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COHOMOGENEITY ONE RIEMANNIAN MANIFOLDS OF CONSTANT POSITIVE CURVATURE

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Abstract. In this paper we study non-simply connected Riemannian manifolds of constant positive curvature which have an orbit of codimension one under the action of a connected closed Lie subgroup of isometries. When the action is reducible we characterize the orbits explicitly. We also prove that in some cases the manifold is homogeneous.

Introduction

Two isometric actions of two Lie groups G and G' on a Riemannian manifold M are called *orbit equivalent* if there exists an isometry of M which maps G-orbits onto G'-orbits. An isometric action of a connected Lie group G on a Riemannian manifold M is called *polar* if there exists a closed embedded smooth submanifold of M which intersects all orbits orthogonally. Such a submanifold is called a *section* of the group action. A polar action is called *hyperpolar* if the sections are flat in the induced metric. A classification of polar representations $G \longrightarrow SO(n)$ was given by Dadok [4]. He showed that every polar representation is orbit equivalent to an *s-representation* , i.e., isotropy representation of a Riemannian symmetric space. An important special case of hyperpolar actions are cohomogeneity one actions (actions whose principal orbits have codimension one) with closed normal geodesics. Cohomogeneity one Riemannian manifolds have been investigated by several authors(see [1], [2] for example). C. Searle [7] provided a complete classification, up to diffeomorphism, of such manifolds when they are compact, of positive curvature and of dimension less than seven and the seven dimensional case has been studied by F. Podestà and L. Verdiani in [6]. A. Kollross obtained a classification of hyperpolar and cohomogeneity one actions on the irreducible Riemannian symmetric spaces of compact type (cf. [5]). The classification of the general case in positive curvature is still open. In this paper we study cohomogeneity one actions on non-simply connected Riemannian manifolds of constant positive curvature, i.e., without loss of generality we may assume that $M = S^n/\Gamma$, where $\Gamma = \pi_1(M)$. The main result of the paper are Theorem 2.5, Propositions 2.2 and 2.4.

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1. Preliminaries

A cohomogeneity one action on sphere, which is a cohomogeneity two action on the corresponding Euclidean space, is orbit equivalent to isotropy representation of a Riemannian symmetric space of rank two (note that the cohomogeneity of an s-representation is the rank of the corresponding symmetric space). Since a totally geodesic submanifold of $\mathbb{R}^n$ is flat, the polar actions on $\mathbb{R}^n$ are automatically hyperpolar. We recall the following two theorems of Dadok [4]. The first theorem classifies hyperpolar representations $H \longrightarrow SO(n)$ on $\mathbb{R}^n$ and the second one describes the reducible polar actions on $\mathbb{R}^n$.

Theorem 1.1 ([4]). *Suppose H is a compact, connected Lie group and $\rho : H \longrightarrow SO(n)$ is a representation such that the action of H on $\mathbb{R}^n$ is hyperpolar. Then there exists a symmetric space G/K and a linear isometry $A : \mathbb{R}^n \longrightarrow \mathfrak{p}$ such that A maps H-orbits isometrically onto K-orbits of the s-representation on $\mathfrak{p}$ ($\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the cartan decomposition).*

Theorem 1.2 ([4]). *Suppose H is a compact, connected Lie group and $\rho : H \longrightarrow SO(V)$ is a polar representation and $V = V_1 \oplus V_2$ is a direct sum decomposition of V into H-invariant subspaces. Let Σ be a section for V, $\Sigma_i = \Sigma \cap V_i$ and let $H_1 = Z_o(\Sigma_2, H)$ and $H_2 = Z_o(\Sigma_1, H)$ denote the identity component of the centralizers in H of Σ_2 and Σ_1 respectively. Then*

(a) $\Sigma = \Sigma_1 \oplus \Sigma_2$,
(b) *the H_i-action on V_i is polar with Σ_i as section,*
(c) *if $s = s_1 + s_2 \in \Sigma_1 \oplus \Sigma_2$ then $H \cdot s = H_1 \cdot s_1 \times H_2 \cdot s_2$.*

