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THE RULE OF TRAJECTORY STRUCTURE AND GLOBAL ASYMPTOTIC STABILITY FOR A FOURTH-ORDER RATIONAL DIFFERENCE EQUATION

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ABSTRACT. In this paper, the following fourth-order rational difference equation

$$x_{n+1} = \frac{x_n^b + x_{n-2}x_{n-3}^b + a}{x_n^b x_{n-2} + x_{n-3}^b + a}, \quad n = 0, 1, 2, \ldots,$$

where $a, b \in [0, \infty)$ and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$, is considered and the rule of its trajectory structure is described clearly out. Mainly, the lengths of positive and negative semicycles of its nontrivial solutions are found to occur periodically with prime period 15. The rule is $1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-$ in a period, by which the positive equilibrium point of the equation is verified to be globally asymptotically stable.

1. Introduction

It is extremely difficult to understand thoroughly the global behaviors of solutions of rational difference equations although they have simple forms (or expressions). One can refer to [1-12], especially [1, 6] for examples to illustrate this.

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order

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greater than one come from the results for rational difference equations. For this, see, for example, the papers in the journal of "Advances in Difference Equations" and the references cited therein. Furthermore, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Recently, S. Kalabuăić and M. R. S. Kulenović [5] considered the rate of convergence of solutions of the following second-order rational difference equation

$$(E_1) \qquad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots,$$

with nonnegative parameters $\alpha, \beta, \gamma, A, B, C$ and nonnegative initial conditions x_{-1}, x_0.

M. R. S. Kulenović et al [7] investigated the global behavior of solutions of the following second-order rational difference equation

$$(E_2) \qquad x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots,$$

where the parameters α, β, A, B, C and the initial conditions x_{-1}, x_0 are nonnegative.

Tim Nesemann [12] utilized the *Strong Negative Feedback Property* of [2] to study the global asymptotic stability of the following third-order rational difference equation

$$(E_3) \qquad x_{n+1} = \frac{x_{n-1} + x_n x_{n-2}}{x_n x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots,$$

where the initial values $x_{-2}, x_{-1}, x_0 \in (0,\ \infty)$.

In this paper we consider the following fourth-order rational difference equation

$$(1) \qquad x_{n+1} = \frac{x_n^b + x_{n-2}x_{n-3}^b + a}{x_n^b x_{n-2} + x_{n-3}^b + a}, \quad n = 0, 1, 2, \ldots,$$

where $a, b \in [0, \infty)$ and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0,\ \infty)$.

It is key for us to find that the lengths of positive and negative semicycles of nontrivial solutions of equation (1) occur periodically with prime period 15. The rule is $1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-$ in a period, by which the positive equilibrium point of the equation is verified to be globally asymptotically stable. Our main idea is to analyze the perturbation of the initial values to the influence of the trajectory structure rule.

According to our knowledge, equation (1) has not been studied so far. Therefore, to study its qualitative properties is theoretically meaningful.

It is easy to see that the positive equilibrium $\bar{x}$ of equation (1) satisfies

$$\bar{x} = \frac{\bar{x}^b + \bar{x}\bar{x}^b + a}{\bar{x}^b\bar{x} + \bar{x}^b + a},$$

from which one can see that equation (1) has a unique positive equilibrium $\bar{x} = 1$.

When $b = 0$, equation (1) is trivial. Hence, we assume in the sequel that $b > 0$.

In the following, we state some main definitions used in this paper.

Definition 1.1. A positive semicycle of a solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to the equilibrium $\bar{x}$, with $l \geq -3$ and $m \leq \infty$ such that

$$\text{either} \quad l = -3 \quad \text{or} \quad l > -3 \quad \text{and} \quad x_{l-1} < \bar{x}$$

and

$$\text{either} \quad m = \infty \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \bar{x}.$$

A negative semicycle of a solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than $\bar{x}$, with $l \geq -3$ and $m \leq \infty$ such that

$$\text{either} \quad l = -3 \quad \text{or} \quad l > -3 \quad \text{and} \quad x_{l-1} \geq \bar{x}$$

and

$$\text{either} \quad m = \infty \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \bar{x}.$$

The length of a semicycle is the number of the total terms contained in it.

