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A NEW SYSTEM OF GENERALIZED NONLINEAR MIXED QUASIVARIATIONAL INEQUALITIES AND ITERATIVE ALGORITHMS IN HILBERT SPACES

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Abstract. We introduce a new system of generalized nonlinear mixed quasivariational inequalities and prove the existence and uniqueness of the solution for the system in Hilbert spaces. The main result of this paper is an extension and improvement of the well-known corresponding results in Kim-Kim [16], Noor [21]-[23] and Verma [24]-[26].

1. Introduction and preliminaries

In recent years, many classical variational inequalities and complementarity problems have been extended and generalized to study a large variety of problems arising in mechanics, physics, optimization and control theory, nonlinear programming problems, economics, transportation, equilibrium problems and engineering sciences, etc. (see, [1], [2], [4]-[26])

In 2001, Verma [25] introduced and studied a new system of nonlinear variational inequalities based on a new system of iterative algorithms. And also, Verma [26] investigated the approximation-solvability of a new system of nonlinear quasivariational inequalities in Hilbert spaces.

In 2004, Kim-Kim [16] introduced and studied a new system of generalized nonlinear mixed variational inequalities bases on a new system of iterative algorithms in Hilbert spaces.

In this paper, we introduce a new system of generalized nonlinear mixed quasivariational inequalities and prove the existence and uniqueness of solution for the systems in Hilbert spaces. We also construct some new iterative algorithms for the problems and give the convergence analysis of the iterative sequences generated by the algorithms.

Throughout this paper, let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let $A, B, S, T, g_1, g_2 : H \to H$ be single-valued mappings, $\phi_1, \phi_2 : H \to \mathbb{R} \cup \{+\infty\}$ be proper convex lower semicontinuous

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functions and K be a closed convex subset of H. We consider the following problem:

Find $x^*, y^* \in H$ such that $g_1(x^*),\ g_2(y^*) \in K$ and

$$(1.1)\quad \begin{cases} \langle \rho(A(y^*) + S(y^*)) + g_1(x^*) - g_2(y^*), x - g_1(x^*)\rangle \\ \geq \rho\phi_1(g_1(x^*)) - \rho\phi_1(x), \\ \langle \gamma(B(x^*) + T(x^*)) + g_2(y^*) - g_1(x^*), x - g_2(y^*)\rangle \\ \geq \gamma\phi_2(g_2(y^*)) - \gamma\phi_2(x) \end{cases}$$

for all $x \in H$, which is called the *new system of generalized nonlinear mixed quasivariational inequalities*, where $\rho > 0$ and $\gamma > 0$ are constants.

Special Cases of the problem (1.1):

(I) If $A = B = 0$, then the problem (1.1) reduces the following:
Find $x^*, y^* \in H$ such that $g_1(x^*), g_2(y^*) \in K$ and

$$(1.2)\quad \begin{cases} \langle \rho S(y^*) + g_1(x^*) - g_2(y^*), x - g_1(x^*)\rangle \geq \rho\phi_1(g_1(x^*)) - \rho\phi_1(x), \\ \langle \gamma T(x^*) + g_2(y^*) - g_1(x^*), x - g_2(y^*)\rangle \geq \gamma\phi_2(g_2(y^*)) - \gamma\phi_2(x) \end{cases}$$

for all $x \in H$, which is called the *system of nonlinear mixed quasivariational inequalities.*

(II) If $\phi_1 = \phi_2 = \delta_K$(the indicator function of a nonempty closed convex subset K), then the problem (1.1) reduces the following:
Find $x^*, y^* \in K$ such that $g_1(x^*),\ g_2(y^*) \in K$ and

$$(1.3)\quad \begin{cases} \langle \rho(A(y^*) + S(y^*)) + g_1(x^*) - g_2(y^*), x - g_1(x^*)\rangle \geq 0, \\ \langle \gamma(B(x^*) + T(x^*)) + g_2(y^*) - g_1(x^*), x - g_2(y^*)\rangle \geq 0 \end{cases}$$

for all $x \in K$, which is called the *system of generalized nonlinear quasivariational inequalities.*

(III) If $\phi_1 = \phi_2 = \delta_K$(the indicator function of a nonempty closed convex subset K) and $A = B = 0$, then the problem (1.1) reduces the following:
Find $x^*, y^* \in K$ such that $g_1(x^*), g_2(y^*) \in K$ and

$$(1.4)\quad \begin{cases} \langle \rho S(y^*) + g_1(x^*) - g_2(y^*), x - g_1(x^*)\rangle \geq 0, \\ \langle \gamma T(x^*) + g_2(y^*) - g_1(x^*), x - g_2(y^*)\rangle \geq 0 \end{cases}$$

for all $x \in K$, which is called the *system of nonlinear quasivariational inequalities.*

Now, we give some definitions and lemmas for the main theorems.

