J. Korean Math. Soc. **44** (2007), No. 4, pp. 809–822

ENDPOINT ESTIMATES FOR MAXIMAL COMMUTATORS IN NON-HOMOGENEOUS SPACES

Guoen Hu, Yan Meng, and Dachun Yang

Abstract. Certain weak type endpoint estimates are established for maximal commutators generated by Calderón-Zygmund operators and $\mathrm{Osc}_{\exp L^r}(\mu)$ functions for $r \geq 1$ under the condition that the underlying measure only satisfies some growth condition, where the kernels of Calderón-Zgymund operators only satisfy the standard size condition and some Hörmander type regularity condition, and $\mathrm{Osc}_{\exp L^r}(\mu)$ are the spaces of Orlicz type satisfying that $\mathrm{Osc}_{\exp L^r}(\mu) = \mathrm{RBMO}(\mu)$ if $r = 1$ and $\mathrm{Osc}_{\exp L^r}(\mu) \subset \mathrm{RBMO}(\mu)$ if $r > 1$.

1. Introduction

It is well known that the doubling condition of the underlying measure is a key assumption in the analysis on spaces of homogeneous type. We recall that μ is said to satisfy the doubling condition if there is a constant $C > 0$ such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all $x \in \mathbb{R}^d$ and $r > 0$, where $B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$. However, during the last several years, many classical results have been proved still valid if the underlying measure μ is a positive Radon measure on $\mathbb{R}^d$, which only satisfies the following growth condition that there exist constants $C_0 > 0$ and $n \in (0, d]$ such that for all $x \in \mathbb{R}^d$ and $r > 0$,

$$\mu(B(x, r)) \leq C_0 r^n; \tag{1.1}$$

see [5, 6, 7, 10, 11, 12]. The Euclidean space $\mathbb{R}^d$ equipped with a Radon measure that only satisfies (1.1) is called a non-homogeneous space since μ may not be doubling. The motivation for developing the analysis on non-homogeneous spaces and some examples of non-doubling measures can be found in [14]. We only point out that the analysis on non-homogeneous spaces played an essential role in solving the long-standing Painlevé's problem by Tolsa in [13].

Received January 11, 2006.

2000 *Mathematics Subject Classification.* Primary 47B47; Secondary 42B20, 43A99.

Key words and phrases. Calderón-Zygmund operator, $\mathrm{Osc}_{\exp L^r}(\mu)$, maximal commutator, endpoint estimate.

The third author was supported by Program for New Century Excellent Talents in University (NCET-04-0142) of China.

The purpose of this paper is to establish some weak type endpoint estimates for the maximal commutators associated to Calderón-Zygmund operators, whose kernels satisfy the standard size condition and some weaker regularity condition, with $\mathrm{Osc}_{\exp L^r}(\mu)$ functions, where $r \geq 1$. Before stating our results, we first recall some necessary notation and definitions.

Throughout this paper, by a cube $Q \subset \mathbb{R}^d$, we mean a closed cube whose sides are parallel to the axes and centered at some point of $\mathrm{supp}(\mu)$, and we denote its side length by $l(Q)$ and its center by x_Q. Let α and β be positive constants such that $\alpha > 1$ and $\beta > \alpha^n$. For a cube Q, we say that Q is (α, β)-doubling if $\mu(\alpha Q) \leq \beta\mu(Q)$, where αQ denotes the cube concentric with Q and having side length $\alpha l(Q)$. In what follows, for definiteness, if α and β are not specified, by a doubling cube we mean a $(2,\, 2^{d+1})$-doubling cube. Especially, for any given cube Q, we denote by $\widetilde{Q}$ the smallest doubling cube in the family $\{2^k Q\}_{k\geq 0}$. For two cubes $Q_1 \subset Q_2$, set

$$K_{Q_1, Q_2} = 1 + \sum_{k=1}^{N_{Q_1,Q_2}} \frac{\mu(2^k Q_1)}{[l(2^k Q_1)]^n},$$

where N_{Q_1,Q_2} is the first positive integer k such that $l(2^k Q_1) \geq l(Q_2)$; see [9] for some basic properties of K_{Q_1,Q_2}.

Definition. For $r \geq 1$, a locally integrable function f is said to belong to the space $\mathrm{Osc}_{\exp L^r}(\mu)$ if there is a constant $C_1 > 0$ such that

(i) for any Q,

$$\begin{aligned}&\left\|f - m_{\widetilde{Q}}(f)\right\|_{\exp L^r,\, Q}\\ &= \inf\left\{\lambda > 0 : \frac{1}{\mu(2Q)}\int_Q \exp\left(\frac{|f - m_{\widetilde{Q}}(f)|}{\lambda}\right)^r d\mu \leq 2\right\} \leq C_1,\end{aligned}$$

(ii) for any two doubling cubes $Q_1 \subset Q_2$, $|m_{Q_1}(f) - m_{Q_2}(f)| \leq C_1 K_{Q_1,Q_2}$,

where $m_{\widetilde{Q}}(f)$ is the mean value of f on $\widetilde{Q}$, namely, $m_{\widetilde{Q}}(f) = \frac{1}{\mu(\widetilde{Q})}\int_{\widetilde{Q}} f(x)\, d\mu(x)$. The minimal constant C_1 satisfying (i) and (ii) is defined to be the $\mathrm{Osc}_{\exp L^r}(\mu)$ norm of f and denoted by $\|f\|_{\mathrm{Osc}_{\exp L^r}(\mu)}$.

The space $\mathrm{Osc}_{\exp L^r}(\mu)$ is an analogy of the classical $\mathrm{Osc}_{\exp L^r}(\mathbb{R}^d)$ space which was introduced by Pérez and Trujillo-González in [8]. Obviously, for any $r_2 > r_1 > 1$, $\mathrm{Osc}_{\exp L^{r_2}}(\mu) \subset \mathrm{Osc}_{\exp L^{r_1}}(\mu) \subset \mathrm{RBMO}(\mu)$. Moreover, from the John-Nirenberg inequality established by Tolsa in [9], it follows that $\mathrm{Osc}_{\exp L^1}(\mu)$ is just the space $\mathrm{RBMO}(\mu)$ of Tolsa in [9].

