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LIL FOR KERNEL ESTIMATOR OF ERROR DISTRIBUTION IN REGRESSION MODEL

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Abstract. This paper considers the problem of estimating the error distribution function in nonparametric regression models. Sufficient conditions are given under which the kernel estimator of the error distribution function based on nonparametric residuals satisfies the law of iterated logarithm.

1. Introduction

In nonparametric regression models, the main focus in statistical literature over the last five decades has been the estimation of the regression function. Relatively little is known about the estimation of the error distribution functions in these models. It is often of interest and of practical importance to know the nature of the error distribution after estimating a regression function. The focus of this paper is to investigate the law of iterated logarithm (LIL) of the kernel estimator of distribution function (d.f.) based on nonparametric residuals.

In parametric regression and autoregressive models several authors have studied these estimators. The weak convergence of the empirical processes based on residuals in parametric regression models is discussed in Koul [5, 6, 8, 9], Loynes [11], Portnoy [14], and Mammen [12], while Boldin [1], Koul [7], and Koul and Osiander [10] discuss this for parametric autoregressive models. The uniform consistency of error density in these models is discussed in Koul [8] in linear regression and autoregression models. Cheng [2] considered the consistency of the histogram error density function and the empirical error distribution estimation in nonparametric regression.

2. Model and estimators

In this section, we introduce the model, the error d.f. estimators. Accordingly, let X and Y be one-dimensional random variables, with X taking values

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in $[0,1]$. In the regression model, we are concerned with here one observes n independent identically distributed (i.i.d.) copies $(X_i, Y_i); 1 \le i \le n$ from (X, Y), such that for some real valued function $m(x)$, $x \in [0,1]$,

$$\epsilon_i = Y_i - m(X_i), i = 1, 2, \ldots, n$$

are i.i.d. random variables with an unknown d.f. G and density g. We further assume that $E(\epsilon|X) = 0, a.s.$, so that $m(x) = E(Y|X = x)$, where ϵ is a copy of ϵ_1. Let $m_n(x)$ denote the well-known Nadaraya [13] kernel regression estimator:

$$m_n(x) := \frac{\sum_{i=1}^n Y_i L((x - X_i)/h_n)}{\sum_{i=1}^n L((x - X_i)/h_n)}, \quad x \in [0,1],$$

where h_n denotes the bandwidth sequence of positive numbers tending to zero, and L is the kernel density function.

Let $\hat{\epsilon}_i := Y_i - m_n(X_i)$, $i = 1, 2, \ldots$ denote the nonparametric residuals and let a_n be another sequence of positive numbers tending to zero. The kernel density and distribution function estimators based on these residuals are given by, respectively

$$\tilde{g}_n(t) = \frac{1}{na_n} \sum_{i=1}^n k(\frac{t - \hat{\epsilon}_i}{a_n}) = \frac{1}{a_n} \int_{-\infty}^{\infty} k(\frac{t-u}{a_n}) d\hat{G}_n(u)$$

and

$$\tilde{G}_n(x) = \int_{-\infty}^x \tilde{g}_n(t)dt = \int_{-\infty}^{\infty} K(\frac{x-t}{a_n}) d\hat{G}_n(t) = \frac{1}{n} \sum_{i=1}^n K(\frac{t - \hat{\epsilon}_i}{a_n}),$$

where $K(x) = \int_{-\infty}^x k(t)dt$, $k(\cdot)$ is kernel density function and

$$\hat{G}_n(t) = \frac{1}{n} \sum_{i=1}^n I(\hat{\epsilon}_i \le t).$$

Denote $d_n(x) := m_n(x) - m(x), x \in [0,1]$. Under some conditions, Härdle et al. [4] have shown that there exists a constant $C > 0$ such that

$$\sup_{x \in [0,1]} |d_n(x)| < C\sqrt{\log n / nh_n}, \text{ all large } n, \ a.s. \tag{2.1}$$