Definition 1.1. Let $T, g : H \to H$ be mappings.

(1) $T : H \to H$ is said to be *k-strongly monotone* if there exists a constant $k > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq k\|x - y\|^2$$

for all $x, y \in H$. This implies that

$$\|T(x) - T(y)\| \geq k\|x - y\|,$$

that is, T is k-expanding and when $k = 1$, it is expanding.

(2) $T : H \to H$ is called *g-k-strongly monotone* if there exists a constant $k > 0$ such that

$$\langle T(x) - T(y), g(x) - g(y) \rangle \geq k\|g(x) - g(y)\|^2$$

for all $x, y \in H$. This implies that

$$\|T(x) - T(y)\| \geq k\|g(x) - g(y)\|,$$

that is, T is g-k-expanding and when $k = 1$, it is g-expanding.

(3) $T : H \to H$ is said to be *s-Lipschitz continuous* if there exists a constant $s > 0$ such that

$$\|T(x) - T(y)\| \leq s\|x - y\|$$

for all $x, y \in H$.

(4) $T : H \to H$ is called *g-s-Lipschitz continuous* if there exists a constant $s > 0$ such that

$$\|T(x) - T(y)\| \leq s\|g(x) - g(y)\|$$

for all $x, y \in H$.

Lemma 1.1 ([16]). *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of nonnegative numbers satisfying the following condition: there exists n_0 such that*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n \tag{1.5}$$

for all $n \geq n_0$, where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$, $\sum_{n=0}^{\infty} c_n < \infty$. Then

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 1.2 ([3]). *For any given $u \in H$, a point $z \in H$ satisfies*

$$\langle u - z, v - u \rangle \geq \rho\phi(u) - \rho\phi(v)$$

for all $v \in H$ if and only if

$$u = J_\phi^\rho(z),$$

where $J_\phi^\rho = (I + \rho\partial\phi)^{-1}$ and $\partial\phi$ denotes the subdifferential of a proper convex lower semicontinuous function $\phi : H \to \mathbb{R} \cup \{+\infty\}$.

Remark 1.1. It is well known that J_ϕ^ρ is nonexpansive (see [3]).

Lemma 1.3. *For any given $x^*, y^* \in H$, (x^*, y^*) is a solution of the problem (1.1) if and only if*

$$\begin{cases} g_1(x^*) = J_{\phi_1}^\rho(g_2(y^*) - \rho(A(y^*) + S(y^*))), \\ g_2(y^*) = J_{\phi_2}^\gamma(g_1(x^*) - \gamma(B(x^*) + T(x^*))). \end{cases}$$

Proof. It is easy to prove that Lemma 1.3 holds from Lemma 1.2. □

2. Existence and uniqueness

Now, we shall show the existence and uniqueness of solution for the problems (1.1).

Theorem 2.1. *Let $S : H \to H$ be a g_2-k_2-strongly monotone and g_2-s_2-Lipschitz continuous mapping, $T : H \to H$ be a g_1-k_1-strongly monotone and g_1-s_1-Lipschitz continuous mapping, $A : H \to H$ be a g_2-l_2-Lipschitz continuous mapping and $B : H \to H$ be a g_1-l_1-Lipschitz continuous mapping. Suppose that $g_1, g_2 : H \to H$ are invertible. If*

$$\begin{cases} 0 < \rho < \dfrac{2(k_2 - l_2)}{s_2^2 - l_2^2}, & l_2 < k_2, \\ 0 < \gamma < \dfrac{2(k_1 - l_1)}{s_1^2 - l_1^2}, & l_1 < k_1, \end{cases} \tag{2.1}$$

then the problem (1.1) *has a unique solution* (x^*, y^*).