Let $K \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,\, y) : x = y\})$ and there exists a constant $C > 0$ such that for all $x,\, y \in \mathbb{R}^d$ with $x \neq y$,

$$|K(x,\, y)| \leq C|x - y|^{-n}, \tag{1.2}$$

and for all $y,\, y' \in \mathbb{R}^d$,

$$\int_{|x-y|\geq 2|y-y'|} \{|K(x,\,y) - K(x,\,y')| + |K(y,\,x) - K(y',\,x)|\}\, d\mu(x) \leq C. \tag{1.3}$$

For any $\epsilon > 0$, define the truncated operators T_ϵ by

$$T_\epsilon(f)(x) = \int_{|x-y|>\epsilon} K(x,\,y) f(y)\, d\mu(y), \tag{1.4}$$

and the maximal Calderón-Zygmund operator T^* by

$$T^*(f)(x) = \sup_{\epsilon>0} |T_\epsilon(f)(x)|\,. \tag{1.5}$$

It is well known that if the operators T_ϵ are bounded on $L^2(\mu)$ uniformly for $\epsilon > 0$, then there is an operator T which is the weak limit as $\epsilon \to 0$ of some subsequence of the uniformly bounded operators T_ϵ. The operator T is also bounded on $L^2(\mu)$ and satisfies that for $f \in L^2(\mu)$ with $\operatorname{supp} f \neq \mathbb{R}^d$, and almost all $x \in \mathbb{R}^d \setminus \operatorname{supp} f$,

$$T(f)(x) = \int_{\mathbb{R}^d} K(x,\,y) f(y)\, d\mu(y).$$

For $m \in \mathbb{N}$ and $b_1, b_2, \ldots, b_m \in \mathrm{RBMO}(\mu)$, define the multilinear commutator $T_{\vec{b}}$ by

$$T_{\vec{b}} f(x) = [b_m,\, [b_{m-1}, \ldots, [b_1,\, T] \cdots]]\,(f)(x),$$

where $\vec{b} = (b_1,\, b_2, \ldots, b_m)$ and $[b_1,\, T]$ is defined by

$$[b_1,\, T](f)(x) = b_1(x) T(f)(x) - T(b_1 f)(x). \tag{1.6}$$

For the case of $m = 1$, we denote $T_{\vec{b}}$ simply by T_b. When the kernel K satisfies the size condition (1.2) and the standard regularity condition: there exist two constants $\alpha \in (0,\,1]$ and $C > 0$ such that for all $x,\, y,\, y' \in \mathbb{R}^d$ with $|x-y| \geq 2|y-y'|$ and $x \neq y$,

$$|K(x,\,y) - K(x,\,y')| + |K(y,\,x) - K(y',\,x)| \leq C \frac{|y-y'|^\alpha}{|x-y|^{n+\alpha}}, \tag{1.7}$$

Tolsa [9] proved that if the operators T_ϵ are bounded on $L^2(\mu)$ uniformly for $\epsilon > 0$, then T_b is bounded on $L^p(\mu)$ for any $p \in (1,\,\infty)$. In [1], we generalized this result of Tolsa and proved that if K satisfies (1.2) and (1.7) and the operators T_ϵ are bounded on $L^2(\mu)$ uniformly for $\epsilon > 0$, then for any $m \in \mathbb{N}$, $T_{\vec{b}}$ is also bounded on $L^p(\mu)$ with $p \in (1,\,\infty)$, and satisfies a weak type endpoint estimate, namely, there exists a constant $C > 0$ such that for all $\lambda > 0$ and all bounded functions f with compact support,

$$\mu\Big(\Big\{x \in \mathbb{R}^d : |T_{\vec{b}}(f)(x)| > \lambda\Big\}\Big) \leq C\varphi_{1/r}\Big(\prod_{i=1}^m \|b_i\|_{\mathrm{Osc}_{\exp L^{r_i}}(\mu)}\Big) \int_{\mathbb{R}^d} \varphi_{1/r}\Big(\frac{|f(x)|}{\lambda}\Big) d\mu(x),$$

where $C > 0$ is a constant, $1/r = \sum_{i=1}^{m} 1/r_i$ and $\varphi_\sigma(t) = t\log^\sigma(2+t)$ for $\sigma,\, t > 0$.

We now define the maximal commutator associated with the operator $T_{\vec{b}}$. For $m \in \mathbb{N}$ and $b_1, b_2, \ldots, b_m \in \mathrm{RBMO}(\mu)$, the maximal commutator $T_{\vec{b}}^*$ is defined by

$$
\begin{aligned}
(1.8) \qquad T_{\vec{b}}^*(f)(x) &= \sup_{\epsilon>0} \left|T_{\epsilon,\vec{b}}(f)(x)\right| \\
&= \sup_{\epsilon>0} \left| \int_{|x-y|>\epsilon} \prod_{i=1}^{m} [b_i(x) - b_i(y)] K(x,\, y) f(y)\, d\mu(y) \right|.
\end{aligned}
$$

Repeating the proof of Lemma 4.1 in [4], we can prove that if K satisfies (1.2) and the following Hörmander-type condition

$$
\begin{aligned}
(1.9) \qquad \sup_{y,\, y' \in \mathbb{R}^d,\, r \geq |y-y'|} \sum_{l=1}^{\infty} l^m \int_{2^l r < |x-y| \leq 2^{l+1} r} \Big\{ &|K(x,\, y) - K(x,\, y')| \\
&+ |K(y,\, x) - K(y',\, x)| \Big\}\, d\mu(x) < \infty,
\end{aligned}
$$

and if the operators T_ϵ are bounded on $L^2(\mu)$ uniformly on $\epsilon > 0$, then for $b_i \in \mathrm{RBMO}(\mu)$ with $i = 1,\, 2, \ldots, m$, the operator $T_{\vec{b}}^*$ is bounded on $L^p(\mu)$ for any $p \in (1,\, \infty)$. It was also proved in [3] that if K satisfies (1.2) and (1.7), and if the operators T_ϵ are bounded on $L^2(\mu)$ uniformly for $\epsilon > 0$, then for $m = 1$, the maximal commutator $T_{\vec{b}}^*$ satisfies the weak type endpoint estimate, namely, there exists a constant $C > 0$ such that for all $\lambda > 0$ and all bounded functions f with compact support,

$$
\begin{aligned}
&\mu\Big(\Big\{x \in \mathbb{R}^d : |T_{\vec{b}}^*(f)(x)| > \lambda\Big\}\Big) \\
&\leq C\varphi_{1/r}\Big(\|b_1\|_{\mathrm{Osc}_{\exp L^r}(\mu)}\Big) \int_{\mathbb{R}^d} \varphi_{1/r}\Big(\frac{|f(x)|}{\lambda}\Big)\, d\mu(x).
\end{aligned}
$$

In this paper, we will further prove that if K satisfies (1.2) and (1.9), then for any $m \in \mathbb{N}$, the maximal commutator $T_{\vec{b}}^*$ enjoys the same endpoint estimate. Our result can be stated as follows.