Cheng [2] discussed that under some conditions,

$$\sup_{x \in [0,1]} |d_n(x)| = O_p(\sqrt{\log n / nh_n}) \text{ as } n \to \infty, \tag{2.2}$$

where

$$h_n \to 0 \text{ and } \sqrt{\log n / nh_n} \to 0. \tag{2.3}$$

Based on the assumptions of (2.2), (2.3) and (2.1), (2.3), respectively, Cheng [2] investigated the convergence rates of $\hat{G}_n(x)$ and consistency of $\hat{g}_n(x)$, where

$$\begin{aligned}\hat{g}_n(x) &= \frac{1}{2na_n}\sum_{i=1}^{n} I(x-a_n < \hat{\epsilon}_i \leq x+a_n)) \\ &:= \frac{1}{2a_n}[\hat{G}_n(x+a_n) - \hat{G}_n(x-a_n)],\ x \in R.\end{aligned}$$

Motived by (2.1), we give the following assumptions on $d_n(x)$:
(A.1) $\sup_{x\in[0,1]} |d_n(x)| = O(\beta_n)$ *a.s.* $n \to \infty$, where $0 < \beta_n \to 0$.

Remark 2.1. By (A.1), we know that there exists a constant $c_0 > 0$ such that $\sup_{x\in[0,1]} |d_n(x)| < c_0\beta_n$ *a.s.* for all large n. Define $R_n = \{\sup_{x\in[0,1]} |d_n(x)| \leq c_0\beta_n\}$. Then $P(\cap_{i=n}^{\infty} R_i) \to 1$.

In this paper, based on the assumption (A.1), we discuss the LIL of Chung-Smirnov's type up-limit, Kolmogorov-Smirnov's and Cramer-Von Mises's type down-limit of $\tilde{G}_n(x)$.

The rest of this paper is organized as follows. Section 3 gives main results and the proofs of main results are provided in section 4.

In the following section, all limits are taken as the sample size n tends to ∞, unless specified otherwise. C and c will represent positive constants whose value may change from one place to another.

3. Main results

In this section, let $\lambda > 0$ and $l > 0$, assume that G is Lipschitz continuous of order λ, $(A.1)$ is fulfilled, and that $k(\cdot)$ is a probability density function with $\int_R |t|^\lambda k(t)dt < \infty$.

Theorem 3.1. *Let* $(n/\log\log n)^{1/2}a_n^\lambda \to 0$ *and* $(\log\beta_n^{-\lambda})/\log\log n \to \infty$.
(1) *If* $(n/\log\log n)^{1/2}\beta_n^\lambda \to 0$, *then*

$$\limsup_{n\to\infty}\left(\frac{2n}{\log\log n}\right)^{1/2}\sup_x |\tilde{G}_n(x) - G(x)| = 1 \quad a.s. \tag{3.4}$$

(2) *If g is Lipschitz continuous of order l and* $(n/\log\log n)^{1/2}\beta_n^{l+1} \to 0$, *then* (3.4) *remains true.*

Theorem 3.2. *Let* $(n/\log\log n)^{1/2}a_n^\lambda \to 0$ *and* $(\log\beta_n^{-\lambda})/\log\log n \to \infty$.
(1) *If* $(n/\log\log n)^{1/2}\beta_n^\lambda \to 0$, *then*

$$\limsup_{n\to\infty}\left(\frac{n}{2\log\log n}\right)^{1/2}\left[\int_R (\tilde{G}_n(x) - G(x))^2 dG(x)\right]^{1/2} = \pi^{-1} \quad a.s. \tag{3.5}$$

(2) *If g is Lipschitz continuous of order l and* $(n/\log\log n)^{1/2}\beta_n^{l+1} \to 0$, *then* (3.5) *remains true.*

Theorem 3.3. *Let* $(n\log\log n)^{1/2}a_n^{\lambda}\to 0$ *and* $(\log\beta_n^{-\lambda})/\log\log n\to\infty$.
(1) *If* $(n\log\log n)^{1/2}\beta_n^{\lambda}\to 0$, *then*

$$\liminf_{n\to\infty}(n\log\log n)^{1/2}\sup_x|\tilde{G}_n(x)-G(x)|=8^{-1/2}\pi \quad a.s. \tag{3.6}$$

(2) *If* g *is Lipschitz continuous of order* l *and* $(n\log\log n)^{1/2}\beta_n^{l+1}\to 0$, *then* (3.6) *remains true.*