Note that H and $H_1 \times H_2$ are not equal in general. For example consider the action of $H = SU(4) = Spin(6)$ on $\mathbb{R}^{14} = \mathbb{R}^8 \oplus \mathbb{R}^6$ by $\mu_4 + \rho_6$ (μ_n and ρ_n are the standard representations of the groups $SU(n)$ and $SO(n)$ respectively). Then $H_1 = Spin(5) = Sp(2)$, $H_2 = SU(3)$ and the action of $SU(4)$ on $\mathbb{R}^{14}$ is orbit equivalent to the action of $Sp(2) \times SU(3)$ on $\mathbb{R}^8 \oplus \mathbb{R}^6$.

Suppose that the Lie group G acts isometrically on S^n. We say that the action of G on S^n is reducible (resp, irreducible) if the corresponding action of G on $\mathbb{R}^{n+1}$ is reducible (resp, irreducible). Recall that cohomogeneity one actions on the sphere $S^n \subseteq \mathbb{R}^{n+1}$ are hyperpolar (they are hyperpolar on $\mathbb{R}^{n+1}$ as well). Also recall that if a connected Lie group G acts by cohomogeneity one on a connected, compact Riemannian manifold M of positive sectional curvature, then the orbit space is homeomorphic to $[0, 1]$, therefore M has exactly two singular orbits.

It is known that irreducible cohomogeneity one actions on spheres are s-representations of irreducible Riemannian symmetric spaces of rank two (see [8]).

In main result we need the following theorem (see [9] pages 88,89) which gives all homogeneous manifolds of constant positive curvature. For definition of groups D^*_m, T^*, O^* and I^* see chapter 2 of [9].

Theorem 1.3 ([9]). *Let M^n be a connected homogeneous Riemannian manifold of dimension n and constant curvature $K > 0$.*

(a) *M^n is isometric to the manifold S^n/Γ, where* (i) *$\mathbf{F}$ is a field $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ (quaternions),* (ii) *S^n is the sphere $|x| = K^{-1/2}$ in a left hermitian vector space V over $\mathbf{F}$, where V has real dimension $n+1$,* (iii) *Γ is a finite multiplicative group of elements of norm 1 in $\mathbf{F}$ which is not contained in a proper subfield F_1, $\mathbb{R} \subseteq F_1 \subseteq \mathbf{F}$, and* (iv) *$\Gamma$ acts on S^n by $\mathbf{F}$-scalar multiplication of vectors.*

Conversely, all the manifolds listed are n-dimensional Riemannian homogeneous manifolds of constant curvature $K > 0$.

(b) *M^n is determined up to isometry by the fundamental group $\pi_1(M^n)$. The only cases are* (i) *$M^n = S^n$;* (ii) *$M^n = \mathbb{R}P^n = S^n/\{\pm I\}$;* (iii) *$n+1 \equiv 0 \pmod 2$ while $M^n = S^n/\mathbb{Z}_m$ with $m > 2$;* (iv) *$n+1 \equiv 0 \pmod 4$ while $M^n = S^n/D^*_m$ with $m > 2$ or $M^n = S^n/T^*$ or $M^n = S^n/O^*$ or $M^n = S^n/I^*$.*

2. Main results

Assume that G acts by cohomogeneity one on S^n and the action is reducible, by Theorem 1.2 there is a G-invariant decomposition of $\mathbb{R}^{n+1}$ as $= V_1 \oplus V_2$. If G has a fixed point $x_0 \in S^n$, then one of the V_i's, say V_1, has dimension one and G fixes the line V_1 point-wise. So there exist two singular orbits $\{x_0\}$, $\{-x_0\}$ and the principal orbits are spheres. If G does not have any fixed point, i.e., $\dim V_i - 1 = m_i > 0$, $i = 1,2$ then the two singular orbits are $S_i = S^{m_i} \subseteq V_i$, $i = 1,2$. Since $S^{m_1}(c_1) \times S^{m_2}(c_2)$, $c_1^2 + c_2^2 = 1$, is invariant under G, the principal orbits are isometric to $S^{m_1}(c_1) \times S^{m_2}(c_2)$, $c_1^2 + c_2^2 = 1$. So we get the following proposition.