Definition 1.2. A solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1) is said to be eventually trivial if x_n is eventually equal to $\bar{x} = 1$; Otherwise, the solution is said to be nontrivial.

For the other concepts in this paper, see [1, 6].

2. Two lemmas

Before to draw a qualitatively clear picture for the positive solutions of equation (1), we first establish two basic lemmas which will play a key role in the proof of our main results.

Lemma 2.1. *A positive solution $\{x_n\}_{n=-3}^{\infty}$ of equation* (1) *is eventually equal to 1 if and only if*

$$(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1)(x_0 - x_{-3}) = 0. \tag{2}$$

Proof. Assume that (2) holds. Then according to equation (1), it is easy to see that the following conclusions hold.

i) If $x_{-2} = 1$, then $x_n = 1$ for $n \geq 4$;

ii) If $x_{-1} = 1$, then $x_n = 1$ for $n \geq 5$;

iii) If $x_0 = 1$, then $x_n = 1$ for $n \geq 6$;

vi) If $x_0 = x_{-3}$, then $x_n = 1$ for $n \geq 7$.

Conversely, assume that

$$(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1)(x_0 - x_{-3}) \neq 0. \tag{3}$$

Then one can show that

$$x_n \neq 1 \quad \text{for any} \quad n \geq 1.$$

Assume the contrary that for some $N \geq 1$,

$$x_N = 1 \quad \text{and that} \quad x_n \neq 1 \quad \text{for} \quad -2 \leq n \leq N-1. \tag{4}$$

Clearly,

$$1 = x_N = \frac{x_{N-1}^b + x_{N-3}x_{N-4}^b + a}{x_{N-1}^b x_{N-3} + x_{N-4}^b + a},$$

which implies $x_{N-1} = x_{N-4}$ and by (3), $N \geq 2$. Thus, from

$$x_{N-4} = x_{N-1} = \frac{x_{N-2}^b + x_{N-4}x_{N-5}^b + a}{x_{N-2}^b x_{N-4} + x_{N-5}^b + a},$$

we can derive $(x_{N-4} - 1)(x_{N-5}^b(x_{N-4} + 1) + a) = 0$, which contradicts (4). □

Remark 2.1. If the initial conditions do not satisfy equality (2), then, for any solution $\{x_n\}$ of equation (1), $x_n \neq 1$ for $n \geq -2$ and $x_n \neq x_{n-3}$ for $n \geq 0$.

Lemma 2.2. *Let $\{x_n\}_{n=-3}^{\infty}$ be a nontrivial positive solution of equation* (1). *Then the following conclusions are true:*

(a) $(x_{n+1} - 1)(x_{n-2} - 1)(x_n^b - x_{n-3}^b) < 0$ *for* $n \geq 0$;
(b) $(x_{n+1} - x_{n-2})(x_{n-2} - 1) < 0$ *for* $n \geq 0$;
(c) $(x_{n+1} - 1)(x_{n-2} - 1)(x_{n-3} - 1) > 0$ *for* $n \geq 1$.

Proof. It follows in light of equation (1) that

$$x_{n+1} - 1 = -\frac{(x_n^b - x_{n-3}^b)(x_{n-2} - 1)}{x_n^b x_{n-2} + x_{n-3}^b + a}, \quad n = 0, 1, 2, \ldots$$

and

$$x_{n+1} - x_{n-2} = \frac{(1 - x_{n-2})(x_n^b(1 + x_{n-2}) + a)}{x_n^b x_{n-2} + x_{n-3}^b + a}, \quad n = 0, 1, 2, \ldots,$$

from which Inequalities (a) and (b) follow. Inequality (b) implies

$$(x_n - x_{n-3})(x_{n-3} - 1) < 0 \qquad \text{for } n \geq 1,$$

which, together with Inequality (a) and by noticing $(x_n^b - x_{n-3}^b)(x_n - x_{n-3}) > 0$ when $b > 0$, indicates Inequality (c). And so the proof is complete. □

3. Main results and their proofs

First we analyze the structure of the semicycles of nontrivial solutions of equation (1). Here, we confine us to consider the situation of the strictly oscillatory solution of equation (1).