Theorem 3.4. *Let* $n^{1/2}(\log\log n)^{3/2}a_n^{\lambda}\to 0$ *and* $(\log\beta_n^{-\lambda})/\log\log n\to\infty$.
(1) *If* $(n\log\log n)^{1/2}\beta_n^{\lambda}\to 0$, *then*

$$\liminf_{n\to\infty}(n\log\log n)^{1/2}[\int_R(\tilde{G}_n(x)-G(x))^2dG(x)]^{1/2}=8^{-1/2} \quad a.s. \tag{3.7}$$

(2) *If* g *is Lipschitz continuous of order* l *and* $(n\log\log n)^{1/2}\beta_n^{l+1}\to 0$, *then* (3.7) *remains true.*

4. Proof of main results

In this section, set $G_n(x):=\frac{1}{n}\sum_{i=1}^n K(\frac{x-\epsilon_i}{a_n})$ and $F_n(x)=\frac{1}{n}\sum_{i=1}^n I(\epsilon_i\le x)$. $\alpha_n(t)=\sqrt{n}(E_n(t)-t)$ $t\in[0,1]$, $E_n(t)=\frac{1}{n}\sum_{i=1}^n I(U_i\le t)$, $U_i=G(\epsilon_i), i=1,2,\ldots$.

Proof of Theorem 3.1. We write

$$\tilde{G}_n(x)-G(x)=[\tilde{G}_n(x)-G_n(x)]+[G_n(x)-G(x)]. \tag{4.8}$$

Thus, it suffices to show that

$$\limsup_{n\to\infty}(\frac{2n}{\log\log n})^{1/2}\sup_x|\tilde{G}_n(x)-G_n(x)|=0 \quad a.s. \tag{4.9}$$

and

$$\limsup_{n\to\infty}(\frac{2n}{\log\log n})^{1/2}\sup_x|G_n(x)-G(x)|=1 \quad a.s. \tag{4.10}$$

We first prove (4.9). By $\int_R k(u)du=1$, we have

$$\begin{aligned}
(4.11)\quad & \sup_x|\tilde{G}_n(x)-G_n(x)| \\
= & \sup_x|\int_R k(u)[\hat{G}_n(x-ua_n)-F_n(x-ua_n)]du| \\
\le & \sup_x|\hat{G}_n(x)-F_n(x)| \\
\le & \sup_x\frac{1}{n}|\sum_{i=1}^n[I(\epsilon_i\le x+d_n(X_i))-G(x+d_n(X_i))-I(\epsilon_i\le x)+G(x)]| \\
& +\sup_x\frac{1}{n}|\sum_{i=1}^n[G(x+d_n(X_i))-G(x)]| \\
:= & I_{n1}+I_{n2}.
\end{aligned}$$

By Remark 2.1, we need only to consider (4.9) on R_n. Using the monotonicity of G, we see that on R_n,

$$I_{n2} \leq \sup_x [G(x + c_0\beta_n) - G(x - c_0\beta_n)].$$

Since G is Lipschitz continuous of order λ, $I_{n2} \leq O(\beta_n^\lambda)$.

If g is Lipschitz continuous of order l, then

$$\begin{aligned} I_{n2} &\leq \sup_x \int_{-c_0\beta_n}^{c_0\beta_n} g(x+u)du \\ &\leq \sup_x [\int_0^{c_0\beta_n} |g(x+u) - g(x)|du + \int_0^{c_0\beta_n} |g(x-u) - g(x)|du] \\ &\leq O(1) \int_0^{c_0\beta_n} u^l du = O(\beta_n^{l+1}). \end{aligned}$$

Note that, on R_n, for any $x \in R$, by the monotonicity of the function I and G, we have

$$\begin{aligned} & \frac{1}{n}\sum_{i=1}^n [I(\epsilon_i \leq x + d_n(X_i)) - G(x + d_n(X_i)) - I(\epsilon_i \leq x) + G(x)] \\ \leq\ & \frac{1}{n}\sum_{i=1}^n [I(\epsilon_i \leq x + c_0\beta_n) - G(x + c_0\beta_n) - I(\epsilon_i \leq x) + G(x)] \\ & \qquad + [G(x + c_0\beta_n) - G(x - c_0\beta_n)] \\ =\ & n^{-1/2}[\alpha_n(G(x + c_0\beta_n)) - \alpha_n(G(x))] + [G(x + c_0\beta_n) - G(x - c_0\beta_n)] \end{aligned}$$

and similarly, we also obtain that

$$\begin{aligned} & \frac{1}{n}\sum_{i=1}^n [I(\epsilon_i \leq x + d_n(X_i)) - G(x + d_n(X_i)) - I(\epsilon_i \leq x) + G(x)] \\ \geq\ & n^{-1/2}[\alpha_n(G(x - c_0\beta_n)) - \alpha_n(G(x))] - [G(x + c_0\beta_n) - G(x - c_0\beta_n)]. \end{aligned}$$

