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ANTI-SYMPLECTIC INVOLUTIONS ON NON-KÄHLER SYMPLECTIC 4-MANIFOLDS

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Abstract. In this note we construct an anti-symplectic involution on the non-Kähler, symplectic 4-manifold which is constructed by Thurston and show that the quotient of the Thurston's 4-manifold is not symplectic.

Also we construct a non-Kähler, symplectic 4-manifold using the Gomph's symplectic sum method and an anti-symplectic involution on the non-Kähler, symplectic 4-manifold.

1. Introduction

Let (X, ω) be a closed, symplectic, 4-manifold with a symplectic structure ω. A smooth map $\sigma : X \to X$ is an anti-symplectic involution if and only if $\sigma^*\omega = -\omega$ and $\sigma^2 = \mathrm{Id}$. If X is a Kähler surface, and σ is anti-holomorphic, that is, $\sigma_* \circ J = -J \circ \sigma_*$ for the complex structure J on X, we can find a Kähler form ω on X such that $\sigma^*\omega = -\omega$. A typical example of an anti-holomorphic involution is a complex conjugation over a complex algebraic surface.

S. Akbulut in [9] conjectured that if X is a simply-connected, closed, symplectic 4-manifold and $\sigma : X \to X$ is an anti-symplectic involution with a smooth non-empty embedded surface as a fixed point set, then the quotient X/σ is completely decomposable, *i.e.*,

$$X/\sigma \cong \sharp r\mathbb{CP}^2 \sharp s(\overline{\mathbb{CP}}^2) \quad \text{or} \quad \sharp n(S^2 \times S^2), \quad \text{for some} \quad r, s, n \in \mathbb{N}.$$

In [1], S. Akbulut showed that if X is a complex algebraic surface and σ is the complex conjugation with a real algebraic surface as fixed point set then X/σ is completely decomposable for many cases.

For a long time, it had been asked whether every closed, symplectic manifold has also a Kähler structure. W. Thurston in [15] produced some examples of a symplectic manifolds which are not Kähler. He constructed symplectic 4-manifolds with the first Betti number $b_1 = 3$. This raised the question of

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whether non-Kähler, symplectic manifolds could be simply-connected. D. McDuff in [10] constructed simply-connected examples with b_3 odd in dimensions ≥ 10, but the question remained open in lower dimensions, notably in dimension 4. About this problem, R. E. Gomph in [7] constructed various types of infinite families of simply-connected symplectic manifolds, including families in dimensions 4, 6, and 8, which are non-Kähler for a variety of different reasons.

As far as we know, no one constructs examples of anti-symplectic involutions on non-Kähler, symplectic 4-manifolds. In Section 2, using the Thurston's 4-manifold and Gomph's symplectic sum method, we construct some examples of anti-symplectic involutions on non-Kähler, symplectic 4-manifolds and show that the quotient of the Thurston's 4-manifold is not symplectic.

These constructions will be useful to understand the anti-symplectic involutions over symplectic 4-manifolds associated with the Akbulut's conjecture.

2. Construction of anti-symplectic involutions

Recall the Thurston's non-Kähler, symplectic 4-manifold [15]. Let $\Gamma = \mathbb{Z}^2 \times \mathbb{Z}^2$ be the group with the non-commutative group operation such that $(j', k') \circ (j, k) = (j + j', A_{j'}k + k')$, where $j = (j_1, j_2), k = (k_1, k_2), j' = (j_1', j_2'), k' = (k_1', k_2')$ and

$$A_{j'} = \begin{pmatrix} 1 & j_2' \\ 0 & 1 \end{pmatrix}.$$

The group Γ acts on $\mathbb{R}^4$ via

$$\Gamma = \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathrm{Diff}(\mathbb{R}^4) : (j, k) \to \rho_{jk}$$

for all $(j, k) \in \mathbb{Z}^2 \times \mathbb{Z}^2$, where $\rho_{jk}(x, y) = (x + j, A_j y + k)$.

$\mathbb{R}^4$ can be identified with $\mathbb{C}^2$ in such a way that the almost complex structure J_0 corresponds to the multiplication by $i = \sqrt{-1}$.

