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PARTIAL SECOND ORDER MOCK THETA FUNCTIONS, THEIR EXPANSIONS AND PADÉ APPROXIMANTS

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Abstract. By proving a summation formula, we enumerate the expansions for the mock theta functions of order 2 in terms of partial mock theta functions of order 2, 3 and 6. We show a relation between Ramanujan's $\mu(q)$-function and his sixth order mock theta functions. In addition, we also give the continued fraction representation for $\mu(q)$ and 2nd order mock theta functions and Padé approximants.

1. Introduction

Ramanujan's last letter to Hardy [10, pp. 354–355] introduced mock theta functions. He listed 17 such functions and assigned them orders of 3, 5 and 7. Watson [12] found three more mock theta functions of order 3; two more of order 5 were found in the "lost" notebook [8]. In his last letter to Hardy, Ramanujan explained the mock theta function.

A mock theta function is a function $f(q)$ defined by a q-series, convergent for $|q| < 1$ which satisfies the following two conditions:

(0) For every root of unity ζ, there is a θ-function $\theta_\zeta(q)$ such that the difference $f(q) - \theta_\zeta(q)$ is bounded as $q \to \zeta$ radially.

(1) There is no single θ-function which works for all ζ i.e., for every θ-function $\theta(q)$ there is some root of unity ζ for which $f(q) - \theta(q)$ is unbounded as $q \to \zeta$ radially.

Gordon and McIntosh [7] developed a method for obtaining mock theta functions from ordinary theta functions by performing certain operations on their q-series expansions. Using this method, they constructed 8 new mock theta functions and called them of order eight.

In 1981, Andrews [2] defined two functions and recently McIntosh [8] considered these two functions and called them of order 2. Using an identity [7, (2.1)], he gave alternative definitions for these two second order mock theta functions. We show that these alternative definitions can be easily obtained by

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using the transformation formula given by Fine in chapter 1 of his remarkable monograph [5], (see also [6, p 11]). By using Heine's transformation, we give yet another definition for these two second order mock theta functions. By using Fine's transformation, these two second order mock theta functions reduce to ${}_2\phi_1$ basic hypergeometric series. Naturally, different transformations of ${}_2\phi_1$ series provide different alternative forms.

In some of Ramanujan's unpublished work, (now published as "Lost" Notebook) a number of identities and expansion formulae for partial sums of theta functions are mentioned without proof. A study of these partial sums, identities and expansions has been made by Andrews. Andrews observed that some of these partial theta function identities have interesting number-theoretic interpretations. These previous studies by mathematicians of partial theta functions motivated us to define and study partial mock theta functions. We define partial mock theta function as the partial sum of the infinite series representing these functions.

In section 3, we give alternative definition for these second order mock theta functions and Ramanujan's $\mu(q)$ function using Heine's transformation.

In section 5, we give a general expansion formula.

In sections 6, 7 and 8 we give relations between mock theta functions and partial mock theta functions of order 2, 3 and 6 and Ramanujan's function $\mu(q)$. $\mu(q)$ appears in a number of identities given by Ramanujan [9]. Andrews [2] has studied this function in detail; its relation with sixth order mock theta function is very interesting.

In section 9, we show a continued fraction representation for second order mock theta functions and $\mu(q)$ and also Padé approximants. Padé approximants [4] has very wide applicability in mathematical approximation and analytical function theory. Padé approximants are a particular type of rational function approximations to the value of a function. The key idea is to match the Taylor series expansion of a function with its rational algebraic expansion, as far as possible.

2. Notation

The following q-notations have been used. For $|q^k| < 1$,

$$\begin{aligned}
(a;q^k)_n &= \prod_{j=0}^{n-1}(1-aq^{kj}), \quad n \geq 1 \\
(a;q^k)_0 &= 1, \\
(a;q^k)_\infty &= \prod_{j=0}^{\infty}(1-aq^{kj}), \\
(a)_n &= (a;q)_n, \\
(a_1,a_2,\ldots,a_m;q^k)_n &= (a_1;q^k)_n(a_2;q^k)_n\cdots(a_m;q^k)_n.
\end{aligned}$$