Thus,

$$\begin{aligned} I_{n1} \leq\ & n^{-1/2} \max\{\sup_x |\alpha_n(G(x + c_0\beta_n)) - \alpha_n(G(x))|, \\ & \qquad \sup_x |\alpha_n(G(x - c_0\beta_n)) - \alpha_n(G(x))|\} \\ & + \sup_x [G(x + c_0\beta_n) - G(x - c_0\beta_n)]. \end{aligned} \tag{4.12}$$

Now, we verify that, for $H \in R$,

$$\begin{aligned} & \sup_x |\alpha_n(G(x + H\beta_n)) - \alpha_n(G(x))| \\ =\ & O(n^{-1/2}\log^2 n) + O((\beta_n^\lambda \log \beta_n^{-\lambda})^{1/2}). \end{aligned} \tag{4.13}$$

In fact, since $\log \beta_n^{-\lambda}/\log\log n \to \infty$, by Theorem 4.4.3 and Theorem 1.15.2 of Csörgö and Révész [3], there exists a Kiefer process $\{K(y,t), 0 \leq y \leq 1, 0 \leq$

$t < \infty\}$ such that

$$\begin{aligned}
&\sup_x |\alpha_n(G(x+H\beta_n)) - \alpha_n(G(x))| \le \sup_{|t-s|\le C\beta_n^\lambda} |\alpha_n(t) - \alpha_n(s)| \\
\le\ & \sup_{|t-s|\le C\beta_n^\lambda} |\alpha_n(t) - n^{-1/2}K(t,n)| + \sup_{|t-s|\le C\beta_n^\lambda} |\alpha_n(s) - n^{-1/2}K(s,n)| \\
&+ n^{-1/2} \sup_{|t-s|\le C\beta_n^\lambda} |K(s,n) - K(t,n)| \\
=\ & O(n^{-1/2}\log^2 n) + O((\beta_n^\lambda \log \beta_n^{-\lambda})^{1/2}),
\end{aligned}$$

here applying $|G(x+H\beta_n) - G(x)| \le C\beta_n^\lambda$. Therefore, (4.13) is proved. Hence, by (4.11)-(4.13), we prove (4.9) from the proof for I_{n2} and assumptions.

Next, we prove (4.10). We observe that

$$\begin{aligned}
G_n(x) - G(x) &= \int_R K(\frac{x-t}{a_n}) dF_n(t) - G(x) \\
&= \int_R [F_n(x - a_n y) - F_n(x)]k(y)dy + [F_n(x) - G(x)] \\
&:= L_1(x) + L_2(x). \qquad (4.14)
\end{aligned}$$

By Theorem 5.1.1 of Csörgö and Révész [3], we have

$$\limsup_{n\to\infty} (\frac{2n}{\log\log n})^{1/2} \sup_x |L_2(x)| = 1 \quad a.s. \qquad (4.15)$$

Hence, to prove (4.10), it suffices to show that

$$\limsup_{n\to\infty} (\frac{2n}{\log\log n})^{1/2} \sup_x |L_1(x)| = 0 \quad a.s. \qquad (4.16)$$

Note that

$$\begin{aligned}
\sup_x |L_1(x)| &\le n^{-1/2} \sup_x |\int_R [\alpha_n(G(x-a_n y)) - \alpha_n(G(x))]k(y)dy| \\
&\quad + \sup_x |\int_R [G(x-a_n y) - G(x)]k(y)dy| \\
&\le n^{-1/2} \sup_x |\int_{M_n}^{\infty} [\alpha_n(G(x-a_n y)) - \alpha_n(G(x))]k(y)dy| \\
&\quad + n^{-1/2} \sup_x |\int_{-M_n}^{M_n} [\alpha_n(G(x-a_n y)) - \alpha_n(G(x))]k(y)dy| \\
&\quad + n^{-1/2} \sup_x |\int_{-\infty}^{-M_n} [\alpha_n(G(x-a_n y)) - \alpha_n(G(x))]k(y)dy| \\
&\quad + O(a_n^\lambda) \int_R |y|^\lambda k(y)dy \\
&:= n^{-1/2}(L_{11} + L_{12} + L_{13}) + O(a_n^\lambda),
\end{aligned}$$