Let ω_0 be a Kähler form on $\mathbb{R}^4$. Then we can say that for all $(x, y) \in \mathbb{C}^2$, $x = x_1 + ix_2, y = y_1 + iy_2$ and

$$\omega_0 = \frac{i}{2}(dxd\bar{x} + dyd\bar{y}) = dx_1dx_2 + dy_1dy_2.$$

Then the diffeomorphism ρ_{jk} preserves the canonical symplectic structure ω_0 on $\mathbb{R}^4$ and the quotient $X = \mathbb{R}^4/\Gamma$ is a compact, symplectic manifold and $\pi_1(X) = \Gamma$. The homology group $H_1(X; \mathbb{Z}) = \pi_1(X)/[\Gamma, \Gamma] \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and so the first Betti number $\beta_1(X) = 3$. Since odd dimensional betti numbers are even over a Kähler surface, $X = \mathbb{R}^4/\Gamma$ is a non-Kähler, symplectic 4-manifold. For details, see [15].

Proposition 2.1. *On the non-Kähler, symplectic 4-manifold $X = \mathbb{R}^4/\Gamma$, the involution $\sigma : X \to X$ defined by $\sigma([x_1, x_2, y_1, y_2]) = [x_1, -x_2, y_1, -y_2]$, for all $[x_1, x_2, y_1, y_2] \in X$, is anti-symplectic for a symplectic structure ω on X.*

Proof. For the Kähler structure ω_0 on $\mathbb{R}^4$ and for all $(j,k) \in \Gamma$, since we have

$$\begin{aligned}\rho_{jk}^*\omega_0 &= \rho_{jk}^*(dx_1dx_2 + dy_1dy_2)\\ &= d(x_1+j_1)d(x_2+j_2) + d(y_1 + j_2y_2 + k_1)d(y_2+k_2)\\ &= dx_1dx_2 + dy_1dy_2 + dj_1(dx_2+dj_2) + dj_2(-dx_1 + y_2(dy_2+dk_2))\\ &\quad + dk_1(dy_2+dk_2) + dk_2(-dy_1 - j_2dy_2),\end{aligned}$$

the Kähler form ω_0 on $\mathbb{R}^4$ descends to a symplectic structure $\omega = dx_1dx_2 + dy_1dy_2$ on $X = \mathbb{R}^4/\Gamma$.

Consider an involution $\sigma_0 : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\sigma_0(x_1,x_2,y_1,y_2) = (x_1,-x_2,y_1,-y_2) \quad \text{for all} \quad (x_1,x_2,y_1,y_2) \in \mathbb{R}^4.$$

Then we have $\sigma_0^*\omega_0 = -\omega_0$ and σ_0 is an anti-sympletic involution for the Kähler form ω_0.

For all $j = (j_1,j_2), k = (k_1,k_2), j' = (j_1',j_2'), k' = (k_1',k_2') \in \Gamma$, and for all $(x_1,x_2,y_1,y_2) \in \mathbb{R}^4$, we have

$$\begin{aligned}\sigma_0 \circ \rho_{jk}(x_1,x_2,y_1,y_2) &= (x_1+j_1, -x_2-j_2, y_1+j_2y_2+k_1, -y_2-k_2),\\ \rho_{j'k'} \circ \sigma_0(x_1,x_2,y_1,y_2) &= (x_1+j_1', -x_2+j_2', y_1+y_2j_2'+k_1', -y_2+k_2').\end{aligned}$$

Thus we conclude that $\rho_{j'k'}\circ\sigma_0(x_1,x_2,y_1,y_2)$ and $\sigma_0\circ\rho_{jk}(x_1,x_2,y_1,y_2)$ descend to the same element $[x_1,-x_2,y_1,-y_2]$ on X and so there is a well-defined involution σ over $X = \mathbb{R}^4/\Gamma$ defined by

$$\sigma : X \to X : [x_1,x_2,y_1,y_2] \mapsto [x_1,-x_2,y_1,-y_2].$$

Since $\sigma^*\omega = \sigma^*(dx_1dx_2 + dy_1dy_2) = -dx_1dx_2 - dy_1dy_2 = -\omega$, σ is an anti-symplectic involution for the symplectic structure ω on the Thurston's non-Kähler, symplectic 4-manifold $\mathbb{R}^4/\Gamma$. □

Let $X' = X/\sigma$ be the quotient of $X = \mathbb{R}^4/\Gamma$ under the anti-symplectic involution σ in Proposition 2.1.