Proof. First, in order to prove the existence of the solution. Define a mapping $F : H \to H$ as follows :

$$\begin{aligned} F(x) = J_{\phi_1}^{\rho}[&J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x)))) \\ &- \rho(A+S)(g_2^{-1}(J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x))))))] \end{aligned} \tag{2.2}$$

for each $x \in H$. Since $J_{\phi_1}^{\rho}$ is nonexpansive, for any x, $y \in H$, we have

$$\begin{aligned} &\|F(x) - F(y)\| \\ &= \|J_{\phi_1}^{\rho}[J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x)))) \\ &\quad - \rho(A+S)(g_2^{-1}(J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x))))))] \\ &\quad - J_{\phi_1}^{\rho}[J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y)))) \\ &\quad - \rho(A+S)(g_2^{-1}(J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y))))))]\| \\ &\le \|J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x)))) \\ &\quad - J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y)))) \\ &\quad - \rho\{S(g_2^{-1}(J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x)))))) \\ &\quad - S(g_2^{-1}(J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y))))))\}\| \\ &\quad + \rho\|A(g_2^{-1}(J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x)))))) \\ &\quad - A(g_2^{-1}(J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y))))))\|. \end{aligned} \tag{2.3}$$

It follows from the conditions of S, T, A, B, g_1, g_2, that

$$\begin{aligned} &\|J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x)))) \\ &\quad - J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y)))) \end{aligned}$$

$$
\begin{aligned}
&- \rho\{S(g_2^{-1}(J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x))))))\\
&\quad - S(g_2^{-1}(J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y))))))\}\|^2\\
= {} & \|J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x))))\\
&\quad - J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y))))\|^2\\
&- 2\rho\langle J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x))))\\
&\qquad - J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y)))),\\
&\qquad S(g_2^{-1}(J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x))))))\\
&\qquad - S(g_2^{-1}(J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y))))))\rangle\\
&+ \rho^2\|S(g_2^{-1}(J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x))))))\\
&\qquad - S(g_2^{-1}(J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y))))))\|^2\\
\le {} & \|J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x))))\\
&\quad - J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y))))\|^2\\
&- 2\rho k_2\|J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x))))\\
&\qquad - J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y))))\|^2\\
&+ \rho^2 s_2^2\|J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x))))\\
&\qquad - J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y))))\|^2\\
= {} & (1 - 2\rho k_2 + \rho^2 s_2^2)\|J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x))))\\
&\quad - J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y))))\|^2\\
\le {} & (1 - 2\rho k_2 + \rho^2 s_2^2)\|x - y - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x))\\
&\quad - (B(g_1^{-1}(y)) + T(g_1^{-1}(y))))\|^2\\
\le {} & (1 - 2\rho k_2 + \rho^2 s_2^2)\\
&\quad \cdot [(\|x - y\|^2 - 2\gamma\langle x - y, T(g_1^{-1}(x)) - T(g_1^{-1}(y))\rangle\\
&\quad + \gamma^2\|T(g_1^{-1}(x)) - T(g_1^{-1}(y))\|^2)^{\frac{1}{2}} + \gamma l_1\|x - y\|]^2\\
\le {} & (1 - 2\rho k_2 + \rho^2 s_2^2)\Big(\sqrt{1 - 2\gamma k_1 + \gamma^2 s_1^2} + \gamma l_1\Big)^2\|x - y\|^2
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
&\rho\|A(g_2^{-1}(J_{\phi_2}^{\gamma}(x - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x))))))\\
&\quad - A(g_2^{-1}(J_{\phi_2}^{\gamma}(y - \gamma(B(g_1^{-1}(y)) + T(g_1^{-1}(y))))))\|\\
\le {} & \rho l_2\|x - y - \gamma(B(g_1^{-1}(x)) + T(g_1^{-1}(x))\\
&\quad - (B(g_1^{-1}(y)) + T(g_1^{-1}(y))))\|\\
\le {} & \rho l_2\{\|x - y - \gamma(T(g_1^{-1}(x)) - T(g_1^{-1}(y)))\| + \gamma l_1\|x - y\|\}
\end{aligned}
\tag{2.5}
$$

$$\leq \rho l_2\Big(\sqrt{1-2\gamma k_1+\gamma^2 s_1^2}+\gamma l_1\Big)\|x-y\|.$$