Theorem 1.1. *Let $m \in \mathbb{N}$, $r_i \geq 1$ and $b_i \in \mathrm{Osc}_{\exp L^{r_i}}(\mu)$ for $i = 1,\, 2, \ldots, m$, and $T_{\vec{b}}^*$ be the same as in* (1.8) *with the kernel K satisfying* (1.2) *and* (1.9). *If the operators T_ϵ are bounded on $L^2(\mu)$ uniformly for $\epsilon > 0$, then there exists a constant $C > 0$ such that for all $\lambda > 0$ and all bounded functions f with compact support,*

$$
\begin{aligned}
&\mu\Big(\Big\{x \in \mathbb{R}^d : |T_{\vec{b}}^*(f)(x)| > \lambda\Big\}\Big) \\
(1.10) \qquad &\leq C\varphi_{1/r}\Big(\prod_{i=1}^{m} \|b_i\|_{\mathrm{Osc}_{\exp L^{r_i}}(\mu)}\Big) \int_{\mathbb{R}^d} \varphi_{1/r}\Big(\frac{|f(x)|}{\lambda}\Big)\, d\mu(x).
\end{aligned}
$$

Throughout this paper, for any index $p \in [1,\, \infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$. For $f \sim g$, we mean that the ratio f/g is bounded and bounded away from zero by constants independent of the relevant variables in f and g. Similar is $f \lesssim g$. Constants with subscripts, such as C_0, are positive constants independent of the main parameters involved but whose values may not be the same at each occurrence.

2. Proof of Theorem 1.1

We begin with a generalization of the Hölder inequality. For $r > 0$, a cube Q and an appropriate function f, define

$$\|f\|_{L(\log L)^r, Q} = \inf\left\{\lambda > 0 : \frac{1}{\mu(2Q)} \int_Q \frac{|f(x)|}{\lambda} \log^r\left(2 + \frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1\right\},$$

and

$$\|f\|_{\exp L^r, Q} = \inf\left\{\lambda > 0 : \frac{1}{\mu(2Q)} \int_Q \exp\left(\frac{|f(x)|}{\lambda}\right)^r d\mu(x) \leq 2\right\}.$$

Then for any cube Q,

$$(2.1) \qquad \frac{1}{\mu(2Q)} \int_Q \left|\prod_{i=1}^m b_i(x) f(x)\right| d\mu(x) \leq C \prod_{i=1}^m \|b_i\|_{\exp L^{r_i}, Q} \|f\|_{L(\log L)^{1/r}, Q},$$

where $r_i \geq 1$ and $1/r = \sum_{i=1}^m 1/r_i$; see Lemma 3.2 in [8] and the related references there.

Lemma 2.1. *Let $m \in \mathbb{N}$, $r_i \geq 1$ and $b_i \in \mathrm{Osc}_{\exp L^{r_i}}(\mu)$ for $i = 1, 2, \ldots, m$, and $M_{\vec{b}}$ be defined by*

$$(2.2) \qquad M_{\vec{b}}(f)(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y-x|\leq r} \prod_{i=1}^m |b_i(x) - b_i(y)| |f(y)| \, d\mu(y).$$

Then there exists a constant $C > 0$ such that for all $\lambda > 0$ and all bounded functions f with compact support,

$$\mu\left(\{x \in \mathbb{R}^d : M_{\vec{b}}(f)(x) > \lambda\}\right) \leq C\varphi_{1/r}\Big(\prod_{i=1}^m \|b_i\|_{\mathrm{Osc}_{\exp L^{r_i}}(\mu)}\Big) \int_{\mathbb{R}^d} \varphi_{1/r}\left(\frac{|f(x)|}{\lambda}\right) d\mu(x),$$

where r and $\varphi_{1/r}$ are the same as in Theorem 1.1.

For the special case $m = 1$, Lemma 2.1 was proved in [3]. For $m \geq 2$, Lemma 2.1 can be proved in a similar way. We omit the details for brevity.

Proof of Theorem 1.1. By the homogeneity, we may assume that for $i = 1, \ldots, m$, $\|b_i\|_{\mathrm{Osc}_{\exp L^{r_i}}(\mu)} = 1$. We carry out the argument by induction on m.

Step I. $m = 1$. In this case, we denote $T^*_{\vec{b}}$ simply by T^*_b. For each fixed bounded function f with compact support and each $\lambda > 0$ (with $\lambda > 2^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|$ if $\|\mu\| < \infty$; note that if $\|\mu\| < \infty$ and $\lambda \leq 2^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|$, then the inequality (1.10) is trivial), applying the Calderón-Zygmund decomposition to f at level λ (see [9]), we can obtain a sequence of cubes $\{Q_j\}_{j\in\mathbb{N}}$ with bounded overlaps (that is, $\sum_j \chi_{Q_j}(x) \lesssim 1$) such that

(a) $\frac{\lambda}{2^{d+1}} < \frac{1}{\mu(2Q_j)} \int_{Q_j} |f(x)|\,d\mu(x)$;
(b) $\frac{1}{\mu(2\eta Q_j)} \int_{\eta Q_j} |f(x)|\,d\mu(x) \le \frac{\lambda}{2^{d+1}}$ for any $\eta > 2$;
(c) $|f(x)| \le \lambda$ μ-a. e. on $\mathbb{R}^d \setminus \cup_j Q_j$;
(d) for each fixed j, let R_j be the smallest $(6, 6^{n+1})$-doubling cube of the form $6^k Q_j$, $k \ge 1$. Set $w_j = \chi_{Q_j} / \sum_k \chi_{Q_k}$. Then there is a function θ_j with $\operatorname{supp}\theta_j \subset R_j$ and satisfying

$$\int_{\mathbb{R}^d} \theta_j(x)\,d\mu(x) = \int_{Q_j} f(x) w_j(x)\,d\mu(x),\ \|\theta_j\|_{L^\infty(\mu)}\mu(R_j) \lesssim \int_{Q_j} |f(x)|\,d\mu(x),$$

and $\sum_j |\theta_j(x)| \lesssim \lambda$.