A generalized basic hypergeometric series with base q_1 is defined as

$$
\begin{aligned}
&{}_A\phi_{A-1}[a_1,a_2,\ldots,a_A;b_1,b_2,\ldots,b_{A-1};q_1,z] \\
&= \sum_{n=0}^{\infty} \frac{(a_1;q_1)_n\cdots(a_A;q_1)_n z^n}{(b_1;q_1)_n\cdots(b_{A-1};q_1)_n(q_1;q_1)_n}, \quad |z|<1.
\end{aligned}
$$

3. Alternative form

Andrews [2] defined the second order mock theta functions as

$$A(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^{(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}}, \tag{3.1}$$

$$B(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^{n}(-q;q^2)_n}{(q;q^2)_{n+1}}, \tag{3.2}$$

and

$$\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2}. \tag{3.3}$$

The last function $\mu(q)$ appears in Ramanujan's "Lost" Notebook. The equality of the two series in (3.1) and (3.2) were shown by McIntosh[8], by using the identity [8, (2.1)].

However, we can prove the equality easily by using the following transformation formula of Fine [5] (see [6, p. 11]):

$$(1-x)_2\phi_1(q,a;b;x) = \sum_{n=0}^{\infty} \frac{(b/a)_n(-ax)^n q^{n(n-1)/2}}{(b)_n(xq)_n}. \tag{3.4}$$

Making $q \to q^2$ and taking $a=-q^2$, $b=q^3$, $x=q$ and $a=-q$, $b=q^3$, $x=q$ respectively, in (3.4), we get the equality in (3.1) and (3.2), respectively.

Letting $q \to q^2$ and taking $a=-q$, $b=-q^2$, $x=-1$ in (3.4), we get

$$\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2} = 2\sum_{n=0}^{\infty} \frac{(-1)^n(-q;q^2)_n}{(-q^2;q^2)_n}, \tag{3.5}$$

which gives an alternate definition for $\mu(q)$. The second series in (3.5) does not converge in the usual sense; however, it converges in the Cesàro sense [3, p.205] which in this case is equal to the limit of the average of consecutive partial sums.

By (3.1),

$$A(q) = \frac{q}{1-q}\,{}_2\phi_1(q^2,-q^2;q^3;q^2,q), \tag{3.6}$$

$$B(q) = \frac{1}{1-q}\,{}_2\phi_1(q^2,-q;q^3;q^2,q), \tag{3.7}$$

and

$$\mu(q) = 2\,{}_2\phi_1(q^2, -q; -q^2; q^2, -1). \tag{3.8}$$

By using the following Heine's transformation formula, we shall give yet another alternative form for these functions:

$$(x, e)_\infty \phi(a, b; e; x) = (b, ax)_\infty \phi(e/b, x; ax; b). \tag{3.9}$$

Letting $q \to q^2$ and taking $a = -q^2$, $b = q^2$, $e = q^3$ and $x = q$ in (3.9) and using (3.6), we get

$$A(q) = \frac{q(q^2; q^2)_\infty(-q^3; q^2)_\infty}{(q; q^2)_\infty^2} {}_2\phi_1(q, q; -q^3; q^2, q^2). \tag{3.10}$$

By interchanging the sign in the values of a and b in the above we have

$$A(q) = \frac{q(-q^2; q^2)_\infty}{(1-q)(q; q^2)_\infty} {}_2\phi_1(-q, q; q^3; q^2, -q^2). \tag{3.11}$$

Similarly for $B(q)$, let $q \to q^2$ and take $a = -q$, $b = q^2$, $e = q^3$ and $x = q$ in (3.9) and use (3.7) to get

$$B(q) = \frac{(q^2; q^4)_\infty}{(q; q^2)_\infty^2} {}_2\phi_1(q, q; -q^2; q^2, q^2). \tag{3.12}$$

For $\mu(q)$, let $q \to q^2$ and take $a = q^2$, $b = -q$, $e = -q^2$, $x = -1$ in (3.9) and use (3.8) to get

$$\mu(q) = \frac{(-q; q^2)_\infty}{(-q^2; q^2)_\infty} {}_2\phi_1(-1, q; -q^2; q^2, -q). \tag{3.13}$$

4. Partial mock theta functions

Partial mock theta functions are defined by taking the partial sums from 0 to N, and we denote by putting a suffix N in the notation for the mock theta functions.