From (2.3)-(2.5), we have

$$\|F(x)-F(y)\| \leq \theta_1\theta_2\|x-y\| \tag{2.6}$$

for all $x, y \in H$, where

$$\theta_1 = \sqrt{1-2\rho k_2+\rho^2 s_2^2}+\rho l_2, \quad \theta_2 = \sqrt{1-2\gamma k_1+\gamma^2 s_1^2}+\gamma l_1. \tag{2.7}$$

It follows from (2.1) that $\theta_1 < 1$ and $\theta_2 < 1$. Thus (2.6) implies that F is a contractive mapping and so there exists a point $g_1(x^*) \in H$ such that $g_1(x^*) = F(g_1(x^*))$. Therefore, from the definition of F, we have

$$g_1(x^*) = J^{\rho}_{\phi_1}(g_2(y^*) - \rho(A+S)(y^*))$$

and

$$g_2(y^*) = J^{\gamma}_{\phi_2}(g_1(x^*) - \gamma(B+T)(x^*)).$$

By Lemma 1.3, we know that (x^*, y^*) is a solution of the problem (1.1).

Next, we show the uniqueness of the solution. Let (u^*, v^*) be an another solution of the problem (1.1). Then we have

$$g_1(u^*) = J^{\rho}_{\phi_1}(g_2(v^*) - \rho(A+S)(v^*))$$

and

$$g_2(v^*) = J^{\gamma}_{\phi_2}(g_1(u^*) - \gamma(B+T)(u^*)).$$

As in the proof of (2.6), we have

$$\|g_1(x^*) - g_1(u^*)\|^2 \leq \theta_1\theta_2\|g_1(x^*) - g_1(u^*)\|^2.$$

Since $\theta_1 < 1$ and $\theta_2 < 1$, it follows that $g_1(x^*) = g_1(u^*)$ and so $g_2(y^*) = g_2(v^*)$. This completes the proof. □

If $A = B = 0$ in Theorem 2.1, then we have the following theorem as a special case:

Theorem 2.2. *Let S and T be the same as in Theorem 2.1. If $0 < \rho < \frac{2k_2}{s_2^2}$ and $0 < \gamma < \frac{2k_1}{s_1^2}$, then the problem (1.2) has a unique solution (x^*, y^*).*

If $\phi_1 = \phi_2 = \delta_K$ in Theorem 2.1, then we have the following theorem as a special case:

Theorem 2.3. *Let K be a nonempty closed convex subset of a Hilbert space H. Suppose that A, B, S and T are the same in Theorem 2.1. If the condition (2.1) holds, then the problem (1.3) has a unique solution (x^*, y^*).*

If $\phi_1 = \phi_2 = \delta_K$ and $A = B = 0$ in Theorem 2.1, then we have the following theorem as a special case:

Theorem 2.4. *Let K be a nonempty closed convex subset of a Hilbert space H. Suppose that S and T are the same in Theorem 2.1. If $0 < \rho < \frac{2k_2}{s_2^2}$ and $0 < \gamma < \frac{2k_1}{s_1^2}$, then the problem (1.4) has a unique solution (x^*, y^*).*

Remark 2.1. (1) If $g_1 = g_2 = I$ (the identity mapping) and $\phi_1 = \phi_2 = \phi$ (a proper convex lower semicontinuous function) in (1.1), then the problem (1.1) reduces to find $x^*, y^* \in H$ such that

$$\begin{cases} \langle \rho(A(y^*) + S(y^*)) + x^* - y^*, x - x^* \rangle \geq \rho\phi(x^*) - \rho\phi(x), \\ \langle \gamma(B(x^*) + T(x^*)) + y^* - x^*, x - y^* \rangle \geq \gamma\phi(y^*) - \gamma\phi(x) \end{cases}$$

for all $x \in H$, which is defined by Kim-Kim [16] and is called *a new system of generalized nonlinear mixed variational inequalities*. Hence Theorem 2.1 is the extension of the result of Kim-Kim [16].

(2) Theorems 2.2, 2.3 and 2.4 are extensions and improvements of the corresponding results in Noor [21]-[23] and Verma [24]-[26].

3. Algorithms and convergence

In this section, we construct some new iterative algorithms for the problems (1.1)-(1.4). We also give the convergence analysis of the iterative sequences generated by the algorithms.