Set $g(x) = f(x)\chi_{\mathbb{R}^d\setminus\cup_j Q_j}(x) + \sum_j \theta_j(x)$ and

$$h(x) = f(x) - g(x) = \sum_j \{f(x)w_j(x) - \theta_j(x)\} = \sum_j h_j(x).$$

Obviously,

$$T_b^*(f)(x) \le T_b^*(g)(x) + T_b^*(h)(x).$$

Note that $\|g\|_{L^\infty(\mu)} \lesssim \lambda$, and $\|g\|_{L^1(\mu)} \lesssim \|f\|_{L^1(\mu)}$. The $L^2(\mu)$-boundedness of T_b^* tells us that

$$\begin{aligned}(2.3)\qquad \mu\left(\left\{x \in \mathbb{R}^d : T_b^*(g)(x) > \lambda\right\}\right) &\lesssim \lambda^{-2} \int_{\mathbb{R}^d} |g(x)|^2\,d\mu(x)\\ &\lesssim \lambda^{-1} \int_{\mathbb{R}^d} |f(x)|\,d\mu(x).\end{aligned}$$

Taking into account the fact that

$$(2.4)\qquad \mu\left(\cup_j 2Q_j\right) \lesssim \lambda^{-1} \int_{\mathbb{R}^d} |f(x)|\,d\mu(x),$$

we see that the proof of Theorem 1.1 can be reduced to proving that

$$(2.5)\qquad \mu\left(\left\{x \in \mathbb{R}^d \setminus \cup_j 2Q_j : T_b^*(h)(x) > \lambda\right\}\right) \lesssim \int_{\mathbb{R}^d} \varphi_{1/r}\Big(\frac{|f(x)|}{\lambda}\Big)\,d\mu(x).$$

To this end, by the vanishing moment conditions of h_j for $j \in \mathbb{N}$, we can write

$$\begin{aligned}&T_b^*(h)(x)\chi_{\mathbb{R}^d\setminus\cup_j 2Q_j}(x)\\ &= \sup_{\epsilon>0}\Bigg|\sum_j \int_{\mathbb{R}^d} \Big\{K_\epsilon(x, y)\,[b(x) - b(y)]\\ &\quad -K_\epsilon(x, x_{Q_j})\left[b(x) - m_{\tilde{Q}_j}(b)\right]\Big\} h_j(y)\,d\mu(y)\Bigg|\chi_{\mathbb{R}^d\setminus\cup_j 2Q_j}(x)\end{aligned}$$

$$
\begin{aligned}
&\le \sum_j \Big|b(x) - m_{\widetilde{Q}_j}(b)\Big| \\
&\quad \times \sup_{\epsilon>0}\Big|\int_{\mathbb{R}^d}\big[K_\epsilon(x,\,y) - K_\epsilon(x,\,x_{Q_j})\big]h_j(y)\,d\mu(y)\Big|\chi_{\mathbb{R}^d\setminus\bigcup_j 2Q_j}(x) \\
&\quad + T^*\Big(\sum_j \Big[b - m_{\widetilde{Q}_j}(b)\Big]\,h_j\Big)(x) \\
&= \mathrm{E}(x) + \mathrm{F}(x),
\end{aligned}
$$

where T^* is defined by (1.5), and K_ϵ for $\epsilon > 0$ is defined by

$$K_\epsilon(x,\,y) = K(x,\,y)\chi_{\{|x-y|>\epsilon\}}(x,\,y).$$

Theorem 1.1 in [2] tells us that if T_ϵ are bounded on $L^2(\mu)$ uniformly on $\epsilon > 0$, and K satisfies (1.2) and (1.3), then T^* is bounded from $L^1(\mu)$ to weak $L^1(\mu)$. Thus,

$$
\begin{aligned}
\mu\left(\{x \in \mathbb{R}^d : |\mathrm{F}(x)| > \lambda\}\right) &\lesssim \lambda^{-1}\sum_j \int_{\mathbb{R}^d}\Big|b(x) - m_{\widetilde{Q}_j}(b)\Big|\,|f(x)w_j(x)|\,d\mu(x) \\
&\quad + \lambda^{-1}\sum_j \int_{\mathbb{R}^d}\Big|b(x) - m_{\widetilde{Q}_j}(b)\Big|\,|\theta_j(x)|\,d\mu(x) \\
&= \mathrm{G} + \mathrm{H}.
\end{aligned}
$$

Note that R_j is also $(2,\,2^{n+1})$-doubling and $R_j = \widetilde{R_j}$. A trivial computation gives us that $K_{\widetilde{Q}_j,\,R_j} \lesssim 1$. Thus,

$$
\begin{aligned}
\mathrm{H} &\lesssim \lambda^{-1}\sum_j \|\theta_j\|_{L^\infty(\mu)}\left\{\int_{R_j}\big|b(x) - m_{R_j}(b)\big|\,d\mu(x)\right. \\
&\quad \left. + \mu(R_j)\Big|m_{R_j}(b) - m_{\widetilde{Q}_j}(b)\Big|\right\} \\
&\lesssim \lambda^{-1}\sum_j \|\theta_j\|_{L^\infty(\mu)}\mu(R_j) \\
&\lesssim \lambda^{-1}\int_{\mathbb{R}^d}|f(x)|\,d\mu(x).
\end{aligned}
$$

On the other hand, by the generalization of Hölder inequality (2.1), we obtain

$$
\begin{aligned}
\mathrm{G} &\lesssim \lambda^{-1}\sum_j \mu(2Q_j)\|f\|_{L(\log L)^{1/r},\,Q_j}\|b - m_{\widetilde{Q}_j}(b)\|_{\exp L^r,\,Q_j} \\
&\lesssim \lambda^{-1}\sum_j \mu(2Q_j)\inf_{t>0}\left\{t + \frac{t}{\mu(2Q_j)}\int_{Q_j}\frac{|f(x)|}{t}\log^{1/r}\Big(2 + \frac{|f(x)|}{t}\Big)d\mu(x)\right\} \\
&\lesssim \int_{\mathbb{R}^d}\frac{|f(x)|}{\lambda}\log^{1/r}\Big(2 + \frac{|f(x)|}{\lambda}\Big)\,d\mu(x).
\end{aligned}
$$

It remains to estimate $\mathrm{E}(x)$. Note that for $y \in R_j$ and $x \in \mathbb{R}^d \setminus 2R_j$, $|x - x_{Q_j}| \sim |x - y|$. A straightforward computation indicates

$$\begin{aligned}
&\sup_{\epsilon>0} \int_{\mathbb{R}^d} \left|K_\epsilon(x, y) - K_\epsilon(x, x_{Q_j})\right| |h_j(y)| \, d\mu(y) \\
&\le \int_{\mathbb{R}^d} \left|K(x, y) - K(x, x_{Q_j})\right| |h_j(y)| \, d\mu(y) \\
&\quad + \sup_{\epsilon>0} \int_{\mathbb{R}^d} \Big\{ |K(x, y)| \chi_{\{|x-y|>\epsilon\} \cap \{|x-x_{Q_j}| \le \epsilon\}}(x, y) \\
&\quad + |K(x, x_{Q_j})| \chi_{\{|x-y| \le \epsilon\} \cap \{|x-x_{Q_j}|>\epsilon\}}(x, y) \Big\} |h_j(y)| \, d\mu(y) \\
&\lesssim \int_{\mathbb{R}^d} \left|K(x, y) - K(x, x_{Q_j})\right| |h_j(y)| \, d\mu(y) + M(h_j)(x),
\end{aligned}$$