Proposition 2.1. *Let G be a connected closed Lie subgroup of isometries of S^n which acts by cohomogeneity one on S^n and its action is reducible, then one of the following cases can happen.*

(a) *There are two zero dimensional singular orbits $\{x_0\}$ and $\{-x_0\}$. Each principal orbit is isometric to $S^{n-1}(c)$, where $0 < c \leq 1$ varies with orbits and c is equal to 1 for only one orbit. If $0 < c < 1$, there are two principal orbits isometric to $S^{n-1}(c)$.*

(b) *There are two singular orbits D_1, D_2 isometric to S^{m_1} and S^{m_2}, where $m_1 + m_2 = n-1$, $m_1, m_2 > 0$ and each principal orbit is isometric to $S^{m_1}(c_1) \times S^{m_2}(c_2)$, where $c_1^2 + c_2^2 = 1$. c_1, c_2 depend on orbits and for each $c_1, c_2 > 0$ satisfying the relation $c_1^2 + c_2^2 = 1$, there is exactly one principal orbit isometric to $S^{m_1}(c_1) \times S^{m_2}(c_2)$.*

From now on we assume that M is a complete, connected, non-simply connected and n-dimensional Riemannian manifold of constant curvature 1, so its universal covering is S^n hence $M = S^n/\Gamma$, where Γ ($\cong \pi_1(M)$) is the group of deck transformations of $S^n \longrightarrow M$. We identify $\pi_1(M)$ with Γ which acts freely on S^n. Recall that if a connected Lie group G acts on M then there exists a covering group G' of G which acts on S^n and $\ker(G' \longrightarrow G)$ is a subgroup of Γ. As a consequence G' commutes with Γ. First we study reducible actions. By using Proposition 2.1 we get

Proposition 2.2. *Let M be a complete non-simply connected Riemannian manifold whose universal covering is S^n and is of cohomogeneity one under the action of a connected closed Lie subgroup $G \subseteq Iso(M)$. If the action of G' on S^n is reducible then one of the following cases occurs:*

(a) *$M = \mathbb{R}P^n$ and there are two singular orbits, one is zero dimensional and the other one is isometric to RP^{n-1}. Each principal orbit is isometric to $S^{n-1}(c)$, where $0 < c < 1$ varies with orbits.*

(b) *There are two singular orbits isometric to $\frac{S^{m_1}}{\Gamma}$ and $\frac{S^{m_2}}{\Gamma}$, where m_1 and m_2 are positive integers with $m_1 + m_2 = n - 1$. Each principal orbit is isometric to $\frac{S^{m_1}(c_1) \times S^{m_2}(c_2)}{\Gamma}$, $c_1^2 + c_2^2 = 1$ (c_1, c_2 varies with orbits).*

Proof. Let Γ be the deck transformations of $S^n \longrightarrow M$. By Proposition 2.1 we have the following two cases.

(a) Case (a) of Proposition 2.1 happens for the action of G' on S^n. In this case for $f \in \Gamma$ either $f(x_0) = x_0$ or $f(x_0) = -x_0$, hence $\Gamma = \{\pm I\}$, therefore $M = S^n/\{\pm I\} = \mathbb{R}P^n$.

(b) Case (b) of Proposition 2.1 happens for the action of G' on S^n. Since each $f \in \Gamma$ decomposes as $\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$, Γ sends each orbit to itself hence the covering map $S^n \longrightarrow M$ maps each orbit to its quotient by Γ. □

A variant of Propositions 2.1 and 2.2 has appeared in a preprint of R. Mirzaie and S. M. B. Kashani, which we benefited from. We observed in Proposition 2.2 that if M^n is a cohomogeneity one Riemannian manifold of constant curvature 1 then there are orbits in M isometric to $\frac{S^{m_i}}{\Gamma}$, $i = 1, 2$, $\Gamma = \pi_1(M)$. Using Theorem 1.3 we get the following corollary.