Theorem 3.1. *Let $\{x_n\}_{n=-3}^{\infty}$ be any strictly oscillatory solution of equation* (1). *Then, the lengths of positive and negative semicycles of the solution periodically successively occur with prime period 15. And in a period, the rule is* $1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-$.

Proof. By Lemma 2.2 (c), one can see that the length of a negative semicycle is not larger than 3, whereas, the length of a positive semicycle is at most 4. Based on the strictly oscillatory character of the solution, we see, for some integer $p \geq 0$, one of the following four cases must occur:

Case 1: $x_{p-3} > 1, x_{p-2} < 1, x_{p-1} > 1$ and $x_p > 1$;

Case 2: $x_{p-3} > 1, x_{p-2} < 1, x_{p-1} > 1$ and $x_p < 1$;

Case 3: $x_{p-3} > 1, x_{p-2} < 1, x_{p-1} < 1$ and $x_p > 1$;

Case 4: $x_{p-3} > 1, x_{p-2} < 1, x_{p-1} < 1$ and $x_p < 1$.

If Case 1 occurs, it follows from Lemma 2.2 (c) that $x_{p+1} < 1$, $x_{p+2} < 1$, $x_{p+3} > 1$, $x_{p+4} < 1, x_{p+5} > 1$, $x_{p+6} < 1$, $x_{p+7} < 1$, $x_{p+8} < 1, x_{p+9} < 1$, $x_{p+10} > 1$, $x_{p+11} > 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} > 1$, $x_{p+15} > 1$, $x_{p+16} < 1$, $x_{p+17} < 1$, $x_{p+18} > 1$, $x_{p+19} < 1$, $x_{p+20} > 1$, $x_{p+21} < 1$, $x_{p+22} < 1$, $x_{p+23} < 1$, $x_{p+24} < 1$, $x_{p+25} > 1$, $x_{p+26} > 1$, $x_{p+27} > 1$, $x_{p+28} < 1$, $x_{p+29} > 1$, $x_{p+30} > 1$, $x_{p+31} < 1$, $x_{p+32} < 1$, $x_{p+33} > 1$, $x_{p+34} < 1$, $x_{p+35} > 1$, $x_{p+36} < 1$, $x_{p+37} < 1$, $x_{p+38} < 1$, $x_{p+39} < 1$, $x_{p+40} > 1$, $x_{p+41} > 1$, $x_{p+42} > 1$, $x_{p+43} < 1$, $x_{p+44} > 1$, $x_{p+45} > 1$, $x_{p+46} < 1$, $x_{p+47} < 1$, $x_{p+48} > 1$, $x_{p+49} < 1$, $x_{p+50} > 1$, $x_{p+51} < 1$, $x_{p+52} < 1$, $x_{p+53} < 1$, $x_{p+54} < 1$, $x_{p+55} > 1$, $x_{p+56} > 1$, $x_{p+57} > 1$, $x_{p+58} < 1$, $x_{p+59} > 1$, $x_{p+60} > 1$, $x_{p+61} < 1$, $x_{p+62} < 1$, $x_{p+63} > 1$, $x_{p+64} < 1$, $x_{p+65} > 1$, $x_{p+66} < 1$, $x_{p+67} < 1$, $x_{p+68} < 1$, $x_{p+69} < 1$, ...

It means that the rule for the lengths of positive and negative semicycles of the solution of equation (1) to successively occur is $\ldots, 1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, \ldots$.