Proposition 2.2. *The quotient $X' = X/\sigma$ is not a symplectic 4-manifold.*

Proof. By Proposition 2.1, there is an anti-symplectic involution $\sigma : X \to X$ defined by

$$\sigma([x_1,x_2,y_1,y_2]) = [x_1,-x_2,y_1,-y_2].$$

Then the fixed point set X^σ of σ is

$$X^\sigma = \{[x_1,x_2,y_1,y_2] \in X \mid [x_1,x_2,y_1,y_2] = [x_1,-x_2,y_1,-y_2]\} \cong T^2.$$

Since the Euler characteristic $\chi(X)$ and the signature $\text{sign}(X)$ of X are

$$\chi(X) = \text{sign}(X) = 0,$$

X is a spin 4-manifold with $b_2^+(X) = 2$ and the canonical class K_X of X satisfies $K_X^2 = 0$.

Since $X = \mathbb{R}^4/\Gamma$ is a double cover of X' branched along the torus T^2, by [16] we have

$$\chi(X) = 2\chi(X') - \chi(T^2), \quad \text{sign}(X) = 2\text{sign}(X') - T^2 \cdot T^2.$$

Then $\chi(X') = \text{sign}(X') = 0$ and so $b_2^+(X') = 0$. Thus we conclude that there is no symplectic structure over the quotient X'. □

Remark. Let X be a smooth 4-manifold on which a finite group G with $|G| = p$ acts smoothly with a 2-dimensional Riemann surface Σ as its fixed point set and let X' be its quotient space. Let $\pi : X \to X'$ be the projection map. Then the quotient X' has a smooth structure that π is smooth.

The Euler characteristic $\chi(X')$ and the signature $\text{sign}(X')$ of X' are

$$\chi(X') = \frac{1}{p}(\chi(X) + (p-1)\chi(\Sigma)), \quad \text{sign}(X') = \frac{1}{p}(\text{sign}(X) + \frac{p^2-1}{3}\Sigma \cdot \Sigma).$$

For details, see [3].

For the second construction of an anti-symplectic involution on a non-Kähler, symplectic 4-manifold, we need to introduce a Dolgachev surface.

A Dolgachev surface is the result of performing two logarithmic transformations on the fibers of the basic elliptic surface $E(1)$ which is $\mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}}^2$ as being equipped with an elliptic fibration.

From now on let X_1 and X_2 be simply-connected Dolgachev surfaces given by relatively prime multiplicities $p_i, q_i \geq 1$, $i = 1, 2$.

Let J_i be the complex structure on X_i and $\pi_i : X_i \to \mathbb{CP}^1$ be the elliptic fibration, $\imath = 1, 2$.

Let D_ϵ be a disk in $\mathbb{R}^2$ with radius $\epsilon = \frac{1}{\sqrt{\pi}}$. We identify a small tubular neighborhood $N(F_i)$ of a generic fiber F_i (Kähler torus) of X_i with $T^2 \times D_\epsilon$ so that the fibration correspond to projection onto D_ϵ and the canonical orientations of T^2 and D_ϵ map to the complex orientation.

Consider a complex conjugation $\sigma_i : X_i \to X_i$ such that $\sigma_i(F_i) = F_i'$ and $\pi_i \circ \sigma_i = \pi \circ c$, where $c : \mathbb{CP}^1 \to \mathbb{CP}^1$ is the complex conjugation and F_i' is another generic fiber with $N(F_i) \cap N(F_i') = \emptyset$, $i = 1, 2$.

Then there is a Kähler form ω_i on X_i such that $\sigma_i^* \omega_i = -\omega_i$, $i = 1, 2$.

Remark. We can assume that there is a complex conjugation σ_i on the Dolgachev surface X_i which satisfies the above conditions, $i = 1, 2$.

For example, we take two generic cubics p_0 and p_1 in $\mathbb{CP}^2$ (intersecting each other in distinct 9 points $P_1, \ldots, P_9$) and construct the corresponding pencil of curves $\{t_0 p_0 + t_1 p_1 | \; [t_0 : t_1] \in \mathbb{CP}^1\}$. Then $\{t_0 p_0 + t_1 p_1 | \; [t_0 : t_1] \in \mathbb{CP}^1\}$ is a one-sheet cover of $\mathbb{CP}^2 - \{P_1, \ldots, P_9\}$.

For all $Q \in \mathbb{CP}^2 - \{P_1, \ldots, P_9\}$, there is unique cubic $t_0 p_0 + t_1 p_1$ which passes through Q and we define a map $f : \mathbb{CP}^2 - \{P_1, \ldots, P_9\} \to \mathbb{CP}^1$ by $f(Q) = [t_0 : t_1]$. For details, see [8].