Corollary 2.3. *Let M^n be a cohomogeneity one Riemannian G-manifold of constant curvature one. Assume that the action of G' on $\tilde{M} = S^n$ is reducible. Then $\pi_1(M)$ is one of the groups $\mathbb{Z}_m$, D_m^*, T^*, O^*, I^*.*

Via the group Γ we study (the actions of) G and G' in more detail. Let $\psi = \psi_1 \oplus \psi_2$ be the reducible action of G' on $\mathbb{R}^{n+1} = V_1 \oplus V_2$. We assume that $\dim V_i - 1 = m_i > 0$, i.e., G has no fixed point in S^n. Recall that G' is a subgroup of the centralizer of Γ in $O(n+1)$ and $G = \frac{G' \cdot \Gamma}{\Gamma}$.

We need the list of compact connected Lie groups which act effectively and transitively on sphere. A. Borel have classified them (cf. [3]) and obtained the following list which we call it the Borel list.

$SO(n)$, $SU(n)$, $U(n)$, $Sp(n)$, $T^1 \cdot Sp(n)$, $Sp(1) \cdot Sp(n)$, $Spin(7)$, $Spin(9)$, G_2.

Proposition 2.4. *With assumptions as in Proposition 2.2, one of the following holds*

(i) *If* $\Gamma = \mathbb{Z}_2$ *then* $\psi_i(G')$, $i = 1, 2$ *can be any group of the Borel list.*

(ii) *If* $\Gamma = \mathbb{Z}_m$, $m > 2$ *then* $\psi_i(G')$, $i = 1, 2$ *can be one of the following groups:*

$$SU(k),\ Sp(k),\ U(k),\ T^1 \cdot Sp(k),\ Sp(1) \cdot Sp(k)$$

(iii) *If* $\Gamma = D_m^*$, $m > 2$ *or* T^* *or* O^* *or* I^* *then* $\psi_i(G')$ *can be* $Sp(k)$.

Proof. Case (i) is obvious. In case (ii), by Theorem 1.3, $n+1$ is even, since the action of Γ on S^n is fixed point free, $m_1 + 1$ and $m_2 + 1$ are even too and Γ acts on V_i by $\mathbb{C}$-scalar multiplication of vectors. G' commutes with Γ and among the groups of the Borel list only groups indicated in (ii) of proposition (whose actions are complex) commute with Γ, so (ii) is obtained. In case (iii) $n+1, m_1+1, m_2+1 \equiv 0 (\mathrm{mod}\ 4)$ and Γ acts on V_i by $\mathbb{H}$-scalar multiplication of vectors. The groups $SO(k)$, $SU(k)$, $U(k)$, $Spin(7)$, $Spin(9)$, G_2 do not commute with Γ, also the action of the group $Sp(1) \cdot Sp(k)$ does not commute with multiplication by quaternions. Hence $\psi_i(G')$ can be only $Sp(k)$. $\square$

Remark. In case (i) $\Gamma = \{\pm I\}$ and $M = \mathbb{R}P^n$. If $-I \in G'$ then $G = G'/\Gamma$, if $-I \notin G'$ then $G' = G$. Note that if $n+1$ is odd, only the case (i) can happen and if $n + 1 \equiv 2 (\mathrm{mod}\ 4)$ the cases (i) and (ii) can happen.

According to the above Proposition it may happen that the actions of two Lie groups G_1 and G_2 on S^n are orbit equivalent but the quotient manifolds $M = S^n/\Gamma$ are different as $\frac{G_i \cdot \Gamma}{\Gamma}$ manifold, $i = 1, 2$ since Γ depends on G_1 and G_2. For example the actions of the groups $G_1 = Sp(2) \times Sp(3)$, $G_2 = SU(4) \times U(6)$ and $G_3 = SO(8) \times SO(12)$ on S^{19} are orbit equivalent but for G_3 the group Γ can be $\mathbb{Z}_2$, for G_2 the group Γ can be $\mathbb{Z}_m$ and for G_1 the group Γ can be $\mathbb{Z}_m$ or $D_p^*, p > 2$ or T^* or O^* or I^*.