where M is the Hardy-Littlewood maximal operator defined by

$$Mh(x) = \sup_{Q \ni x} \frac{1}{[l(Q)]^n} \int_Q |h(y)| \, d\mu(y).$$

Therefore,

$$\begin{aligned}
\mathrm{E}(x) &\lesssim \sum_j \left|b(x) - m_{\widetilde{Q}_j}(b)\right| \int_{\mathbb{R}^d} \left|K(x, y) - K(x, x_{Q_j})\right| |h_j(y)| \, d\mu(y) \chi_{\mathbb{R}^d \setminus 2R_j}(x) \\
&\quad + M_{\tilde{b}}\Big(\sum_j |h_j|\Big)(x) + M\Big(\sum_j |b - m_{\widetilde{Q}_j}(b)||h_j|\Big)(x) \\
&\quad + \sum_j \left|b(x) - m_{\widetilde{Q}_j}(b)\right| T^*(h_j)(x) \chi_{2R_j \setminus 2Q_j}(x) \\
&\quad + \sum_j \left|b(x) - m_{\widetilde{Q}_j}(b)\right| \sup_{\epsilon>0} \int_{\mathbb{R}^d} \left|K_\epsilon(x, x_{Q_j}) h_j(y)\right| \, d\mu(y) \chi_{2R_j \setminus 2Q_j}(x) \\
&= \mathrm{E}_1(x) + \mathrm{E}_2(x) + \mathrm{E}_3(x) + \mathrm{E}_4(x) + \mathrm{E}_5(x).
\end{aligned}$$

An application of Lemma 2.1 gives us that

$$\begin{aligned}
\mu\left(\{x \in \mathbb{R}^d : \mathrm{E}_2(x) > \lambda\}\right) &\lesssim \int_{\mathbb{R}^d} \varphi_{1/r}\Big(\frac{\sum_j |f\omega_j(x)|}{\lambda}\Big) \, d\mu(x) \\
&\quad + \int_{\mathbb{R}^d} \varphi_{1/r}\Big(\frac{\sum_j |\theta_j(x)|}{\lambda}\Big) \, d\mu(x) \\
&= \mathrm{I} + \mathrm{J}.
\end{aligned}$$

Obviously,

$$\mathrm{I} \lesssim \int_{\mathbb{R}^d} \varphi_{1/r}\Big(\frac{|f(x)|}{\lambda}\Big) \, d\mu(x).$$

Recall that $\sum_j |\theta_j(x)| \lesssim \lambda$. It then follows that

$$\mathrm{J} \lesssim \lambda^{-1} \sum_j \|\theta_j\|_{L^\infty(\mu)} \mu(R_j) \lesssim \lambda^{-1} \int_{\mathbb{R}^d} |f(x)| \, d\mu(x),$$

and so

$$\mu\left(\left\{x \in \mathbb{R}^d : \mathrm{E}_2(x) > \lambda\right\}\right) \lesssim \int_{\mathbb{R}^d} \varphi_{1/r}\Big(\frac{|f(x)|}{\lambda}\Big)\, d\mu(x).$$

Since the Hardy-Littlewood maximal operator M is bounded from $L^1(\mu)$ to weak $L^1(\mu)$, similar to the estimate for $\mathrm{F}(x)$, it follows that

$$\mu\left(\left\{x \in \mathbb{R}^d : \mathrm{E}_3(x) > \lambda\right\}\right) \lesssim \int_{\mathbb{R}^d} \varphi_{1/r}\Big(\frac{|f(x)|}{\lambda}\Big)\, d\mu(x).$$

To estimate $\mathrm{E}_4(x)$, observing that $1 \le k \le N_{2Q_j,\,2R_j}$, $K_{\widetilde{Q_j},\,\widetilde{2^{k+1}Q_j}} \lesssim K_{Q_j,\,R_j} \lesssim 1$, we can easily obtain that

$$\begin{aligned}
&\mu\left(\left\{x \in \mathbb{R}^d : \mathrm{E}_4(x) > \lambda\right\}\right)\\
&\lesssim \frac{1}{\lambda}\sum_j \sum_{k=1}^{N_{2Q_j,\,2R_j}} \int_{2^{k+1}Q_j\setminus 2^kQ_j}\\
&\quad\times \int_{\mathbb{R}^d} \frac{|b(x) - m_{\widetilde{Q_j}}(b)|}{|x-y|^n} \left\{|f(y)w_j(y)| + |\theta_j(y)|\right\}\, d\mu(y)d\mu(x)\\
&\lesssim \frac{1}{\lambda}\sum_j \sum_{k=1}^{N_{2Q_j,\,2R_j}} \frac{\mu(2^{k+2}Q_j)}{l(2^kQ_j)^n} K_{\widetilde{Q_j},\,\widetilde{2^{k+1}Q_j}}\\
&\quad\times \left\{\int_{Q_j} |f(y)|\, d\mu(y) + \|\theta_j\|_{L^\infty(\mu)}\mu(R_j)\right\}\\
&\lesssim \frac{1}{\lambda}\sum_j \int_{Q_j} |f(y)|\, d\mu(y),
\end{aligned}$$

and similarly,

$$\mu\left(\left\{x \in \mathbb{R}^d : \mathrm{E}_5(x) > \lambda\right\}\right) \lesssim \frac{1}{\lambda}\sum_j \int_{Q_j} |f(y)|\, d\mu(y).$$