Over the elliptic fibration $p : \mathbb{CP}^2 \sharp 9 \overline{\mathbb{CP}}^2 \to \mathbb{CP}^1$, we consider a complex conjugation $\sigma_0 : \mathbb{CP}^2 \sharp 9 \overline{\mathbb{CP}}^2 \to \mathbb{CP}^2 \sharp 9 \overline{\mathbb{CP}}^2$ which covers the complex conjugation over $\mathbb{CP}^1$.

The complex conjugation σ_0 sends Q to $\bar{Q}$ and the cubic $\bar{t_0} p_0 + \bar{t_1} p_1$ passes through $\bar{Q}$ and $f(\bar{Q}) = [\bar{t_0} : \bar{t_1}] \in \mathbb{CP}^1$.

Let $[t_0 : t_1]$ be a generic point in $\mathbb{CP}^1$. Then $p^{-1}([t_0 : t_1])$ is a generic torus. If $[\bar{t}_0 : \bar{t}_1]$ is a generic point and $[\bar{t}_0 : \bar{t}_1] \neq [t_0 : t_1]$ then $p^{-1}([\bar{t}_0 : \bar{t}_1])$ is another generic torus F' and for small tubular neighborhoods of F and F', we may assume that $N(F) \cap N(F') = \emptyset$.

Over the blowing up part $\mathbb{CP}^2 \sharp 9 \overline{\mathbb{CP}}^2 - (\mathbb{CP}^2 - \{P_1, \ldots, P_9\})$, we consider a canonical complex conjugation over $\overline{\mathbb{CP}}^2$.

Since the Dolgachev surface X_i is the result of performing two logarithmic transformations on the generic fibers of the basic elliptic surface $E(1)$, for the multiplicities of the two logarithmic transformations the complex structure and elliptic fibration extend over the Dolgachev surface X_i, $i = 1, 2$. Thus the complex conjugation σ_i on X_i acts similarly with the action of the σ_0 on $E(1)$, $i = 1, 2$.

Lemma 2.3. *For any such anti-holomorphic involution σ_i, there is a tubular neighborhood $N(F_i)$ of a generic fiber F_i such that σ_i sends $N(F_i)$ to a tubular neighborhood $N(F_i')$ of F_i', $i = 1, 2$.*

Proof. Since $N(F_i) \cong F_i \times D_\epsilon$, each point $x \in N(F_i)$ can be written by $x = ((e^{i\theta_1}, e^{i\theta_2}), re^{i\theta_3}) \in F_i \times D_\epsilon$, where $(e^{i\theta_1}, e^{i\theta_2}) \in F_i$ and $re^{i\theta_3} \in D_\epsilon$ with $0 \leq r \leq \epsilon$, $0 \leq \theta \leq 2\pi$.

For all $x \in N(F_i)$, the complex conjugation σ_i on $N(F_i)$, $i = 1, 2$, acts as

$$\sigma_i(((e^{i\theta_1}, e^{i\theta_2}), re^{i\theta_3})) = (\sigma_i(e^{i\theta_1}, e^{i\theta_2}), re^{-i\theta_3}) = ((e^{-i\theta_1}, e^{-i\theta_2}), re^{-i\theta_3}) \in F_i' \times D_\epsilon \cong N(F_i').$$

□

The fibration on X_i determines a canonical normal framing of F_i, so there is a fiber-orientation reversing bundle isomorphism $\psi_1 : N(F_1) \to N(F_2)$, respecting the given framings and an orientation preserving diffeomorphism $\phi_1 : N(F_1) - F_1 \to N(F_2) - F_2$ by composing ψ_1 with the diffeomorphism

$$f : r \mapsto \sqrt{\epsilon^2 - r^2}, \qquad 0 < r < \epsilon$$

that turns each punctured normal fiber inside out.

Let $X_1 \sharp_{\phi_1} X_2$ be the smooth, closed, oriented 4-manifold obtained from $(X_1 - F_1) \amalg (X_2 - F_2)$ by using ϕ_1 to identify $N(F_1) - F_1$ with $N(F_2) - F_2$.

