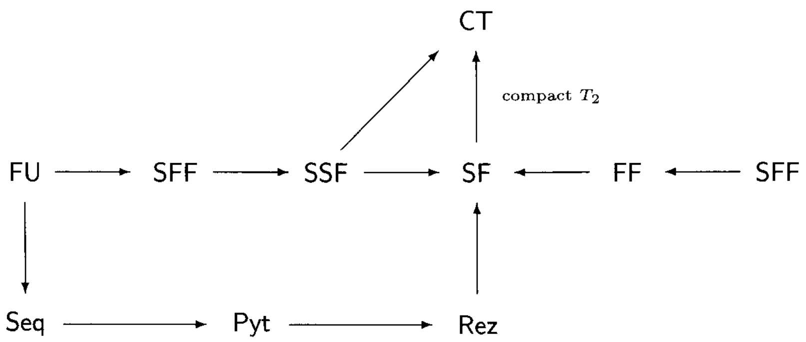
A space X is said to be:

- FF: *filter-Fréchet* if for each $x \in X$ and each $A \in \Omega_x$ there is a sequence $(\mathcal{F}_n : n \in \mathbb{N})$ of filter-bases on A such that:
  - (FF1) For each $n \in \mathbb{N}$, there is an $F_n \in \mathcal{F}_n$ such that $x \notin \overline{F_n}$;
  - (FF2) For each neighborhood U of x there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $F_n \subset U$ for some $F_n \in \mathcal{F}_n$.
- SFF: *strongly filter-Fréchet* if for each $x \in X$ and each $A \in \Omega_x$ there is a sequence $(\mathcal{F}_n : n \in \mathbb{N})$ of filter-bases on A satisfying (FF1) and (FF2) above and the condition
  - (FF3) For each $n \in \mathbb{N}$ there is a countable $F \in \mathcal{F}_n$.
- SSF: *strongly set-Fréchet* if for each $x \in X$ and each $A \in \Omega_x$ there is a sequence $(B_n : n \in \mathbb{N})$ of pairwise disjoint subsets of A such that the following conditions hold:
  - (SF1) $x \notin \overline{B_n}$ for each $n \in \mathbb{N}$;
  - (SF2) each neighborhood U of x intersects all but finitely many sets B_n;

(SF3) each B_n is countable.

SF: *set-Fréchet* if only conditions (SF1) and (SF2) in SSF are satisfied.

The following diagram describes relationships among the mentioned classes of spaces (CT denotes the class of spaces having countable tightness).



The Fréchet-Urysohn property and sequentiality in hyperspaces were studied in [16], [3], [12], the countable tightness property in [16], [9], [3] (and variations on tightness in [4]), the Reznichenko property and the Pytkeev property in [11]. We consider here hyperspaces having the remaining properties in the diagram above. The considered basic spaces X are locally compact (and Hausdorff); in such a case the space $(2^X, \mathsf{F})$ is Hausdorff [7].

Theorem 7. *Let X be a locally compact space. Then* (1) *implies* (2) :

(1) $(2^X, \mathsf{F})$ *is a filter-Fréchet space;*

(2) *For each open set $Y \subset X$ and each k-cover $\mathcal{U}$ of Y there is a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ of filter-bases on $\mathcal{U}$ such that:*

  (*i*) *For each n, there is $\mathcal{C}_n \in \mathcal{B}_n$ which is not a k-cover of Y;*

  (*ii*) *For each compact subset K of Y there is $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, then there exists $\mathcal{H}_n \in \mathcal{B}_n$ satisfying $K \subset H$ for every $H \in \mathcal{H}_n$.*