For $\mathrm{E}_1(x)$, another application of the generalized Hölder inequality (2.1) yields

$$\begin{aligned}
&\int_{\mathbb{R}^d\setminus 2R_j} \left|b(x) - m_{\widetilde{Q_j}}(b)\right| \left|K(x,\,y) - K(x,\,x_{Q_j})\right|\, d\mu(x)\\
&\le \sum_{k=1}^{\infty} \left|m_{\widetilde{2^{k+1}R_j}}(b) - m_{\widetilde{Q_j}}(b)\right| \int_{2^{k+1}R_j\setminus 2^kR_j} \left|K(x,\,y) - K(x,\,x_{Q_j})\right|\, d\mu(x)\\
&\quad+ \sum_{k=1}^{\infty} \int_{2^{k+1}R_j\setminus 2^kR_j} \left|b(x) - m_{\widetilde{2^{k+1}R_j}}(b)\right| \left|K(x,\,y) - K(x,\,x_{Q_j})\right|\, d\mu(x)\\
&\lesssim \sum_{k=1}^{\infty} K_{\widetilde{Q_j},\,\widetilde{2^{k+1}R_j}} \int_{2^{k+1}R_j\setminus 2^kR_j} \left|K(x,\,y) - K(x,\,x_{Q_j})\right|\, d\mu(x)
\end{aligned}$$

$$+\sum_{k=1}^{\infty}\mu\left(2^{k+2}R_j\right)\left\|b-m_{\widetilde{2^{k+1}R_j}}(b)\right\|_{\exp L^r(\mu),\,2^{k+1}R_j}$$
$$\times\left\|\left\{K(\cdot,\,y)-K(\cdot,\,x_{Q_j})\right\}\chi_{2^{k+1}R_j\setminus 2^kR_j}(\cdot)\right\|_{L(\log L)^{1/r}(\mu),\,2^{k+1}R_j}.$$

Let

$$\lambda_k=\left[\mu\left(2^{k+2}R_j\right)\right]^{-1}\left(k\int_{2^{k+1}R_j\setminus 2^kR_j}\left|K(x,\,y)-K(x,\,x_{Q_j})\right|\,d\mu(x)+2^{-k}\right).$$

By (1.2), we then have that for $y\in R_j$,

$$\begin{aligned}&\frac{1}{\mu(2^{k+2}R_j)}\int_{2^{k+1}R_j\setminus 2^kR_j}\frac{|K(x,\,y)-K(x,\,x_{Q_j})|}{\lambda_k}\\&\quad\times\log^{1/r}\Big(2+\frac{|K(x,\,y)-K(x,\,x_{Q_j})|}{\lambda_k}\Big)\,d\mu(x)\\&\lesssim\frac{1}{\mu(2^{k+2}R_j)}\int_{2^{k+1}R_j\setminus 2^kR_j}\frac{|K(x,\,y)-K(x,\,x_{Q_j})|}{\lambda_k}\\&\quad\times\log^{1/r}\left(2+\frac{1}{\lambda_k|x-y|^n}+\frac{1}{\lambda_k|x-x_{Q_j}|^n}\right)d\mu(x)\\&\lesssim\frac{k}{\mu(2^{k+2}R_j)}\int_{2^{k+1}R_j\setminus 2^kR_j}\frac{|K(x,\,y)-K(x,\,x_{Q_j})|}{\lambda_k}\,d\mu(x)\\&\lesssim 1.\end{aligned}$$

Thus,

$$\left\|\left\{K(\cdot,\,y)-K(\cdot,\,x_{Q_j})\right\}\chi_{2^{k+1}R_j\setminus 2^kR_j}(\cdot)\right\|_{L(\log L)^{1/r}(\mu),\,2^{k+1}R_j}\lesssim\lambda_k.$$

This via (1.9) tells us that

$$\begin{aligned}&\int_{\mathbb{R}^d\setminus 2R_j}\left|b(x)-m_{\widetilde{Q}_j}(b)\right|\left|K(x,\,y)-K(x,\,x_{Q_j})\right|\,d\mu(x)\\&\lesssim\sum_{k=1}^{\infty}\left(k\int_{2^{k+1}R_j\setminus 2^kR_j}\left|K(x,\,y)-K(x,\,x_{Q_j})\right|\,d\mu(x)+2^{-k}\right)\\&\lesssim 1.\end{aligned}$$

Therefore,

$$\begin{aligned}\mu\left(\left\{x\in\mathbb{R}^d:\,\mathrm{E}_1(x)>\lambda\right\}\right)&\lesssim\frac{1}{\lambda}\sum_j\int_{R_j}\int_{\mathbb{R}^d\setminus 2R_j}\left|b(x)-m_{\widetilde{Q}_j}(b)\right|\\&\qquad\times\left|K(x,\,y)-K(x,\,x_{Q_j})\right|\left|h_j(y)\right|d\mu(x)\,d\mu(y)\\&\lesssim\frac{1}{\lambda}\sum_j\int_{R_j}|h_j(y)|\,d\mu(y)\\&\lesssim\frac{1}{\lambda}\int_{\mathbb{R}^d}|f(y)|\,d\mu(y),\end{aligned}$$

which along with the estimates for $E_2(x)$, $E_3(x)$, $E_4(x)$ and $E_5(x)$ gives the desired estimate for $E(x)$. Combining the estimates for $E(x)$ and $F(x)$ yields the estimate (2.5) and then completes the proof of $m = 1$.

Step II. $m \geq 2$. In this case, we need more notation. For $0 \leq i \leq m$, we denote by C_i^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(i)\}$ of $\{1, 2, \dots, m\}$ with i different elements. For $\sigma \in C_i^m$, the complementary sequence σ' is given by $\sigma' = \{1, 2, \dots, m\} \setminus \sigma$. If $\sigma = \emptyset$, we set $\sigma' = \{1, \dots, m\}$. For any i-tuple $r = (r_1, r_2, \dots, r_i)$, we write $1/r_\sigma = 1/r_{\sigma(1)} + \cdots + 1/r_{\sigma(i)}$ and $1/r_{\sigma'} = 1/r - 1/r_\sigma$, where $1/r = 1/r_1 + \cdots + 1/r_m$. Let $\vec{b} = (b_1, b_2 \dots, b_m)$ be a finite family of locally integrable functions. For $\sigma \in C_i^m$, we set $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(i)})$ and the product $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(i)}$. For $\sigma = \emptyset$, we define $b_\sigma = 1$. With this notation, we write

$$\|\vec{b}_\sigma\|_{\mathrm{Osc}_{\exp L^{r_\sigma}}(\mu)} = \|b_{\sigma(1)}\|_{\mathrm{Osc}_{\exp L^{r_{\sigma(1)}}}(\mu)} \cdots \|b_{\sigma(i)}\|_{\mathrm{Osc}_{\exp L^{r_{\sigma(i)}}}(\mu)}.$$

For $i \in \{1, \dots, m\}$ and $\sigma \in C_i^m$, we set

$$[b(y) - b(z)]_\sigma = \left[b_{\sigma(1)}(y) - b_{\sigma(1)}(z)\right] \cdots \left[b_{\sigma(i)}(y) - b_{\sigma(i)}(z)\right]$$

and

$$\left[m_{\tilde{Q}}(b) - b(y)\right]_\sigma = \left[m_{\tilde{Q}}(b_{\sigma(1)}) - b_{\sigma(1)}(y)\right] \cdots \left[m_{\tilde{Q}}(b_{\sigma(i)}) - b_{\sigma(i)}(y)\right],$$

where Q is any cube in $\mathbb{R}^d$ and $y,\, z \in \mathbb{R}^d$. For any $\sigma \in C_i^m$, define

$$T^*_{\vec{b}_\sigma}(f)(x) = \left|\sup_{\epsilon>0} \int_{\mathbb{R}^d} K_\epsilon(x,\, y) \prod_{j=1}^{i} \left[b_{\sigma(j)}(x) - b_{\sigma(j)}(y)\right] f(y)\, d\mu(y)\right|.$$

When $\sigma = \{1, \dots, m\}$, we denote $T^*_{\vec{b}_\sigma}$ simply by $T^*_{\vec{b}}$.