If Case 2 occurs, then Lemma 2.2 (c) implies that $x_{p+1} < 1$, $x_{p+2} < 1$, $x_{p+3} < 1$, $x_{p+4} > 1$, $x_{p+5} > 1$, $x_{p+6} > 1$, $x_{p+7} < 1$, $x_{p+8} > 1, x_{p+9} > 1$, $x_{p+10} < 1$, $x_{p+11} < 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} > 1$, $x_{p+15} < 1$, $x_{p+16} < 1$, $x_{p+17} < 1$, $x_{p+18} < 1$, $x_{p+19} > 1$, $x_{p+20} > 1$, $x_{p+21} > 1$, $x_{p+22} < 1$, $x_{p+23} > 1$, $x_{p+24} > 1$, $x_{p+25} < 1$, $x_{p+26} < 1$, $x_{p+27} > 1$, $x_{p+28} < 1$, $x_{p+29} > 1$, $x_{p+30} < 1$, $x_{p+31} < 1$, $x_{p+32} < 1$, $x_{p+33} < 1$, $x_{p+34} > 1$, $x_{p+35} > 1$, $x_{p+36} > 1$, $x_{p+37} < 1$, $x_{p+38} > 1$, $x_{p+39} > 1$, $x_{p+40} < 1$, $x_{p+41} < 1$, $x_{p+42} > 1$, $x_{p+43} < 1$, $x_{p+44} > 1$, $x_{p+45} < 1$, $x_{p+46} < 1$, $x_{p+47} < 1$, $x_{p+48} < 1$, $x_{p+49} > 1$, $x_{p+50} > 1$, $x_{p+51} > 1$, ... This shows the rule for the numbers of terms of positive and negative semicycles of the solution of equation (1) to successively occur still is $4^-, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, \ldots$.

When Case 3 or Case 4 happens, a similar deduction leads to that $x_{p+1} < 1$, $x_{p+2} > 1$, $x_{p+3} < 1$, $x_{p+4} < 1, x_{p+5} < 1$, $x_{p+6} < 1$, $x_{p+7} > 1$, $x_{p+8} > 1, x_{p+9} > 1$, $x_{p+10} < 1$, $x_{p+11} > 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} < 1$, $x_{p+15} > 1$, $x_{p+16} < 1$, $x_{p+17} > 1$, $x_{p+18} < 1$, $x_{p+19} < 1$, $x_{p+20} < 1$, $x_{p+21} < 1$, $x_{p+22} > 1$, $x_{p+23} > 1$, $x_{p+24} > 1$, $x_{p+25} < 1$, $x_{p+26} > 1$, $x_{p+27} > 1$, $x_{p+28} < 1$, $x_{p+29} < 1$, $x_{p+30} > 1$, $x_{p+31} < 1$, $x_{p+32} > 1$, $x_{p+33} < 1$, $x_{p+34} < 1$, $x_{p+35} < 1$, $x_{p+36} < 1$, $x_{p+37} > 1$, $x_{p+38} > 1$, $x_{p+39} > 1$, $x_{p+40} < 1$, $x_{p+41} > 1$, $x_{p+42} > 1$, $x_{p+43} < 1$, $\ldots$,

or

$x_{p+1} < 1$, $x_{p+2} > 1$, $x_{p+3} > 1$, $x_{p+4} > 1, x_{p+5} < 1$, $x_{p+6} > 1$, $x_{p+7} > 1$, $x_{p+8} < 1, x_{p+9} < 1$, $x_{p+10} > 1$, $x_{p+11} < 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} < 1$, $x_{p+15} < 1$, $x_{p+16} < 1$, $x_{p+17} > 1$, $x_{p+18} > 1$, $x_{p+19} > 1$, $x_{p+20} < 1$, $x_{p+21} > 1$, $x_{p+22} > 1$, $x_{p+23} < 1$, $x_{p+24} < 1$, $x_{p+25} > 1$, $x_{p+26} < 1$, $x_{p+27} > 1$, $x_{p+28} < 1$, $x_{p+29} < 1$, $x_{p+30} < 1$, $x_{p+31} < 1$, $x_{p+32} > 1$, $x_{p+33} > 1$, $x_{p+34} > 1$, $x_{p+35} < 1$, $x_{p+36} > 1$, $x_{p+37} > 1$, $x_{p+38} < 1$, $x_{p+39} < 1$, $x_{p+40} > 1$, $x_{p+41} < 1$, $x_{p+42} > 1$, $x_{p+43} < 1, x_{p+44} < 1$, $x_{p+45} < 1$, $x_{p+46} < 1, \ldots$.