Proof. Let $\mathcal{U}$ be a k-cover of an open subset Y of X. Then $Y^c \in 2^X$ and the subset $\mathcal{A} := \mathcal{U}^c$ of 2^X satisfies $\mathcal{A} \in \Omega_{Y^c}$. Indeed, let $W = (K^c)^+ \cap (\cap_{i \leq m} V_i^-)$ be an F-neighborhood of Y^c. Since $\mathcal{U}$ is a k-cover of Y and $K \subset Y$, there is a $U \in \mathcal{U}$ with $K \subset U$. From $Y^c \cap V_i \neq \emptyset$ for all $i \leq m$, and $U^c \supset Y^c$ it follows easily that $U^c \in W$. Since $(2^X, \mathsf{F})$ is a filter-Fréchet space, there is a sequence $(\mathcal{F}_n : n \in \mathbb{N})$ of filter-bases on $\mathcal{A}$ that satisfies the conditions (FF1) and (FF2). For each $n \in \mathbb{N}$ let $\mathcal{B}_n$ denote the collection $\{\mathcal{G}^c : \mathcal{G} \in \mathcal{F}_n\}$. Each $\mathcal{B}_n$ is a filter-base on $\mathcal{U}$ because each $\mathcal{F}_n$ is a filter-base on $\mathcal{A}$. We prove that the sequence $(\mathcal{B}_n : n \in \mathbb{N})$ satisfies the conditions (*i*) and (*ii*). For each n, there is an element $\mathcal{S}_n \in \mathcal{F}_n$ such that $Y^c \notin Cl_{\mathsf{F}}(\mathcal{S}_n)$. It follows that $\mathcal{S}_n^c \in \mathcal{B}_n$

is not a k-cover of Y (otherwise, $Y^c \in Cl_{\mathsf{F}}(\mathcal{S}_n)$). To check (ii), let K be a compact subset of Y. Then $(K^c)^+$ is a F-neighborhood of Y^c, hence there is $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there exists some $\mathcal{S}_n \in \mathcal{F}_n$ with $\mathcal{S}_n \subset (K^c)^+$. Consequently, $\mathcal{S}_n^c$ is an element of $\mathcal{B}_n$ that satisfies $K \subset S$ for every $S \in \mathcal{S}_n$. Therefore, (ii) holds. □

Similarly to the proof of Theorem 7 one can prove

Theorem 8. *For a locally compact space X we have* (1) *implies* (2) *below:*

(1) $(2^X, \mathsf{F})$ *is a strongly filter-Fréchet space;*

(2) *For each open set $Y \subset X$ and each k-cover $\mathcal{U}$ of Y there is a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ of filter-bases on $\mathcal{U}$ such that:*
  - (i) *For each n, there is $\mathcal{C}_n \in \mathcal{B}_n$ which is not a k-cover of Y;*
  - (ii) *For each compact K subset of Y there is $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, then there exists $\mathcal{H}_n \in \mathcal{B}_n$ satisfying $K \subset H$ for every $H \in \mathcal{H}_n$;*
  - (iii) *For each $n \in \mathbb{N}$ there is some countable element in $\mathcal{B}_n$.*

The following example shows that in Theorem 8 (2) need not imply (1).

Example 9. [CH] There exists a space X satisfying the condition (2) in the previous theorem, but $(2^X, \mathsf{F})$ is not a strongly filter-Fréchet space.

Let X be the Hausdorff, compact, hereditarily Lindelöf, non hereditarily separable space constructed under CH in [15]. From Corollary 2.16 in [3], it follows that $(2^X, \mathsf{F})$ has uncountable tightness so that it is not SFF.

Let us show that the condition (2) in Theorem 8 holds. Let Y be any open subset of X and let $\mathcal{U}$ be a k-cover of Y. As Y is locally compact and Lindelöf, it is hemicompact (see 3.8.C.(b) in [6]). Let $(K_n : n \in \mathbb{N})$ be an increasing countable family of compact subsets of Y such that each compact subset of Y is contained in some K_n. For each n pick a set $U_n \in \mathcal{U}$ such that $K_n \subset U_n$. Since $\mathcal{U}$ is a k-cover of Y, Y is not a member of $\mathcal{U}$. Thus for each $n \in \{1, \dots, n_0\}$, where n_0 is some element in $\mathbb{N}$, pick a point $x_n \in Y \setminus U_n$ and for each $n \in \mathbb{N}$ define

$$\mathcal{B}_n = \{\{U_i : n \leq i \leq n^*\} : n^* \geq n\}.$$

It is clear that $\{U_i : n \leq i \leq n_1^*\} \subset \{U_i : n \leq i \leq n_2^*\}$ for $n \leq n_1^* \leq n_2^*$, so that the collection $\mathcal{B}_n$ is linearly ordered by inclusion and in particular it is a filter base. We show that the sequence $(\mathcal{B}_n : n \in \mathbb{N})$ satisfies:

- (i) For each n, there is $\mathcal{C}_n \in \mathcal{B}_n$ which is not a k-cover of Y;
- (ii) For each compact set $K \subset Y$ there is $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, then there exists $\mathcal{H}_n \in \mathcal{B}_n$ satisfying $K \subset H$ for every $H \in \mathcal{H}_n$;
- (iii) For each $n \in \mathbb{N}$ there is some countable element in $\mathcal{B}_n$.

Of course, (iii) is satisfied because every element of $\mathcal{B}_n$ is finite. The condition (i) is also true. Indeed, for a given $i \in \mathbb{N}$, we have by the choice of points x_n that no element of $\{U_i : n \leq i \leq n^*\}$ includes the set $\{x_i : n \leq i \leq n^*\}$ which is a compact subset of Y. Finally, let us prove that (ii) holds. Let K be a compact subset of Y. There exists $n_0 \in \mathbb{N}$ such that $K \subset K_{n_0}$. For each $n \geq n_0$ take any element $\mathcal{H}_n := \{U_i : n \leq i \leq \overline{n}\}$ in $\mathcal{B}_n = \{\{U_i : n \leq i \leq n^*\} : n^* \geq n\}$, where $\overline{n}$ is an element of $\mathbb{N}$ with $\overline{n} \geq n$. Then we have $K \subset H$ for each $H \in \mathcal{H}_n$. Indeed, for each $i \in \mathbb{N}$ with $n \leq i \leq \overline{n}$ we have $K \subset K_{n_0} \subset K_n \subset K_i \subset U_i$. □

Theorem 10. *If X is a locally compact space and $(2^X, \mathsf{F})$ has the strong set-Fréchet property, then for each open set $Y \subset X$ and each k-cover $\mathcal{U}$ of Y there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of countable, pairwise disjoint subsets of $\mathcal{U}$ such that:*

- (i) *no $\mathcal{V}_n$ is a k-cover of Y;*
- (ii) *each compact subset K of Y ix contained in an element of $\mathcal{V}_n$ for all but finitely many n.*

Proof. Let Y be an open subset of X and let $\mathcal{U}$ be a k-cover of Y. Then the set $\mathcal{A} := \mathcal{U}^c \subset 2^X$ belongs to Ω_{Y^c} (see the proof of Theorem 7). By (1) choose countable, pairwise disjoint sets $\mathcal{B}_n \subset \mathcal{A}$, $n \in \mathbb{N}$, such that $Y^c \notin Cl_{\mathsf{F}}(\mathcal{B}_n)$ for each $n \in \mathbb{N}$, but each F-neighborhood of Y^c intersects all but finitely many sets $\mathcal{B}_n$, say all $\mathcal{B}_n$ for $n \geq n_0$ for some $n_0 \in \mathbb{N}$. The sets $\mathcal{V}_n := \mathcal{B}_n^c$, $n \in \mathbb{N}$, are countable, pairwise disjoint subsets of $\mathcal{U}$. No $\mathcal{V}_n$ is a k-cover of Y. On the other hand, let K be a compact subset of Y. Then $(K^c)^+$, being a F-neighborhood of Y^c, meets each $\mathcal{B}_n$ for $n \geq n_5$; pick for each such n a $B_n \in \mathcal{B}_n \cap (K^c)^+$. Then for each $n \geq n_0$, $K \subset B_n^c$, hence (ii) holds. □

In a similar way one can prove:

Theorem 11. *If X is a locally compact space and $(2^X, \mathsf{F})$ is a set-Fréchet space, then for each open set $Y \subset X$ and each k-cover $\mathcal{U}$ of Y there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of pairwise disjoint subsets of $\mathcal{U}$ such that:*

- (i) *no $\mathcal{V}_n$ is a k-cover of Y;*
- (ii) *each compact subset of Y is contained in an element of $\mathcal{V}_n$ for all but finitely many n.*

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