where $M_n = \log\log n$. By (4.15) and $\int_R |y|^\lambda k(y)dy < \infty$, we obtain that

$$\begin{aligned}
&\limsup_{n\to\infty}(\log\log n)^{-1/2}L_{11} \\
\le\ & 2\limsup_{n\to\infty}(\log\log n)^{-1/2}\sup_x|\alpha_n(G(x))|\int_{M_n}^{\infty}k(y)dy \\
\le\ & 2\limsup_{n\to\infty}(n/\log\log n)^{1/2}M_n^{-\lambda}\sup_x|F_n(x)-G(x)|\int_R|y|^\lambda k(y)dy = 0 \quad a.s.
\end{aligned}$$

Similarly, $\limsup_{n\to\infty}(\log\log n)^{-1/2}L_{13} = 0,\ a.s.$

Note that $(n/\log\log n)^{1/2}a_n^\lambda \to 0$, so $a_n = (n/\log\log n)^{-1/2\lambda}C_n$, where $C_n \to 0$, and hence $\log(M_na_n)^{-1}/\log\log n \to \infty$. Using (4.13), we have

$$\begin{aligned}
L_{12} &\le \sup_x\sup_{|y|\le M_n}|\alpha_n(G(x-a_ny)) - \alpha_n(G(x))| \\
&= O(n^{-1/2}\log^2 n) + O((M_na_n\log(M_na_n)^{-1})^{1/2}) \quad a.s.
\end{aligned}$$

Since $M_na_n\log(M_na_n)^{-1} \to 0$, $\limsup_{n\to\infty}(\log\log n)^{-1/2}L_{12} = 0,\ a.s.$ So, (4.16) is verified. Therefore, Theorem 3.1 is proved. □

Proof of Theorem 3.2. We observe that

$$\begin{aligned}
&[\int_R(\widetilde{G}_n(x) - G(x))^2dG(x)]^{1/2} \\
\le\ & [\int_R(\widetilde{G}_n(x) - G_n(x))^2dG(x)]^{1/2} + [\int_R(G_n(x)-G(x))^2dG(x)]^{1/2} \\
\le\ & \sup_x|\widetilde{G}_n(x) - G_n(x)| + [\int_R(G_n(x)-G(x))^2dG(x)]^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
&[\int_R(\widetilde{G}_n(x) - G(x))^2dG(x)]^{1/2} \\
\ge\ & [\int_R(G_n(x)-G(x))^2dG(x)]^{1/2} - \sup_x|\widetilde{G}_n(x) - G_n(x)|. \qquad (4.17)
\end{aligned}$$

Therefore, it suffices to show that

$$\limsup_{n\to\infty}(n/2\log\log n)^{1/2}[\int_R(G_n(x)-G(x))^2dG(x)]^{1/2} = \pi^{-1} \quad a.s. \qquad (4.18)$$

and

$$\limsup_{n\to\infty}(n/2\log\log n)^{1/2}\sup_x|\widetilde{G}_n(x) - G_n(x)| = 0 \quad a.s. \qquad (4.19)$$

Clearly, (4.19) holds from (4.9).

Next, we prove (4.18). We obtain from (4.14) that

$$\begin{aligned}
&\int_R(G_n(x)-G(x))^2dG(x) \\
=\ & \int_R L_1^2(x)dG(x) + 2\int_R L_1(x)L_2(x)dG(x) + \int_R L_2^2(x)dG(x).
\end{aligned}$$

Hence, to prove (4.18), it suffices to show that

$$\limsup_{n\to\infty}(n/2\log\log n)^{1/2}[\int_R L_2^2(x)dG(x)]^{1/2} = \pi^{-1} \quad a.s. \tag{4.20}$$

$$\limsup_{n\to\infty}(n/\log\log n)^{1/2}[\int_R L_1^2(x)dG(x)]^{1/2} = 0 \quad a.s. \tag{4.21}$$

$$\limsup_{n\to\infty}(n/\log\log n)|\int_R L_1(x)L_2(x)dG(x)| = 0 \quad a.s. \tag{4.22}$$