Thus, the same regulation is valid for the lengths of positive and negative semicycles which occur successively. The proof is complete. □

Remark 3.1. It is known to all that the four cases in the proof of Theorem 3.1 are caused by the perturbation of the initial around the equilibrium point. So, the theorem 3.1 actually indicates that the perturbation of the initial values may lead to the variation of the trajectory structure rule for the solutions of equation (1).

Next, we state the second main result in this note.

Theorem 3.2. *Assume that $a, b \in [0, \infty)$. Then the positive equilibrium of equation* (1) *is globally asymptotically stable.*

Proof. When $b = 0$, equation (1) is trivial. So, we only consider the case $b > 0$, and prove that the positive equilibrium point $\bar{x}$ of equation (1) is both locally asymptotically stable and globally attractive. The linearized equation of equation(1) about the positive equilibrium $\bar{x} = 1$ is

$$y_{n+1} = 0 \cdot y_n + 0 \cdot y_{n-1} + 0 \cdot y_{n-2} + 0 \cdot y_{n-3}, \quad n = 0, 1, \ldots.$$

By virtue of [6, Remark 1.3.7], $\bar{x}$ is locally asymptotically stable. It remains to verify that every positive solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1) converges to 1 as $n \to \infty$. Namely, we want to prove

$$\lim_{n\to\infty} x_n = \bar{x} = 1. \tag{5}$$

If the initial values of the solution satisfy (2), then Lemma 1 says the solution is eventually equal to 1 and of course, (5) holds. Therefore, we assume in the following that the initial values of the solution do not satisfy (2). Then, by Remark 2.1 we know, for any solution $\{x_n\}$ of equation (1), $x_n \neq 1$ for $n \geq -2$ and $x_n \neq x_{n-3}$ for $n \geq 0$.

If the solution is nonoscillatory about the positive equilibrium point $\bar{x}$ of equation (1), then we know from Lemma 2.2 (c) that the solution is actually an eventually positive one. According to Lemma 2.2 (b), we see that $\{x_{3n}\}$, $\{x_{3n-1}\}$ and $\{x_{3n-2}\}$ are eventually decreasing and bounded from the below by the constant 1. So, the limits

$$\lim_{n\to\infty} x_{3n} = L, \quad \lim_{n\to\infty} x_{3n+1} = M \quad \text{and} \quad \lim_{n\to\infty} x_{3n+2} = N$$

exist and are finite. Note

$$x_{3n+1} = \frac{x_{3n}^b + x_{3n-2}x_{3n-3}^b + a}{x_{3n}^b x_{3n-2} + x_{3n-3}^b + a}, \quad x_{3n+2} = \frac{x_{3n+1}^b + x_{3n-1}x_{3n-2}^b + a}{x_{3n+1}^b x_{3n-1} + x_{3n-2}^b + a}$$

and

$$x_{3n+3} = \frac{x_{3n+2}^b + x_{3n}x_{3n-1}^b + a}{x_{3n+2}^b x_{3n} + x_{3n-1}^b + a},$$

take the limits on both sides of the above equalities and obtain

$$M = \frac{L^b + M \times L^b + a}{L^b \times M + L^b + a}, \quad N = \frac{M^b + N \times M^b + a}{M^b \times N + M^b + a}, \quad \text{and}$$
$$L = \frac{N^b + L \times N^b + a}{N^b \times L + N^b + a}.$$

Solve these equations. We get $L = M = N = 1$, which shows (5) is true.

Thus, it suffices to prove that (5) holds for the solution to be strictly oscillatory.

Consider now $\{x_n\}$ to be strictly oscillatory about the positive equilibrium point $\bar{x}$ of equation (1). By virtue of Theorem 3.1, one understands that the lengths of positive and negative semicycles of the solution periodically successively occur, and in a period, the rule is $1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-$.