Indeed, $X_1 \sharp_{\phi_1} X_2$ can be obtained from the compact manifolds $X_1 - N(F_1)$ and $X_2 - N(F_2)$ by gluing along the boundaries $\partial(X_1 - N(F_1)) = F_1 \times \partial D_\epsilon$ and $\partial(X_2 - N(F_2)) = F_2 \times \partial D_\epsilon$ using the map

$$\mathrm{id}_{T^2} \times (\text{complex conjugation}) : S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1.$$

Then $X_1\sharp_{\phi_1}X_2$ is known to be a simply-connected, elliptic surface with Euler characteristic 24. For details, see Chapter 3 in [8].

However, R. E. Gomph in [6] constructed non-Kähler, simply-connected, symplectic 4-manifolds $X_1\sharp_{\phi_1'}X_2$ by a slight modification ϕ_1' of ϕ_1. He didn't glue by the fiber preserving map ϕ_1. Instead, he composed ϕ_1 with a cyclic permutation of the three factors $F_1 \times S^1 = S^1 \times S^1 \times S^1$ before gluing.

Let $p : F_1 \times S^1 \to F_1 \times S^1$ be the cyclic permutation defined by $p(x_1, x_2, x_3) = (x_3, x_1, x_2)$ for all $(x_1, x_2, x_3) \in F_1 \times S^1$. Then we have

$$\begin{aligned}\phi_1'(x) &= \phi_1'(((e^{i\theta_1}, e^{i\theta_2}), re^{i\theta_3})) = p \circ \phi_1(((e^{i\theta_1}, e^{i\theta_2}), re^{i\theta_3}))\\ &= p(((e^{i\theta_1}, e^{i\theta_2}), \sqrt{\epsilon^2 - r^2}e^{-i\theta_3})) = ((e^{-i\theta_3}, e^{i\theta_1}), \sqrt{\epsilon^2 - r^2}e^{i\theta_2}),\end{aligned}$$

and the twisted gluing map ϕ_1' is given by the matrix

$$N = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

with respect to the corresponding basis for $H^1(F_i \times S^1; \mathbb{Z})$, $i = 1, 2$.

Let $X_1\sharp_{\phi_1'}X_2$ be the smooth, closed, oriented 4-manifold obtained from $(X_1 - F_1) \amalg (X_2 - F_2)$ by using ϕ_1' to identify $N(F_1) - F_1$ and $N(F_2) - F_2$.

By [7], $X_1\sharp_{\phi_1'}X_2$ is the 4- manifold $K(p_1, q_1; 1, 1; p_2, q_2)$ which is simply-connected, not diffeomorphic to any elliptic surfaces for any relatively prime multiplicities $p_i, q_i > 1$, $i = 1, 2$. For details, see [7].

As above ϕ_1, the fibration on X_i determines a canonical normal framing of $\sigma_i(F_i) = F_i'$, so there is a fiber-orientation reversing bundle isomorphism $\psi_2 : N(F_1') \to N(F_2')$, respecting the given framings and an orientation preserving diffeomorphism $\phi_2 : N(F_1') - F_1' \to N(F_2') - F_2'$ by composing ψ_2 with the diffeomorphism f. As the same constructions of ϕ_1', we consider a twisted gluing map $\phi_2' = p \circ \phi_2 : N(F_1') - F_1' \to N(F_2') - F_2'$.

Let $X_1\sharp_{\phi_1', \phi_2'}X_2$ be a smooth, closed, oriented 4-manifold obtained from

$$X_1\sharp_{\phi_1'}X_2 - (F_1' \amalg F_2')$$

by using ϕ_2' to identify $N(F_1') - F_1'$ and $N(F_2') - F_2'$.

Proposition 2.4. *$X_1\sharp_{\phi_1', \phi_2'}X_2$ is a non-Kähler, symplectic 4-manifold.*

Proof. Since $X_1\sharp_{\phi_1', \phi_2'}X_2$ is obtained from the non-Kähler, symplectic 4-manifold $X_1\sharp_{\phi_1'}X_2$ by deleting $N(F_1') \amalg N(F_2')$ and by gluing along the boundaries $F_1' \times \partial D_\epsilon$ and $F_2' \times \partial D_\epsilon$ by the map ϕ_2', $X_1\sharp_{\phi_1', \phi_2'}X_2$ is not diffeomorphic to any elliptic surfaces.

Let ω_i be the Kähler form over the Dolgachev surface X_i, $i = 1, 2$.