Thus

$$\begin{aligned} A_N(q) &= \frac{q(q^2; q^2)_\infty(-q^3; q^2)_\infty}{(q; q^2)_\infty^2} \sum_{n=0}^{N} \frac{(q; q^2)_n (q; q^2)_n q^{2n}}{(q^2; q^2)_n (-q^3; q^2)_n} \\ &= \frac{q(q^2; q^2)_\infty(-q^3; q^2)_\infty}{(q; q^2)_\infty^2} {}_2\phi_1(q, q; -q^3; q^2, q^2)_N. \end{aligned} \tag{4.1}$$

Similarly for the other two functions.

5. Main expansion formula

We shall be using Abel's summation formula [3, p. 194]

$$\sum_{r=0}^{p}\alpha_r\beta_r=\beta_{p+1}\sum_{r=0}^{p}\alpha_r+\sum_{m=0}^{p}(\beta_m-\beta_{m+1})\sum_{r=0}^{m}\alpha_r \tag{5.1}$$

frequently, for giving a relation between partial mock theta functions and mock theta functions. To get, we choose α_r and β_r so that $\sum_{r=0}^{m}\alpha_r$ is some partial mock theta function, and $\sum_{r=0}^{\infty}\alpha_r\beta_r$ is some mock theta function.

6. Relation between partial mock theta function $A_m(q)$ and mock theta functions $A(q)$, $B(q)$

Case 1. Letting $q\to q^2$ and taking

$$\alpha_r=\frac{q(q^2;q^2)_\infty(-q^2;q^2)_\infty(q;q^2)_r^2q^{2r}}{(q;q^2)_\infty^2(-q^3;q^2)_r(q^2;q^2)_r},$$
$$\beta_r=\frac{(-q^3;q^2)_r}{(-q^2;q^2)_r}$$

in (5.1) and using (3.10) and (3.12), we get

$$\begin{aligned}qB_p(q)-\frac{(-q^{2p+4};q^2)_\infty}{(-q^{2p+5};q^2)_\infty}A_p(q)=&\frac{(1-q)(-q^2;q^2)_\infty}{(-q;q^2)_\infty}\\&\times\sum_{m=0}^{p}\frac{(-q;q^2)_{m+1}q^{2m+2}}{(-q^2;q^2)_{m+1}}A_m(q).\end{aligned}\tag{6.1}$$

Making $p\to\infty$, we have

$$qB(q)-A(q)=\frac{(1-q)(-q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(-q;q^2)_{m+1}q^{2m+2}}{(-q^2;q^2)_{m+1}}A_m(q).\tag{6.2}$$

Case 2. Let $q\to q^2$ and taking

$$\alpha_r=\frac{(q^4;q^4)_\infty}{(q;q^2)_\infty^2}\frac{(q;q^2)_r^2q^{2r}}{(-q^2;q^2)_r(q^2;q^2)_r},$$
$$\beta_r=\frac{(-q^2;q^2)_r}{(-q^3;q^2)_r}$$

in (5.1) and using (3.10) and (3.12), we get

$$\begin{aligned}&\frac{q(-q^{2p+5};q^2)_\infty}{(-q^{2p+4};q^2)_\infty}B_p(q)-A_p(q)\\&=\frac{(1-q)(-q;q^2)_\infty}{(1+q)(-1;q^2)_\infty}\sum_{m=0}^{p}\frac{(-1;q^2)_{m+1}q^{2m+2}}{(-q^3;q^2)_{m+1}}B_m(q).\end{aligned}\tag{6.3}$$

Making $p \to \infty$, we have

$$qB(q) - A(q) = \frac{(1-q)(-q;q^2)_\infty}{(1+q)(-1;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-1;q^2)_{m+1} q^{2m+2}}{(-q^3;q^2)_{m+1}} B_m(q) \tag{6.4}$$

7. Relation between partial mock theta functions of order three and partial mock theta functions of order two

Applying Heine's transformation on Fine's representation of the third order mock theta functions, we get interesting relations between partial third order mock theta functions and partial second order mock theta functions.