For simplicity, for some integer $p \geq 0$, we denote by $\{x_p, x_{p+1}, x_{p+2}, x_{p+3}\}^-$ the terms of a negative semicycle of length four, followed by $\{x_{p+4}, x_{p+5}, x_{p+6}\}^+$ a positive semicycle with length three, then a negative semicycle $\{x_{p+7}\}^-$ and a positive semicycle $\{x_{p+8}, x_{p+9}\}^+$, and so on. Namely, the rule for the lengths of negative and positive semicycles to occur successively can be periodically expressed as follows:

$$\{x_{p+15n}, x_{p+15n+1}, x_{p+15n+2}, x_{p+15n+3}\}^-,$$
$$\{x_{p+15n+4}, x_{p+15n+5}, x_{p+15n+6}\}^+, \{x_{p+15n+7}\}^-, \{x_{p+15n+8}, x_{p+15n+9}\}^+,$$
$$\{x_{p+15n+10}, x_{p+15n+11}\}^-, \{x_{p+15n+12}\}^+, \{x_{p+15n+13}\}^-, \{x_{p+15n+14}\}^+,$$

$n = 0, 1, \ldots.$

Then the following results can be easily observed:

(i) $x_{p+15n+3} > x_{p+15n}$; $x_{p+15n+16} > x_{p+15n+13} > x_{p+15n+10} > x_{p+15n+7}$; $x_{p+15n+18} > x_{p+15n+15}$;

(ii) $x_{p+15n+5} > x_{p+15n+8}$; $x_{p+15n+6} > x_{p+15n+9} > x_{p+15n+12}$;

(iii) $x_{p+15n+7}x_{p+15n+4} > 1$; $x_{p+15n+6}x_{p+15n+3} < 1$; $x_{p+15n+15}\ x_{p+15n+12} > 1$;

(iv) $x_{p+15n+4}x_{p+15n+1} < 1$; $x_{p+15n+5}x_{p+15n+2} < 1$; $x_{p+15n+11}x_{p+15n+8} > 1$;

(v) $x_{p+15n+14}x_{p+15n+11} < 1$; $x_{p+15n+17}x_{p+15n+14} > 1$.

Actually, the inequalities (i) and (ii) are followed straightforward from Lemma 2.2 (b). From the following observations

$$\begin{aligned} x_{p+15n+7} &= \frac{x_{p+15n+6}^b + x_{p+15n+4}x_{p+15n+3}^b + a}{x_{p+15n+6}^b x_{p+15n+4} + x_{p+15n+3}^b + a} \\ &> \frac{x_{p+15n+6}^b + x_{p+15n+4}x_{p+15n+3}^b + a}{x_{p+15n+6}^b x_{p+15n+4} + x_{p+15n+4}^2 x_{p+15n+7}^b + a x_{p+15n+4}} \\ &= \frac{1}{x_{p+15n+4}}, \end{aligned}$$

$$\begin{aligned} x_{p+15n+6} &= \frac{x_{p+15n+5}^b + x_{p+15n+3}x_{p+15n+2}^b + a}{x_{p+15n+5}^b x_{p+15n+3} + x_{p+15n+2}^b + a} \\ &< \frac{x_{p+15n+5}^b x_{p+15n+3} + x_{p+15n+2}^b + a}{x_{p+15n+5}^b x_{p+15n+3} + x_{p+15n+3}^2 x_{p+15n+2}^b + a x_{p+15n+3}} \\ &= \frac{1}{x_{p+15n+3}}, \end{aligned}$$

and

$$\begin{aligned} x_{p+15n+15} &= \frac{x_{p+15n+14}^b + x_{p+15n+12}x_{p+15n+11}^b + a}{x_{p+15n+14}^b x_{p+15n+12} + x_{p+15n+11}^b + a} \\ &> \frac{x_{p+15n+14}^b + x_{p+15n+12}x_{p+15n+11}^b + a}{x_{p+15n+14}^b x_{p+15n+12} + x_{p+15n+12}^2 x_{p+15n+11}^b + a x_{p+15n+12}} \\ &= \frac{1}{x_{p+15n+12}}, \end{aligned}$$

we see that (iii) is true.