Now we give the algorithm for solving the problem (1.1) as follows:

Algorithm 3.1. For given $x_0 \in H$, define the iterative sequences $\{g_1(x_n)\}$ and $\{g_2(y_n)\}$ with mixed errors as follows:

$$(3.1) \qquad \begin{cases} g_1(x_{n+1}) = (1 - \alpha_n)g_1(x_n) \\ \qquad\qquad + \alpha_n J^{\rho}_{\phi_1} g(g_2(y_n) - \rho(A(y_n) + S(y_n))) \\ \qquad\qquad + \alpha_n u_n + w_n, \\ g_2(y_n) = J^{\gamma}_{\phi_2}(g_1(x_n) - \gamma(B(x_n) + T(x_n))) + v_n \end{cases}$$

for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\{u_n\}$, $\{w_n\}$, $\{v_n\}$ are sequences in H and $g_1, g_2 : H \to H$ are mappings.

If $A = B = 0$, then Algorithm 3.1 reduces to the following algorithm for solving the problem (1.2).

Algorithm 3.2. For given $x_0 \in H$, define the iterative sequences $\{g_1(x_n)\}$ and $\{g_2(y_n)\}$ with mixed errors as follows:

$$\begin{cases} g_1(x_{n+1}) = (1 - \alpha_n)g_1(x_n) + \alpha_n J^{\rho}_{\phi_1}(g_2(y_n) - \rho S(y_n)) \\ \qquad\qquad + \alpha_n u_n + w_n, \\ g_2(y_n) = J^{\gamma}_{\phi_2}(g_1(x_n) - \gamma T(x_n)) + v_n \end{cases}$$

for all $n \geq 0$, where $\{\alpha_n\}$, $\{u_n\}$, $\{w_n\}$, $\{v_n\}$, g_1 and g_2 are the same as in Algorithm 3.1.

If $\phi_1 = \phi_2 = \delta_K$ in Algorithms 3.1 and 3.2, then the following algorithms for solving the problems (1.3) and (1.4), respectively.

Algorithm 3.3. For given $x_0 \in H$, define the iterative sequences $\{g_1(x_n)\}$ and $\{g_2(y_n)\}$ with mixed errors as follows:

$$\begin{cases} g_1(x_{n+1}) = (1-\alpha_n)g_1(x_n) + \alpha_n P_K(g_2(y_n) - \rho(A(y_n) + S(y_n))) \\ \qquad\qquad + \alpha_n u_n + w_n, \\ g_2(y_n) = P_K(g_1(x_n) - \gamma(B(x_n) + T(x_n))) + v_n \end{cases}$$

for all $n \geq 0$, where $\{\alpha_n\}$, $\{u_n\}$, $\{w_n\}$, $\{v_n\}$, g_1 and g_2 are the same as in Algorithm 3.1.

Algorithm 3.4. For given $x_0 \in H$, define the iterative sequences $\{g_1(x_n)\}$ and $\{g_2(y_n)\}$ with mixed errors as follows:

$$\begin{cases} g_1(x_{n+1}) = (1-\alpha_n)g_1(x_n) + \alpha_n P_K(g_2(y_n) - \rho S(y_n)) \\ \qquad\qquad + \alpha_n u_n + w_n, \\ g_2(y_n) = P_K(g_1(x_n) - \gamma T(x_n)) + v_n \end{cases}$$

for all $n \geq 0$, where $\{\alpha_n\}$, $\{u_n\}$, $\{w_n\}$, $\{v_n\}$, g_1 and g_2 are the same as in Algorithm 3.1.

Now, by using Algorithm 3.1, we prove the following theorem.

Theorem 3.1. *Let A, B, S, T, g_1, g_2 be the same as in Theorem* 2.1 *and $\{g_1(x_n)\}$ be the iterative sequence generated by Algorithm* 3.1 *satisfying the conditions*

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{n=0}^{\infty} \|w_n\| < \infty, \quad \lim_{n\to\infty} \|u_n\| = \lim_{n\to\infty} \|v_n\| = 0. \tag{3.2}$$

If the condition (2.1) *holds, then (x_n, y_n) converges to the unique solution (x^*, y^*) of the problem* (1.1).