By using the Theorem 5.1.3 of Csörgö and Révész [3], (4.20) is obtained. (4.16) implies (4.21). As to (4.22), we obtain from (4.20) and (4.21) that

$$\begin{aligned} & \limsup_{n\to\infty}(\frac{n}{\log\log n})|\int_R L_1(x)L_2(x)dG(x)| \\ \leq\ & \limsup_{n\to\infty}(\frac{n}{\log\log n})^{1/2}[\int_R L_2^2(x)dG(x)]^{1/2} \\ & \cdot \limsup_{n\to\infty}(\frac{n}{\log\log n})^{1/2}[\int_R L_1^2(x)dG(x)]^{1/2} \\ =\ & 0 \quad a.s. \end{aligned}$$

□

Proof of Theorem 3.3. By (4.8), it suffices to show that

$$\limsup_{n\to\infty}(n\log\log n)^{1/2}\sup_x|\tilde{G}_n(x) - G_n(x)| = 0 \quad a.s. \tag{4.23}$$

and

$$\liminf_{n\to\infty}(n\log\log n)^{1/2}\sup_x|G_n(x) - G(x)| = 8^{-1/2}\pi \quad a.s. \tag{4.24}$$

By assumptions, similar to the arguments in (4.9), we can verify (4.23).

Note that (see (4.14)) $G_n(x) - G(x) = L_1(x) + L_2(x)$. By Theorem 5.1.7 of Csörgö and Révész [3], we have

$$\liminf_{n\to\infty}(n\log\log n)^{1/2}\sup_x|L_2(x)| = 8^{-1/2}\pi \quad a.s. \tag{4.25}$$

Similar to the arguments for (4.16) and taking $M_n = (\log\log n)^{2/\lambda}$, we can verify that

$$\limsup_{n\to\infty}(n\log\log n)^{1/2}\sup_x|L_1(x)| = 0 \quad a.s. \tag{4.26}$$

Hence, (4.24) is proved from (4.25) and (4.26). □

Proof of Theorem 3.4. Similar to the proof in Theorem 3.2, it suffices to show that

$$\liminf_{n\to\infty}(n\log\log n)^{1/2}[\int_R (G_n(x) - G(x))^2 dG(x)]^{1/2} = 8^{-1/2} \quad a.s. \tag{4.27}$$

and

$$\limsup_{n\to\infty}(n\log\log n)^{1/2}\sup_x|\widetilde{G}_n(x)-G_n(x)|=0 \quad a.s. \tag{4.28}$$

As the proof for (4.9), we can verify (4.28).

To prove (4.27), similar to the proof for (4.18), we need only to prove the following three equations:

$$\liminf_{n\to\infty}(n\log\log n)^{1/2}[\int_R L_2^2(x)dG(x)]^{1/2}=8^{-1/2} \quad a.s. \tag{4.29}$$

$$\limsup_{n\to\infty}n^{1/2}(\log\log n)^{3/2}[\int_R L_1^2(x)dG(x)]^{1/2}=0 \quad a.s. \tag{4.30}$$

$$\limsup_{n\to\infty}(n\log\log n)|\int_R L_1(x)L_2(x)dG(x)|=0 \quad a.s. \tag{4.31}$$

By using the Theorem 5.1.7 of Csörgö and Révész [3], (4.29) is obtained.

Taking $M_n=(\log\log n)^{3/\lambda}$, similar to the arguments for (4.16), we can verify that (4.30).

As to (4.31), applying (4.30) and Theorem 5.1.1 of Csörgö and Révész [3] (also see (4.15)), we have

$$\begin{aligned}
&\limsup_{n\to\infty}(n\log\log n)|\int_R L_1(x)L_2(x)dG(x)| \\
\le\ &\limsup_{n\to\infty}n^{1/2}(\log\log n)^{3/2}[\int_R L_1^2(x)dG(x)]^{1/2} \\
&\cdot\limsup_{n\to\infty}(n/\log\log n)^{1/2}\sup_x|L_2(x)| \\
=\ &0 \quad a.s.
\end{aligned}$$

□

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