Declarations (iv) and (v) can be deduced easily from

$$\begin{aligned} x_{p+15n+4} &= \frac{x_{p+15n+3}^b + x_{p+15n+1}x_{p+15n}^b + a}{x_{p+15n+3}^b x_{p+15n+1} + x_{p+15n}^b + a} \\ &< \frac{x_{p+15n+3}^b + x_{p+15n+1}x_{p+15n}^b + a}{x_{p+15n+3}^b x_{p+15n+1} + x_{p+15n+1}^2 x_{p+15n}^b + a x_{p+15n+1}} \\ &= \frac{1}{x_{p+15n+1}}, \end{aligned}$$

$$\begin{aligned} x_{p+15n+5} &= \frac{x_{p+15n+4}^b + x_{p+15n+2}x_{p+15n+1}^b + a}{x_{p+15n+4}^b x_{p+15n+2} + x_{p+15n+1}^b + a} \\ &< \frac{x_{p+15n+4}^b + x_{p+15n+2}x_{p+15n+1}^b + a}{x_{p+15n+4}^b x_{p+15n+2} + x_{p+15n+2}^2 x_{p+15n+1}^b + a x_{p+15n+2}} \\ &= \frac{1}{x_{p+15n+2}}, \end{aligned}$$

$$\begin{aligned} x_{p+15n+11} &= \frac{x^b_{p+15n+10} + x_{p+15n+8}x^b_{p+15n+7} + a}{x^b_{p+15n+10}x_{p+15n+8} + x^b_{p+15n+7} + a} \\ &> \frac{x^b_{p+15n+10} + x_{p+15n+8}x^b_{p+15n+7} + a}{x^b_{p+15n+10}x_{p+15n+8} + x^2_{p+15n+8}x^b_{p+15n+7} + ax_{p+15n+8}} \\ &= \frac{1}{x_{p+15n+8}}, \end{aligned}$$

and

$$\begin{aligned} x_{p+15n+14} &= \frac{x^b_{p+15n+13} + x_{p+15n+11}x^b_{p+15n+10} + a}{x^b_{p+15n+13}x_{p+15n+11} + x^b_{p+15n+10} + a} \\ &< \frac{x^b_{p+15n+13} + x_{p+15n+11}x^b_{p+15n+11} + a}{x^b_{p+15n+13}x_{p+15n+11} + x^2_{p+15n+11}x^b_{p+15n+10} + ax_{p+15n+11}} \\ &= \frac{1}{x_{p+15n+11}}, \end{aligned}$$

$$\begin{aligned} x_{p+15n+17} &= \frac{x^b_{p+15n+16} + x_{p+15n+14}x^b_{p+15n+13} + a}{x^b_{p+15n+16}x_{p+15n+14} + x^b_{p+15n+13} + a} \\ &> \frac{x^b_{p+15n+16} + x_{p+15n+14}x^b_{p+15n+13} + a}{x^b_{p+15n+16}x_{p+15n+14} + x^2_{p+15n+14}x^b_{p+15n+13} + ax_{p+15n+14}} \\ &= \frac{1}{x_{p+15n+14}}, \end{aligned}$$

respectively.

Now, it follows from (i) - (iii) that

$$\begin{aligned} &\tfrac{1}{x_{p+15n+3}} > x_{p+15n+6} > x_{p+15n+9} > x_{p+15n+12} > \tfrac{1}{x_{p+15n+15}} > \tfrac{1}{x_{p+15n+18}}, \\ &n = 0, 1, 2, \ldots, \end{aligned} \tag{6}$$

which in turn means that $\{x_{p+15n+3}\}_{n=0}^{\infty}$ is increasing with upper bound 1. So, the limit

$$\lim_{n\to\infty} x_{p+15n+3} = L$$

exists and is finite. Accordingly, by view of (6), we obtain

$$\lim_{n\to\infty} x_{p+15n+15} = L$$

and

$$\lim_{n\to\infty} x_{p+15n+6} = \lim_{n\to\infty} x_{p+15n+9} = \lim_{n\to\infty} x_{p+15n+12} = \frac{1}{L}.$$