Proof. By Theorem 2.1, we know that the problem (1.1) has a unique solution (x^*, y^*). It follows from Lemma 1.3 that

$$cg_1(x^*) = J^{\rho}_{\phi_1}(g_2(y^*) - \rho(A(y^*) + S(y^*)))$$

and

$$g_2(y^*) = J^{\gamma}_{\phi_2}(g_1(x^*) - \gamma(B(x^*) + T(x^*))). \tag{3.4}$$

From (3.1) and (3.3), we have

$$\begin{aligned} &\|g_1(x_{n+1}) - g_1(x^*)\| \\ &= \|(1-\alpha_n)g_1(x_n) + \alpha_n J^{\rho}_{\phi_1}(g_2(y_n) - \rho(A(y_n) + S(y_n))) \\ &\quad + \alpha_n u_n + w_n - g_1(x^*)\| \end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n)\|g_1(x_n)-g_1(x^*)\| \\
&\quad + \alpha_n \|J^{\rho}_{\phi_1}(g_2(y_n)-\rho(A(y_n)+S(y_n))) \\
&\quad - J^{\rho}_{\phi_1}(g_2(y^*)-\rho(A(y^*)+S(y^*)))\| + \alpha_n\|u_n\| + \|w_n\| \\
&\le (1-\alpha_n)\|g_1(x_n)-g_1(x^*)\| + \alpha_n\|g_2(y_n)-g_2(y^*) \\
&\quad - \rho(A(y_n)+S(y_n)-(A(y^*)+S(y^*)))\| + \alpha_n\|u_n\| + \|w_n\| \\
&\le (1-\alpha_n)\|g_1(x_n)-g_1(x^*)\| \\
&\quad + \alpha_n\|g_2(y_n)-g_2(y^*)-\rho(S(y_n)-S(y^*))\| \\
&\quad + \alpha_n \rho l_2 \|g_2(y_n)-g_2(y^*)\| + \alpha_n\|u_n\| + \|w_n\|.
\end{aligned}
\tag{3.5}
$$

Since S is g_2-k_2-strongly monotone and g_2-s_2-Lipschitz continuous, we get

$$
\begin{aligned}
&\|g_2(y_n)-g_2(y^*)-\rho(S(y_n)-S(y^*))\| \\
&\le \sqrt{1-2\rho k_2+\rho^2 s_2^2}\ \|g_2(y_n)-g_2(y^*)\|.
\end{aligned}
\tag{3.6}
$$

Combining (3.5) and (3.6), we have

$$
\begin{aligned}
\|g_1(x_{n+1})-g_1(x^*)\| \le{}& (1-\alpha_n)\|g_1(x_n)-g_1(x^*)\| \\
&+ \alpha_n\theta_1\|g_2(y_n)-g_2(y^*)\| \\
&+ \alpha_n\|u_n\| + \|w_n\|,
\end{aligned}
\tag{3.7}
$$

where $\theta_1 = \sqrt{1-2\rho k_2+\rho^2 s_2^2} + \rho l_2$. Again from (3.1) and (3.4), we have

$$
\begin{aligned}
&\|g_2(y_n)-g_2(y^*)\| \\
&= \|J^{\gamma}_{\phi_2}(g_1(x_n)-\gamma(B(x_n)+T(x_n))) + v_n \\
&\qquad - J^{\gamma}_{\phi_2}(g_1(x^*)-\gamma(B(x^*)+T(x^*)))\| \\
&\le \|g_1(x_n)-g_1(x^*)-\gamma(B(x_n)+T(x_n)-(B(x^*)+T(x^*)))\| \\
&\quad + \|v_n\| \\
&\le \|g_1(x_n)-g_1(x^*)-\gamma(T(x_n)-T(x^*))\| \\
&\quad + \gamma l_1\|g_1(x_n)-g_1(x^*)\| + \|v_n\|.
\end{aligned}
\tag{3.8}
$$

Since T is g_1-k_1-strongly monotone and g_1-s_1-Lipschitz continuous, we have

$$
\begin{aligned}
&\|g_1(x_n)-g_1(x^*)-\gamma(T(x_n)-T(x^*))\| \\
&\le \sqrt{1-2\gamma k_1+\gamma^2 s_1^2}\ \|g_1(x_n)-g_1(x^*)\|.
\end{aligned}
\tag{3.9}
$$