The transformed third order mock theta functions are:

$$\psi(q) = \frac{q(q^2,-q^3;q^2)_\infty}{(q;q^2)_\infty} {}_2\phi_1(q,0;-q^3;q^2,q^2) \tag{7.1}$$

and

$$\omega(q) = \frac{(q^2,q^2)_\infty}{(q;q^2)_\infty^2} {}_2\phi_1(q,q;0;q^2,q^2). \tag{7.2}$$

Case 1. Letting $q \to q^2$ and taking

$$\alpha_r = \frac{q(q^2;q^2)_\infty(-q^3;q^2)_\infty(q;q^2)_r^2 q^{2r}}{(q;q^2)_\infty^2(q^2;q^2)_r(-q^3;q^2)_r},$$
$$\beta_r = \frac{1}{(q;q^2)_r}$$

in (5.1) and using (3.10) and (7.1), we get

$$(q^{2p+3};q^2)_\infty A_p(q) - \psi_p(q) = (q;q^2)_\infty \sum_{m=0}^{p} \frac{q^{2m+1}}{(q;q^2)_{m+1}} A_m(q). \tag{7.3}$$

Making $p \to \infty$, we have

$$A(q) - \psi(q) = (q;q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(q;q^2)_{m+1}} A_m(q). \tag{7.4}$$

Case 2. Letting $q \to q^2$ and taking

$$\alpha_r = \frac{(q^2;q^2)_\infty(q;q^2)_r^2 q^{2r}}{(q;q^2)_\infty^2(q^2;q^2)_r},$$
$$\beta_r = \frac{1}{(-q^3;q^2)_r}$$

in (5.1) and using (3.10) and (7.2), we get

$$A_p(q) - q(-q^{2p+5};q^2)_\infty \omega_p(q) = (-q^3;q^2)_\infty \sum_{m=0}^{p} \frac{q^{2(m+2)}}{(-q^3;q^2)_{m+1}} \omega_m(q). \tag{7.5}$$

Making $p \to \infty$, we have

$$A(q) - q\omega(q) = (-q^3;q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{2(m+2)}}{(-q^2;q^2)_{m+1}} \omega_m(q). \tag{7.6}$$

8. Relation between sixth order mock theta functions, Ramanujan's function $\mu(q)$ and second order mock theta functions

Ramanujan's μ-function found in "Lost" Notebook [8] is

$$\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q^2;q^2)_n^2} \tag{8.1}$$

and the sixth order mock theta functions, also found in "Lost" Notebook, are

$$\Phi_L(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q)_{2n}}, \tag{8.2}$$

$$\psi_L(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2} (q;q^2)_n}{(-q)_{2n+1}}, \tag{8.3}$$

or

$$\psi_L(-q) = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{q^{(n+1)^2} (-q;q^2)_n}{(q^2;q^2)_n (q^3;q^2)_n}. \tag{8.4}$$

We have put a suffix L in the definition to differentiate it from other mock theta functions with the same notation.

The expansion (5.1) gives the following curious expansions:

Case 1. Letting $q \to q^2$ and taking

$$\alpha_r = \frac{(-q;q^2)_r q^{(r+1)^2}}{(1-q)(-q^2;q^2)_r (q^3;q^2)_r}$$
$$\beta_r = \frac{(-q^2;q^2)_r}{(q^3;q^2)_r}$$

in (5.1) and using (3.10) and (8.4), we get

$$\begin{aligned} &2A_p(q) + \frac{(-1;q^2)_{p+1}}{(q;q^2)_{p+2}} \psi_L(-q) \\ &= \frac{(1+q)}{(1-q)} \sum_{m=0}^{p} \frac{(-1;q^2)_{m+1} q^{2m+2}}{(q^3;q^2)_{m+1}} \psi_{Lm}(-q). \end{aligned} \tag{8.5}$$

Making $p \to \infty$, we have

$$2A(q) + \frac{(-1;q^2)_\infty}{(q;q^2)_\infty} \psi_L(-q) = \frac{(1+q)}{(1-q)} \sum_{m=0}^{\infty} \frac{(-1;q^2)_{m+1} q^{2m+2}}{(q^3;q^2)_{m+1}} \psi_{Lm}(-q). \tag{8.6}$$

Case 2. Letting $q \to q^2$ and taking

$$\alpha_r = \frac{(-q;q^2)_r q^{(r+1)^2}}{(1-q)^2(q^3;q^2)_r^2},$$
$$\beta_r = \frac{(q^3;q^2)_r}{(-q^2;q^2)_r}$$

in (5.1) and using (3.10) and (8.4), we get

$$\psi_{Lp}(-q) + \frac{2(q;q^2)_{p+2}}{(-1;q^2)_{p+2}} A_p(q) = -(1+q)\sum_{m=0}^{p} \frac{(q;q^2)_{m+1} q^{2m+2}}{(-q^2;q^2)_{m+1}} A_m(q). \tag{8.7}$$