The following example shows that in Corollary 2.3 the manifold M is not necessarily homogeneous.

Example (1). Let M be the Lens space S^3/Γ, where Γ is generated by

$$\begin{pmatrix} R(1/n) & 0 \\ 0 & R(k/n) \end{pmatrix},$$

where $R(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}$ and $(n, k) = 1$, $n > k > 1$. Since $1 < k < n$, Γ does not act by scalar multiplication of vectors hence by Theorem 1.3 the manifold M is not homogeneous (one can easily see that the dimension of the centralizer of Γ in $O(4)$ is two). The group $SO(2) \times SO(2) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, B \in SO(2) \right\} \subseteq O(4)$ acts by cohomogeneity one on S^3 and Γ is an abelian normal subgroup of this group, so we have the action of the group $G = \frac{SO(2) \times SO(2)}{\Gamma}$ on $M = S^3/\Gamma$ which is of cohomogeneity one.

Now we assume that the action of G' on $\mathbb{R}^{n+1}$ is irreducible. Consider the simple case $\Gamma = \mathbb{Z}_2 = \{\pm I\}$, i.e., $M = \mathbb{R}P^n$. Let $P_1 = G'x_1$ be a singular orbit in S^n. If $-x_1 \notin P_1$ then $P_2 = G'(-x_1)$ is the other singular orbit and Γ maps these two orbits to each other, so $\pi(P_1) = \pi(P_2)$ is a singular orbit in M and the other singular orbit is exceptional. If $-x_1 \in P_1$ then for the second singular orbit $P_2 = G'x_2$, $-x_2 \in P_2$ as well, therefore $Q_i = \pi(P_i)$, $i = 1, 2$ are singular orbits of M. Note that for the reducible action of G' we got only the latter case.

Theorems 6.1.11 and 6.3.1 of [9] give all finite groups Γ which act on sphere without fixed point. Let Γ be of type I introduced in Theorem 6.1.11. It has two generators A and B with relations $A^m = B^q = 1$, $BAB^{-1} = A^r$, $(q(r-1), m) = 1$ and if d is the order of r in the group $U_m = \{a \in \mathbb{N} : (a, m) = 1,\ a \leq m\}$ then $d \mid q$ and q/d is divisible by every prime divisor of d. Note that if $n+1 \equiv 2 (\mathrm{mod}\ 4)$ Γ is just of type I. We assume that $|\Gamma| > 2$ then $n+1 = 2p$. The group Γ has $2d$-dimensional representation $\hat{\pi}_{k,l}$, where $(k, m) = 1 = (l, n)$ as follows:

$$\hat{\pi}_{k,l}(A) = \begin{pmatrix} R(\frac{k}{m}) & & & \\ & R(\frac{kr}{m}) & & \\ & & \ddots & \\ & & & R(\frac{kr^{d-1}}{m}) \end{pmatrix},$$

$$\hat{\pi}_{k,l}(B) = \begin{pmatrix} 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \\ R(l/q') & 0 & \cdots & 0 \end{pmatrix},$$

where $R(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}$ and $q = q'd$. If $p = sd$ then a fixed point free representation of Γ on $\mathbb{R}^{n+1}$ is

$$\varphi = \hat{\pi}_{k_1,l_1} \oplus \cdots \oplus \hat{\pi}_{k_s,l_s} \tag{2.1}$$

(see chapters 5-7 of [9]). If two representations φ_1 and φ_2 of Γ are equivalent then the manifolds M_1 and M_2 are isometric, in fact they are conjugate. So we identify equivalent representations, in particular since $\hat{\pi}_{k_i,l_i}$ and $\hat{\pi}^{-1}_{k_i,l_i}$ are equivalent, we assume that $\hat{\pi}_{k_i,l_i} \neq \hat{\pi}^{-1}_{k_j,l_j}$ for $i \neq j$. When Γ is of type I we get the following theorem.