Combining (3.8) and (3.9), we obtain

$$
\|g_2(y_n)-g_2(y^*)\| \le \theta_2\|g_1(x_n)-g_1(x^*)\| + \|v_n\|, \tag{3.10}
$$

where $\theta_2 = \sqrt{1 - 2\gamma k_1 + \gamma^2 s_1^2} + \gamma l_1$. It follows from (3.7) and (3.10) that

$$\begin{aligned} &\|g_1(x_{n+1}) - g_1(x^*)\| \\ &\le (1-\alpha_n)\|g_1(x_n) - g_1(x^*)\| + \alpha_n\theta_1\theta_2\|g_1(x_n) - g_1(x^*)\| \\ &\quad + \alpha_n\theta_1\|v_n\| + \alpha_n\|u_n\| + \|w_n\| \\ &= \Big(1 - \alpha_n\Big(1 - \theta_1\theta_2\Big)\Big)\|g_1(x_n) - g_1(x^*)\| + \|w_n\| \\ &\quad + \alpha_n(1-\theta_1\theta_2)\cdot\frac{1}{1-\theta_1\theta_2}\Big(\theta_1\|v_n\| + \|u_n\|\Big). \end{aligned} \tag{3.11}$$

Let $a_n = \|g_1(x_n) - g_1(x^*)\|$, $b_n = \frac{1}{1-\theta_1\theta_2}(\theta_1\|v_n\| + \|u_n\|)$, $c_n = \|w_n\|$, $t_n = \alpha_n(1-\theta_1\theta_2)$. Then (3.11) can be rewritten as follows:

$$a_{n+1} \le (1-t_n)a_n + b_n t_n + c_n.$$

From the assumption, we know that $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{t_n\}$ satisfying the conditions of Lemma 1.1. Thus $\lim_{n\to\infty} a_n = 0$ and so $g_1(x_n) \to g_1(x^*)$ as $n \to \infty$. Since $\lim_{n\to\infty} g_1(x_n) = g_1(x^*)$, it follows from (3.2), (3.8) and (3.9) that $g_2(y_n) \to g_2(y^*)$ as $n \to \infty$. Since g_1, g_2 are invertible,

$$\lim_{n\to\infty} x_n = x^*, \quad \lim_{n\to\infty} y_n = y^*.$$

This completes the proof. □

If $A = B = 0$ in Theorem 3.1, then we have the following:

Theorem 3.2. *Let S, T be the same as in Theorem 2.2 and let $\{g_1(x_n)\}$, $\{g_2(y_n)\}$ be the iterative sequences generated by Algorithm 3.2. If $0 < \rho < \frac{2k_2}{s_2^2}$ and $0 < \gamma < \frac{2k_1}{s_1^2}$, then (x_n, y_n) converges to the unique solution (x^*, y^*) of the problem (1.2).*

If $\phi_1 = \phi_2 = \delta_K$ in Theorem 3.1, then we have the following theorem.

Theorem 3.3. *Let K, A, B, S, T be the same as in Theorem 2.3 and let $\{g_1(x_n)\}$, $\{g_2(y_n)\}$ be the iterative sequences generated by Algorithm 3.3. If the condition (2.1) holds, then (x_n, y_n) converges to the unique solution (x^*, y^*) of the problem (1.3).*

If $\phi_1 = \phi_2 = \delta_K$ and $A = B = 0$ in Theorem 3.1, then we have the following theorem.

Theorem 3.4. *Let K, S, T be the same as in Theorem 2.4 and let $\{g_1\ (x_n)\}$, $\{g_2(y_n)\}$ be the iterative sequences generated by Algorithm 3.4. If $0 < \rho < \frac{2k_2}{s_2^2}$ and $0 < \gamma < \frac{2k_1}{s_1^2}$, then (x_n, y_n) converges to the unique solution (x^*, y^*) of the problem (1.4).*

Remark 3.1. (1) If $g_1 = g_2 = I$ and $\phi_1 = \phi_2 = \phi$ in Algorithm 3.1 and Theorem 3.1, then we can easily get the result of Kim-Kim [16], as a special case.

(2) Theorems 3.2, 3.3 and 3.4 are extensions and improvements of the corresponding results in Noor [21]-[23] and Verma [24]-[26].

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