Making $p \to \infty$, we have

$$\psi_L(-q) + \frac{(q;q^2)_\infty}{(-q^2;q^2)_\infty} A(q) = -(1+q)\sum_{m=0}^{\infty} \frac{(q;q^2)_{m+1} q^{2m+2}}{(-q^2;q^2)_{m+1}} A_m(q). \tag{8.8}$$

Case 3. Letting $q \to q^2$ and taking

$$\alpha_r = \frac{(-1)^r q^{r^2}(q;q^2)_r}{(-q^2;q^2)_r^2},$$
$$\beta_r = \frac{(-q^2;q^2)_r}{(-q;q^2)_r},$$

in (5.1) and using (8.1) and (8.2), we get

$$\begin{aligned} &2\Phi_{Lp}(q) - \frac{(-1;q^2)_{p+2}}{(-q;q^2)_{p+1}} \mu_p(q) \\ &= (1-q)\sum_{m=0}^{p} \frac{(-1;q^2)_{m+1} q^{2m+1}}{(-q;q^2)_{m+1}} \mu_m(q). \end{aligned} \tag{8.9}$$

Making $p \to \infty$, we have

$$\begin{aligned} &2\Phi_L(q) - \frac{(-1;q^2)_\infty}{(-q;q^2)_\infty} \mu(q) \\ &= (1-q)\sum_{m=0}^{\infty} \frac{(-1;q^2)_{m+1} q^{2m+1}}{(-q;q^2)_{m+1}} \mu_m(q). \end{aligned} \tag{8.10}$$

Case 4. Letting $q \to q^2$ and taking

$$\alpha_r = \frac{(-1)^r (q;q^2)_r q^{r^2}}{(-q;q^2)_r(-q^2;q^2)_r},$$
$$\beta_r = \frac{(-q;q^2)_r}{(-q^2;q^2)_r}$$

in (5.1) and using (8.1) and (8.2), we get

$$\frac{(-q;q^2)_{p+1}}{(-q^2;q^2)_{p+1}} \Phi_{Lp}(q) - \mu_p(q) = (1-q)\sum_{m=0}^{p} \frac{(-q;q^2)_m q^{2m+1}}{(-q^2;q^2)_{m+1}} \Phi_{Lm}(q). \tag{8.11}$$

Making $p \to \infty$, we have

$$(8.12)\qquad \frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}\Phi_L(q) - \mu(q) = (1-q)\sum_{m=0}^{\infty}\frac{(-q;q^2)_m q^{2m+1}}{(-q^2;q^2)_{m+1}}\Phi_{Lm}(q).$$

9. Continued fraction representation, Padé approximants and convergents for second order mock theta functions

Agarwal [1] has given the continued fraction, Padé approximants and their convergents for ${}_2\phi_1(\alpha, q; \gamma, x)$ as follows:

$$(9.1)\qquad {}_2\phi_1(\alpha, q; \gamma, x) = \cfrac{1}{1 - \cfrac{x\gamma_1}{1 - \cfrac{x\gamma_2}{\ddots}}}$$

where

$$\gamma_{2n+1} = \frac{(1-\alpha q^n)(1-\gamma q^{n-1})q^n}{(1-\gamma q^{2n-1})(1-\gamma q^{2n})},$$

$$(9.2)\qquad \gamma_{2n+2} = \frac{(\alpha - \gamma q^n)(1-q^{n+1})q^n}{(1-\gamma q^{2n})(1-\gamma q^{2n+1})}.$$

The Padé approximants (n, n) and $(n, n+1)$ are

$$R_{n,n} = \frac{P_{n,n}}{Q_{n,n}}$$

where

$$(9.3)\qquad P_{n,n} = \sum_{k=0}^{n}\frac{(\alpha)_k x^k}{(\gamma)_k}\sum_{s=0}^{n-k}\frac{(q^{-n})_s(q^{-n}/\alpha)_s}{(q)_s(q^{1-2n}/\gamma)_s}(x\alpha q/\gamma)^s$$

and

$$(9.4)\qquad Q_{n,n} = {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-n}/\alpha \\ -q^{1-2n}/\gamma \end{matrix}; x\alpha q/\gamma\right),$$

and

$$R_{n,n+1} = \frac{P_{n,n+1}}{Q_{n,n+1}},$$

where

(9.5)