Theorem 2.5. *Let $M = S^n/\Gamma$, where Γ is of type I introduced above. If M is a cohomogeneity one G-manifold and the action of G' on S^n is irreducible then M is a homogeneous manifold.*

Proof. The covering group G' of G, which acts by cohomogeneity one on S^n, is a subgroup of $C^o(\Gamma) := C^o_{O(n+1)}(\varphi(\Gamma))$ = the identity component of the centralizer of $\varphi(\Gamma)$ in $O(n+1)$. By calculating the group $C(\Gamma)$ we obtain some restrictions on G'. If $|\Gamma| = 2$ then $M = \mathbb{R}P^n$, so we assume that $|\Gamma| > 2$. If $d = 1$ then $r = m = 1$ and $\Gamma = \langle B \rangle \cong \mathbb{Z}_q$ has two dimensional representation $\varphi_k = R(k/q)$ and by (2.1) a $2p$-dimensional representation of Γ is of the form

$\varphi = \varphi_{k_1} \oplus \cdots \oplus \varphi_{k_p}$ whose matrix is $\mathrm{diag}(R(k_1/q), \ldots, R(k_p/q))$. Note that φ_{k_i} and $\varphi_{k_i}^{-1}$ are equivalent, so in the decomposition of φ we assume that $\varphi_{k_i} \neq \varphi_{k_j}^{-1}$ for $i \neq j$. By Theorem 1.3 $M = S^n/\Gamma$ is homogeneous if and only if all φ_{k_i}'s are equal and in this case $C^o(\Gamma) \cong U(n)$. If φ_{k_i}'s are not equal and we assume that $\varphi_{k_1} = \cdots = \varphi_{k_l}$, $\varphi_{k_{l+1}} = \cdots = \varphi_{k_p}$, $\varphi_{k_{l+1}} \neq \varphi_{k_l}$ then $C^o(\Gamma) \cong U(l) \times U(p-l)$, hence the action of G' is reducible on $\mathbb{R}^{n+1}$. For other cases of φ_{k_i}'s, in addition to the reducibility of the action of G' on $\mathbb{R}^{n+1}$, the cohomogeneity of the action is greater than one. So the only possible case is that all φ_{k_i}'s are equal, hence M is homogeneous. If $d > 1$ and $p = sd$ then $\varphi = \hat{\pi}_{k_1,l_1} \oplus \cdots \oplus \hat{\pi}_{k_s,l_s}$. If $s = 1$ then $C^o(\Gamma) \cong T^1 = U(1)$ so G' can not act by cohomogeneity one on S^n unless $n = 2$ which implies $|\Gamma| = 2$ and $M = \mathbb{R}P^2$. If $s > 1$ and all $\hat{\pi}_{k_i,l_i}$ are equal then $C(\Gamma)$ is isomorphic to a group of matrices of the form $\mathrm{diag}(D, \ldots, D)$, where D is a $2s \times 2s$-matrix of the form

$$\begin{pmatrix} A & A_1 & A_2 & \cdots & A_{s-1} \\ B_1 & A & A_1 & \cdots & A_{s-2} \\ B_2 & B_1 & A & \ddots & A_{s-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ B_{s-1} & B_{s-2} & & \cdots & A \end{pmatrix}, \text{ where } A,\ A_i \text{ and } B_j \text{ have the form } \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Since this group is a subgroup of $O(n+1)$, its dimension is at most $2s$, therefore $n - \mathrm{dim}G' \geq n - 2s = 2s(d-1) - 1 \geq 3$. If some $\hat{\pi}_{k_i,l_i}$s are not equal (again we assumed $\hat{\pi}_{k_i,l_i} \neq \hat{\pi}_{k_j,l_j}^{-1}$) then $C(\Gamma)$ becomes smaller, so the cohomogeneity of the action of G' on S^n is greater than one. □

It is not easy to describe the structure of orbits of an irreducible action of cohomogeneity one on S^n and $M = S^n/\Gamma$. For principal orbits of irreducible actions of cohomogeneity one on S^n we refer the reader to table II of [8], where the isotropy subgroup of a principal point has been given.