Now let $m \geq 2$ be an integer. We assume that (1.10) holds for any $1 \leq i \leq m-1$ and any subset $\sigma \in C_i^m$. For any fixed f and $\lambda > 2^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|$, let Q_j, R_j, θ_j, w_j, g, h and h_j be the same as in Step I. By an argument similar to the estimates for (2.3) and (2.4), it suffices to verify that

$$\mu\left(\left\{x \in \mathbb{R}^d \setminus \cup_j 2Q_j : |T^*_{\vec{b}} h(x)| > \lambda\right\}\right) \lesssim \int_{\mathbb{R}^d} \varphi_{1/r}\left(\frac{|f(x)|}{\lambda}\right) d\mu(x). \tag{2.6}$$

With the aid of the formula that for $y,\, z \in \mathbb{R}^d$,

$$\prod_{i=1}^{m} \left[m_{\tilde{Q}}(b_i) - b_i(z)\right] = \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} [b(y) - b(z)]_{\sigma'} \left[m_{\tilde{Q}}(b) - b(y)\right]_\sigma$$

and the vanishing moment conditions satisfied by h_j for $j \in \mathbb{N}$, it is easy to see that

$$\begin{aligned}
T_{\vec{b}}^* h(x) = & \sup_{\epsilon>0} \left| \sum_j \int_{\mathbb{R}^d} \left\{ K_\epsilon(x, y) \prod_{i=1}^m [b_i(x) - b_i(y)] \right.\right. \\
& \left.\left. - K_\epsilon(x, x_{Q_j}) \prod_{i=1}^m \left[b_i(x) - m_{\widetilde{Q}_j}(b_i) \right] \right\} h_j(y)\, d\mu(y) \right| \\
\le & \sup_{\epsilon>0} \sum_j \prod_{i=1}^m \left| b_i(x) - m_{\widetilde{Q}_j}(b_i) \right| \\
& \times \int_{\mathbb{R}^d} \left| K_\epsilon(x, y) - K_\epsilon(x, x_{Q_j}) \right| |h_j(y)|\, d\mu(y) \\
& + T^* \Big(\sum_j \prod_{i=1}^m \left[b_i - m_{\widetilde{Q}_j}(b_i) \right] h_j \Big)(x) \\
& + \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} T_{\vec{b}_{\sigma'}}^* \Big(\sum_j \left[b - m_{\widetilde{Q}_j}(b) \right]_\sigma h_j \Big)(x) \\
= & T_{\vec{b}}^{*,\mathrm{I}}(h)(x) + T_{\vec{b}}^{*,\mathrm{II}}(h)(x) + \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} T_{\vec{b}_{\sigma'}}^{*,\mathrm{III}}(h)(x).
\end{aligned}$$

Similar to the estimate for $\mathrm{E}(x)$ in Step I, we have

$$\mu\left(\left\{x \in \mathbb{R}^d \setminus \cup_j 2Q_j : T_{\vec{b}}^{*,\mathrm{I}}(h)(x) > \lambda\right\}\right) \lesssim \int_{\mathbb{R}^d} \varphi_{1/r}\left(\frac{|f(x)|}{\lambda}\right) d\mu(x).$$

On the other hand, the same argument as that for $\mathrm{F}(x)$ in Step I leads to that

$$\mu\left(\left\{x \in \mathbb{R}^d \setminus \cup_j 2Q_j : T_{\vec{b}}^{*,\mathrm{II}}(h)(x) > \lambda\right\}\right) \lesssim \int_{\mathbb{R}^d} \varphi_{1/r}\left(\frac{|f(x)|}{\lambda}\right) d\mu(x).$$

For each fixed i with $1 \le i \le m-1$, our induction hypothesis now states that

$$\begin{aligned}
& \mu\left(\left\{x \in \mathbb{R}^d : T_{\vec{b}_{\sigma'}}^{*,\mathrm{III}}(h)(x) > \lambda\right\}\right) \\
& \lesssim \sum_j \int_{\mathbb{R}^d} \varphi_{1/r_{\sigma'}}\left(\left| \left[b(x) - m_{\widetilde{Q}_j}(b) \right]_\sigma \right| \frac{|w_j(x) f(x)|}{\lambda} \right) d\mu(x) \\
& \quad + \int_{\mathbb{R}^d} \varphi_{1/r_{\sigma'}}\Big(\sum_j \left| \left[b(x) - m_{\widetilde{Q}_j}(b) \right]_\sigma \right| \frac{|\theta_j(x)|}{\lambda} \Big) d\mu(x) \\
& = \mathrm{M}_\sigma + \mathrm{N}_\sigma.
\end{aligned}$$

Applying the inequality

$$\varphi_{1/r_{\sigma'}}(t_0 t_1 \cdots t_i) \lesssim \varphi_{1/r}(t_0) + \exp t_1^{r_{\sigma(1)}} + \cdots + \exp t_i^{r_{\sigma(i)}},\ t_0, t_1, \ldots, t_i > 0$$

(see Lemma 2.2 in [8]), and the fact (a) in the Calderón-Zygmund decomposition, we then deduce that

$$\begin{aligned}
\mathrm{M}_\sigma &\lesssim \sum_j \int_{\mathbb{R}^d} \varphi_{1/r}\Big(\|\vec{b}_\sigma\|_{\mathrm{Osc}_{\exp L^{r_\sigma}}(\mu)} \frac{|\chi_{Q_j}(x) f(x)|}{\lambda}\Big)\, d\mu(x)\\
&\quad + \sum_j \sum_{l=1}^{i} \int_{\mathbb{R}^d} \exp\Big(\frac{|b_{\sigma(l)}(x) - m_{\widetilde{Q}_j}(b_{\sigma(l)})|}{\|b_{\sigma(l)}\|_{\mathrm{Osc}_{\exp L^{r_{\sigma(l)}}}(\mu)}} \chi_{Q_j}(x)\Big)^{r_{\sigma(l)}} d\mu(x)\\
&\lesssim \int_{\mathbb{R}^d} \frac{|f(x)|}{\lambda} \log^{1/r}\Big(2 + \frac{|f(x)|}{\lambda}\Big)\, d\mu(x) + \sum_j \mu(2Q_j)\\
&\lesssim \int_{\mathbb{R}^d} \frac{|f(x)|}{\lambda} \log^{1/r}\Big(2 + \frac{|f(x)|}{\lambda}\Big)\, d\mu(x).
\end{aligned}$$