By using the same argument with Theorem 1.3 in [6], for any choice of the diffeomorphism $\phi_2' : N(F_1') - F_1' \to N(F_2') - F_2'$, there is a symplectic structure

ω on $X_1\sharp_{\phi_1',\phi_2'}X_2$ defined by

$$\omega = \begin{cases} \omega_i & X_i - (N(F_i) \amalg N(F_i')), i = 1,2, \\ \omega_0 + t\eta & (N(F_1) - F_1)\sharp_{\phi_1'}(N(F_2) - F_2), \\ \omega_0 + t\eta' & (N(F_1') - F_1')\sharp_{\phi_2'}(N(F_2') - F_2'), \end{cases}$$

where $t \in (0, t_0]$ for sufficiently small t_0. The η and η' are closed 2-forms compactly supported in $N(F_2)$ and $N(F_2')$ respectively and they are Poincaré dual to $[F_2] \in H_2(X_2;\mathbb{R})$ and $[F_2'] \in H_2(X_2;\mathbb{R})$ respectively.

The Euler characteristic and the signature satisfy

$$\chi(X_1\sharp_{\phi_1',\phi_2'}X_2) = \chi(X_1) + \chi(X_2) = 24,$$
$$\mathrm{sign}(X_1\sharp_{\phi_1',\phi_2'}X_2) = \mathrm{sign}(X_1) + \mathrm{sign}(X_2) = -16.$$

□

Proposition 2.5. *There is an anti-symplectic involution on the non-Kähler, symplectic 4-manifold $X_1\sharp_{\phi_1',\phi_2'}X_2$.*

Proof. For all $x = ((e^{i\theta_1}, e^{i\theta_2}), re^{i\theta_3}) \in N(F_1) - F_1$ and $x' = ((e^{i\vartheta_1}, e^{i\vartheta_2}), re^{i\vartheta_3}) \in N(F_1') - F_1'$, $0 < r < \epsilon$, $0 \le \theta, \vartheta \le 2\pi$, we have

$$\sigma_1(x) = \phi_2'(\sigma_1(x)) \quad \text{on} \quad (N(F_1') - F_1')\sharp_{\phi_2'}(N(F_2') - F_2'),$$
$$\sigma_1(x') = \phi_1'(\sigma_1(x')) \quad \text{on} \quad (N(F_1) - F_1)\sharp_{\phi_1'}(N(F_2) - F_2).$$

To show that there is an anti-symplectic involution on $X_1\sharp_{\phi_1',\phi_2'}X_2$ induced from the complex conjugations σ_i on X_i, $i = 1,2$, we have to prove that

$$\sigma_1(x') = \phi_1'(\sigma_1(x')) = \sigma_2(\phi_2'(x')) \quad \text{over} \quad (N(F_1) - F_1)\sharp_{\phi_1'}(N(F_2) - F_2),$$
$$\sigma_1(x) = \phi_2'(\sigma_1(x)) = \sigma_2(\phi_1'(x)) \quad \text{over} \quad (N(F_1') - F_1')\sharp_{\phi_2'}(N(F_2') - F_2').$$

Since we have

$$\begin{aligned} \phi_2'(\sigma_1(x)) &= \phi_2'(\sigma_1(((e^{i\theta_1}, e^{i\theta_2}), re^{i\theta_3}))) = \phi_2'(((e^{-i\theta_1}, e^{-i\theta_2}), re^{-i\theta_3})) \\ &= ((e^{i\theta_3}, e^{-i\theta_1}), \sqrt{\epsilon^2 - r^2}e^{-i\theta_2}), \sigma_2(\phi_1'(x)) \\ &= \sigma_2(((e^{-i\theta_3}, e^{i\theta_1}), \sqrt{\epsilon^2 - r^2}e^{i\theta_2})) = ((e^{i\theta_3}, e^{-i\theta_1}), \sqrt{\epsilon^2 - r^2}e^{-i\theta_2}), \end{aligned}$$

and

$$\begin{aligned} \phi_1'(\sigma_1(x')) &= \phi_1'(\sigma_1(((e^{i\vartheta_1}, e^{i\vartheta_2}), re^{i\vartheta_3}))) = \phi_1'(((e^{-i\vartheta_1}, e^{-i\vartheta_2}), re^{-i\vartheta_3})) \\ &= ((e^{i\vartheta_3}, e^{-i\vartheta_1}), \sqrt{\epsilon^2 - r^2}e^{-i\vartheta_2}), \sigma_2(\phi_2'(x')) \\ &= \sigma_2(((e^{-i\vartheta_3}, e^{i\vartheta_1}), \sqrt{\epsilon^2 - r^2}e^{i\vartheta_2})) \\ &= ((e^{i\vartheta_3}, e^{-i\vartheta_1}), \sqrt{\epsilon^2 - r^2}e^{-i\vartheta_2}), \end{aligned}$$