Next, combining (ii), (iv) and (v), we have

$$\begin{aligned} &\tfrac{1}{x_{p+15n+2}} > x_{p+15n+5} > x_{p+15n+8} > \tfrac{1}{x_{p+15n+11}} > x_{p+15n+14} > \tfrac{1}{x_{p+15n+17}}, \\ &n = 0, 1, 2, \ldots. \end{aligned} \tag{7}$$

It is easy to know from (7) that $\{x_{p+15n+2}\}_{n=0}^{\infty}$ is increasing with upper bound 1. Hence, the limit $\lim_{n\to\infty} x_{p+15n+2} = M$ exists and is finite. It is clear from (7) that

$$\lim_{n\to\infty} x_{p+15n+11} = M$$

and

$$\lim_{n\to\infty} x_{p+15n+5} = \lim_{n\to\infty} x_{p+15n+8} = \lim_{n\to\infty} x_{p+15n+14} = \frac{1}{M}.$$

Finally, from the second inequality of (i) and the first inequalities of (iii) and (iv), one may get

$$\begin{aligned} x_{p+15n+16} >& x_{p+15n+13} > x_{p+15n+10} > x_{p+15n+7} \\ >& \frac{1}{x_{p+15n+4}} > x_{p+15n+1}, \qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{8}$$

It shows $\{x_{p+15n+1}\}_{n=0}^{\infty}$ increasing with upper bound 1, and hence there is an N such that the following limits hold:

$$\begin{aligned} \lim_{n\to\infty} x_{p+15n+1} &= \lim_{n\to\infty} x_{p+15n+7} = \lim_{n\to\infty} x_{p+15n+10} \\ &= \lim_{n\to\infty} x_{p+15n+13} = N \quad \text{and} \\ \lim_{n\to\infty} x_{p+15n+4} &= \frac{1}{N}. \end{aligned}$$

It suffices to verify that $L = M = N = 1$. To this end, noting

$$\begin{aligned} x_{p+15n+6} &= \frac{x_{p+15n+5}^b + x_{p+15n+3}x_{p+15n+2}^b + a}{x_{p+15n+5}^b x_{p+15n+3} + x_{p+15n+2}^b + a}, \\ x_{p+15n+8} &= \frac{x_{p+15n+7}^b + x_{p+15n+5}x_{p+15n+4}^b + a}{x_{p+15n+7}^b x_{p+15n+5} + x_{p+15n+4}^b + a}, \end{aligned}$$

and

$$x_{p+15n+10} = \frac{x_{p+15n+9}^b + x_{p+15n+7}x_{p+15n+6}^b + a}{x_{p+15n+9}^b x_{p+15n+7} + x_{p+15n+6}^b + a},$$

taking the limits on both sides of the above equalities, we obtain

$$\begin{aligned} \frac{1}{L} &= \frac{(\frac{1}{M})^b + L \times M^b + a}{(\frac{1}{M})^b \times L + M^b + a}, \quad \frac{1}{M} = \frac{N^b + \frac{1}{M} \times (\frac{1}{N})^b + a}{N^b + \frac{1}{M} \times (\frac{1}{N})^b + a}, \\ N &= \frac{(\frac{1}{L})^b + N \times (\frac{1}{L})^b + a}{(\frac{1}{L})^b \times N + (\frac{1}{L})^b + a}. \end{aligned}$$

Solving these equations we can derive $L = M = N = 1$. Up to now, we have shown $\lim_{n\to\infty} x_{p+15n+k} = 1, k = 1, 2, \ldots, 15$. So, the proof for Theorem 3.2 is complete. $\square$

Finally, we present the rule of the trajectory structure of equation (1).

Theorem 3.3. *The rule of the trajectory structure of equation* (1) *is that all of its solutions asymptotically approach its equilibrium, furthermore, any one of its solutions is either*

(1) *eventually trivial; or*

(2) *nonoscillatory and eventually positively* (*i.e.*, $x_n \geq 1$); *or*

(3) *strictly oscillatory with the lengths of positive and negative semicycles periodically successively occurring with prime period 15 and the rule to be* $1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-$ *in a period.*

The proof of Theorem 3.3 follows from Lemma 2.1, Theorem 3.1 and Theorem 3.2 and so is omitted here.

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