$$P_{n,n+1} = \frac{(-x)^n q^{3n(1-n)/2}(q^2;q)_n(q\alpha;q)_n(q\gamma;q)_n}{(q;q)_n(q\gamma;q)_{2n}} \times \sum_{k=0}^{n}\frac{(q^{-n})_k(\alpha)_k(\gamma q^{n+1})_k q^k}{(\gamma)_k(\alpha q)_k(q^2)_k}\,{}_3\phi_2\left(\begin{matrix} q^{-n+k}, \gamma q^{n+k+1}, q \\ q^{2+k}, \alpha q^{1+k} \end{matrix}; q/x\right),$$

$$(9.6)\qquad Q_{n,n+1} = {}_2\phi_1\left(\begin{matrix} q^{-n-1}, q^{-n}/\alpha \\ q^{-2n}/\gamma \end{matrix}; x\alpha q/\gamma\right).$$

We shall write $A(q)$, given in (3.1), as a ${}_2\phi_1$ series and then use these results of Agarwal [1] to give continued fraction representation and Padé approximants for the second order mock theta function $A(q)$.

Rewriting

$$A(q) = \frac{1}{(1-q)}\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n(q^2;q^2)_n q^{n+1}}{(q^3;q^2)_n(q^2;q^2)_n} \tag{9.7}$$
$$= \frac{q}{1-q}{}_2\phi_1(q^2,-q^2;q^3;q^2,q).$$

In (9.1), let $q \to q^2$ and take $\alpha_r = -q^2$, $\gamma = q^3$, $x = q$, to get the continued fraction representation

$$\frac{1-q}{q}A(q) = \cfrac{1}{1-\cfrac{r_1 q}{1-\cfrac{r_2 q}{1-\cfrac{r_3 q}{1..}}}} \tag{9.8}$$

where

$$r_{2n+1} = \frac{(1-q^{2n+1})(1+q^{2n+2})q^{2n}}{(1-q^{4n+1})(1-q^{4n+3})}$$

and

$$r_{2n+2} = \frac{(1+q^{2n+1})(1-q^{2n+2})q^{2n+2}}{(1-q^{4n+3})(1-q^{4n+5})}. \tag{9.9}$$

Taking the above values of α, γ and x in (9.3) and (9.4) the Padé approximants are, if

$$R_{n,n} = \frac{P_{n,n}}{Q_{n,n}},$$

then

$$P_{n,n} = \sum_{k=0}^{n}\frac{(-q^2;q^2)_k q^k}{(q^3;q^2)_k}\sum_{s=0}^{n-k}\frac{(q^{-2n};q^2)_s(-q^{-2n-2};q^2)_s}{(q^2;q^2)_s(q^{-4n-1};q^2)_s}(-q^2)^s \tag{9.10}$$

and

$$Q_{n,n} = {}_2\phi_1\left(\begin{matrix} q^{-2n}, q^{-2n-2} \\ q^{-4n-1} \end{matrix} ; q^2, -q^2\right),$$

and $(n, n+1)$ approximants by (9.5) and (9.6) are

$$P_{n,n+1} = \frac{(-1)^n q^{n(4-3n)}(q^4;q^2)_n(-q^4;q^2)_n(q^5;q^2)_n}{(q^2;q^2)_n(q^5;q^2)_{2n}}$$
$$\times \sum_{k=0}^{n}\frac{(q^{-2n};q^2)_k(-q^2;q^2)_k(q^{2n+5};q^2)_k q^{2k}}{(q^3;q^2)_k(-q^4;q^2)_k(q^4;q^2)_k} \tag{9.11}$$
$$\times {}_3\phi_2\left(\begin{matrix} q^{-2n+2k}, q^{2n+2k+5}, q^2 \\ q^{4+2k}, -q^{4+2k} \end{matrix} ; q^2, q\right)$$

and

$$Q_{n,n+1} = {}_2\phi_1\left(\begin{matrix} q^{-2n-2}, -q^{-2n-2} \\ q^{-4n-3} \end{matrix} ; q^2, -q^2\right). \tag{9.12}$$

Similarly taking the definition of $B(q)$, given in (3.2), and the definition of $\mu(q)$, given in (3.3), we get the continued fraction representation and the Padé approximants for $B(q)$ and $\mu(q)$.

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