To estimate N_σ, let $r_j = \lambda^{-1}|\theta_j|$, and $\Lambda \subset \mathbb{N}$ be a finite index set. The convexity of $\varphi_{1/r_{\sigma'}}$ says that

$$\begin{aligned}
&\varphi_{1/r_{\sigma'}}\Big(\sum_{j\in\Lambda} \Big|\Big[b(x) - m_{\widetilde{Q}_j}(b)\Big]_\sigma\Big| \frac{|\theta_j(x)|}{\lambda}\Big)\\
&\le \sum_{j\in\Lambda} \Big(\frac{r_j}{\sum_{l\in\Lambda} r_l}\Big) \varphi_{1/r_{\sigma'}}\Big(\Big|\Big[b(x) - m_{\widetilde{Q}_j}(b)\Big]_\sigma\Big| \chi_{R_j}(x) \sum_{l\in\Lambda} r_l\Big)\\
&\lesssim \frac{1}{\sum_{l\in\Lambda} r_l} \varphi_{1/r_{\sigma'}}\Big(\sum_{l\in\Lambda} r_l\Big) \sum_{j\in\Lambda} r_j \varphi_{1/r_{\sigma'}}\Big(\big|\big[b(x) - m_{\widetilde{Q}_j}(b)\big]_\sigma\big| \chi_{R_j}(x)\Big)\\
&\lesssim \log^{1/r_{\sigma'}}\Big(2 + \sum_{l\in\Lambda} r_l\Big) \sum_{j\in\Lambda} r_j \varphi_{1/r_{\sigma'}}\Big(\big|\big[b(x) - m_{\widetilde{Q}_j}(b)\big]_\sigma\big| \chi_{R_j}(x)\Big)\\
&\lesssim \sum_{j\in\Lambda} r_j \varphi_{1/r_{\sigma'}}\Big(\big|\big[b(x) - m_{\widetilde{Q}_j}(b)\big]_\sigma\big| \chi_{R_j}(x)\Big).
\end{aligned}$$

This in turn leads to that

$$\begin{aligned}
\mathrm{N}_\sigma &\lesssim \lambda^{-1} \sum_j \|\theta_j\|_{L^\infty(\mu)} \int_{R_j} \varphi_{1/r_{\sigma'}}\Big(\Big|\Big[b(x) - m_{\widetilde{Q}_j}(b)\Big]_\sigma\Big|\Big)\, d\mu(x)\\
&\lesssim \lambda^{-1} \sum_j \|\theta_j\|_{L^\infty(\mu)} \mu(R_j)\\
&\lesssim \lambda^{-1} \int_{\mathbb{R}^d} |f(x)|\, d\mu(x).
\end{aligned}$$

Therefore, for $1 \le i \le m-1$ and $\sigma \in C_i^m$, we have

$$\mu\Big(\Big\{x \in \mathbb{R}^d : T^{*,\mathrm{III}}_{\vec{b}_{\sigma'}}(h)(x) > \lambda\Big\}\Big) \lesssim \int_{\mathbb{R}^d} \varphi_{1/r}\Big(\frac{|f(x)|}{\lambda}\Big)\, d\mu(x),$$

which completes the proof of (2.6), and hence the proof of Theorem 1.1. □

Acknowledgements. The authors would like to express their sincere thanks to the referee for his careful reading and useful suggestions.

References

[1] G. Hu, Y. Meng, and D. Yang, *Multilinear commutators of singular integrals with non doubling measures*, Integral Equations Operator Theory **51** (2005), no. 2, 235–255.

[2] ———, *Estimates for maximal singular integral operators in non-homogeneous spaces*, Proc. Roy. Soc. Edinburgh Sect. A **136** (2006), no. 2, 351–364.

[3] ———, *Endpoint estimate for maximal commutators with non-doubling measures*, Acta Math. Sci. Ser. B Engl. Ed. **26** (2006), no. 2, 271–280.

[4] ———, *Boundedness of some maximal commutators in Hardy-type spaces with non-doubling measures*, Acta Math. Sin. (Engl. Ser.) **23** (2007), no. 6, 1129–1148.

[5] F. Nazarov, S. Treil, and A. Volberg, *Accretive system Tb-theorems on nonhomogeneous spaces*, Duke Math. J. **113** (2002), no. 2, 259–312.

[6] ———, *Tb-theorem on nonhomogeneous spaces*, Acta Math. **190** (2003), 151–239.

[7] J. Orobitg and C. Pérez, *A_p weights for nondoubling measures in $\mathbb{R}^n$ and applications*, Trans. Amer. Math. Soc. **354** (2002), no. 5, 2013–2033.

[8] C. Pérez and R. Trujillo-González, *Sharp weighted estimates for multilinear commutators*, J. London Math. Soc. **65** (2002), no. 3, 672–692.

[9] X. Tolsa, *BMO, H^1, and Calderon-Zygmund operators for non doubling measures*, Math. Ann. **319** (2001), no. 1, 89–149.

[10] ———, *A $T(1)$ theorem for non-doubling measures with atoms*, Proc. London Math. Soc. (3) **82** (2001), no. 1, 195–228.

[11] ———, *Littlewood-Paley theory and the $T(1)$ theorem with non-doubling measures*, Adv. Math. **164** (2001), no. 1, 57–116.

[12] ———, *The space H^1 for nondoubling measures in terms of a grand maximal operator*, Trans. Amer. Math. Soc. **355** (2003), no. 1, 315–348.

[13] ———, *Painlevé's problem and the semiadditivity of analytic capacity*, Acta Math. **190** (2003), 105–149.

[14] J. Verdera, *The fall of the doubling condition in Calderón-Zygmund theory*, Publ. Mat. **Vol. Extra** (2002), 275–292.

Guoen Hu
Department of Applied Mathematics
University of Information Engineering
Zhengzhou 450002, P. R. China
E-mail address: `huguoen@eyou.com`

Yan Meng
School of Information
Renmin University of China
Beijing 100872, P. R. China
E-mail address: `mengyan77@126.com`

Dachun Yang
School of Mathematical Sciences
Beijing Normal University
Beijing 100875, P. R. China
E-mail address: `dcyang@bnu.edu.cn`