The result of Theorem 2.5 is that if $M = S^n/\Gamma$, Γ is of type I, is not homogeneous then it is not of cohomogeneity one under the action of G. Our conjecture is that for other types of the group Γ (introduced in Theorems 6.1.11 and 6.3.1 of [9]) the theorem is valid as well, i.e.,

Conjecture. *Let $M = S^n/\Gamma$ (Γ of types other than I) be a Riemannian manifold of constant curvature 1 and G a closed connected subgroup of $Iso(M)$ such that the action of the covering group G' on S^n is irreducible. If M is not homogeneous then it is not of cohomogeneity one under the action of G.*

In the following examples we have used the fact that if we identify $\mathbb{R}^{nm} \cong \mathbb{R}^n \otimes \mathbb{R}^m$ with $\mathrm{Mat}(n \times m, \mathbb{R})$, the $n \times m$ real matrices, then the standard metric of $\mathbb{R}^{nm}$ is $\langle X, X\rangle = \mathrm{Tr}(XX^t)$, $X \in \mathrm{Mat}(n \times m, \mathbb{R})$ and we have $\langle AXB^t, AXB^t\rangle = \langle X, X\rangle$ for $(A, B) \in SO(n) \times SO(m)$.

The following example is the only one in which a Lie group acts irreducibly and by cohomogeneity one on an even dimensional sphere (see table II of [8]).

Example (2). The action of $G = SO(3)$ on $\mathbb{R}^5 \simeq S_0^2(\mathbb{R}^3)$ given by $\rho(A)X = AXA^t$, $A \in SO(3)$, $X \in S_0^2(\mathbb{R}^3)$ is irreducible and of cohomogeneity one on $S^4 \subseteq \mathbb{R}^5$. One can see, with an easy calculation, that the stabilizer of the point $X = \mathrm{diag}(a, -a, 0)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the stabilizer subgroup of the point $Y = \mathrm{diag}(b, b, -2b)$ is isomorphic to $SO(2) \cdot (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \simeq SO(2) \times \mathbb{Z}_2$. On the other hand the two points Y and $-Y$ are not in the same orbit, so we have two isometric singular orbits $G(Y)$ and $G(-Y)$ and all regular points have stabilizer subgroups isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since $-I_5 \in O(5)$ commutes with the group $G = SO(3) \hookrightarrow SO(5)$ (we identify the group G with its image under ρ), and is not in G we get the action of G on the manifold $S^4/\{\pm I_5\} = \mathbb{R}P^4$ which is of cohomogeneity one. The two singular orbits $G(\pm Y)$ are identified in $\mathbb{R}P^4$ and there is an exceptional orbit correspond to the point $X = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ whose stabilizer subgroup is isomorphic to the dihedral group D_4.

The following is a simple example illustrates Theorem 2.5.

Example (3). Consider the action of the group $G = SO(3) \otimes SO(2) \subseteq SO(6)$ on $\mathbb{R}^3 \otimes \mathbb{R}^2 = \mathbb{R}^6$ given by $(A, B) \cdot X = AXB^t$, $X \in \mathrm{Mat}_{3\times 2}(\mathbb{R})$. The action is irreducible and of cohomogeneity one on S^5. Since $(I_3, -I_2) \cdot X = -X$, in contrast to the previous example, X and $-X$ are in the same orbit. The two singular orbits are $G(X_1)$ and $G(X_2)$, where $X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ whose stabilizer subgroup is isomorphic to $SO(2) \times \mathbb{Z}_2$ and $X_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}$ whose stabilizer subgroup is isomorphic to $SO(2)$. Regular points have stabilizer subgroups isomorphic to $\mathbb{Z}_2$. The group $\Gamma \cong \mathbb{Z}_n$ as a subgroup of $\{(I_3, B) : B \in SO(2)\} \subseteq Z(G)$ commutes with G and we get the action of the group G/Γ on S^5/Γ which is of cohomogeneity one. Note that $M = S^5/\Gamma$ is homogeneous. (In fact $G = SO(3) \times SO(2) = SO(3) \times U(1)$ is a subgroup of $U(3)$ and we can identify $\mathbb{R}^6$ with $\mathbb{C}^3$).

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