we conclude that there is a well-defined involution σ on $X_1\sharp_{\phi_1',\phi_2'}X_2$ such that

$$\sigma = \begin{cases} \sigma_i & X_i - (N(F_i) \amalg N(F_i')) \subset X_1\sharp_{\phi_1',\phi_2'}X_2, i = 1, 2, \\ \sigma_1(x') = \sigma_2(\phi_2'(x')) & (N(F_1) - F_1)\sharp_{\phi_1'}(N(F_2) - F_2), \\ \sigma_1(x) = \sigma_2(\phi_1'(x)) & (N(F_1') - F_1')\sharp_{\phi_2'}(N(F_2') - F_2'), \end{cases}$$

for all $x \in N(F_1) - F_1$ and $x' \in N(F_1') - F_1'$.

For the symplectic structure ω on $X_1\sharp_{\phi_1',\phi_2'}X_2$,

$$\sigma^*\omega = \sigma_i^*\omega_i = -\omega_i = -\omega \text{ on } X_i - (N(F_i) \amalg N(F_i')) \subset X_1\sharp_{\phi_1',\phi_2'}X_2, \quad i = 1, 2.$$

For all $x \in N(F_1) - F_1$, we have

$$\sigma^*\omega(x) = \sigma_1^*\omega_1(x) = -\omega_1(x) = -\omega(x).$$

Also

$$\sigma^*\omega(\phi_1'(x)) = \sigma_2^*(\omega_2 + t\eta)(\phi_1'(x))$$

for all $t \in (0, t_0]$. Since η is Poincaré dual to $[F_2]$ in $H_2(X_2; \mathbb{R})$, η can be written by

$$\eta = dy_3dy_4$$

for all $((y_1, y_2), (y_3, y_4)) \in F_2 \times (D_\epsilon - \{0\})$. Then $y_3 = r\cos\theta$ and $y_4 = r\sin\theta$ for the polar coordinate $(r, \theta) \in D_\epsilon - \{0\}$, $0 < r < \epsilon$, $0 \le \theta \le 2\pi$.

Then we have

$$\begin{aligned} \sigma_2^*(\eta) &= \sigma_2^*(dy_3dy_4) = \sigma_2^* d(r\cos\theta)d(r\sin\theta) \\ &= d(r\cos(-\theta))d(r\sin(-\theta)) = -d(r\cos\theta)d(r\sin\theta) = -dy_3dy_4 = -\eta. \end{aligned}$$

Thus we conclude that

$$\sigma_2^*(\omega_2 + t\eta)(\phi_1'(x)) = -(\omega_2 + t\eta)(\phi_1'(x)) = -\omega(\phi_1'(x)).$$

From the above equations, $\sigma^*\omega = -\omega$ on $(N(F_1) - F_1)\sharp_{\phi_1'}(N(F_2) - F_2)$.

Similarly, for all $x' \in N(F_1') - F_1'$,

$$\begin{aligned} \sigma^*\omega(x') &= \sigma_1^*\omega_1(x') = -\omega_1(x') = -\omega(x'), \\ \sigma^*\omega(\phi_2'(x')) &= \sigma_2^*(\omega_2 + t\eta')(\phi_2'(x')) \\ &= -\omega_2(\phi_2'(x')) - t\eta'(\phi_2'(x')) = -\omega(\phi_2'(x')). \end{aligned}$$

The above equations imply that $\sigma^*\omega = -\omega$ on $(N(F_1') - F_1')\sharp_{\phi_2'}(N(F_2') - F_2')$. Thus σ is an anti-symplectic involution on the non-Kähler, symplectic 4-manifold $X_1\sharp_{\phi_1',\phi_2'}X_2$ for the symplectic structure ω. □

We can contrast with the anti-symplectic involution in Proposition 2.5 with an anti-holomorphic involution over an elliptic surface.

Example 2.1. Let X_i and σ_i be as in Proposition 2.5, $i = 1, 2$. Now consider an elliptic surface $X_1\sharp_{\phi_1}X_2$ obtained by the fiber sum $\phi_1 : N(F_1) - F_1 \to N(F_2) - F_2$ (instead of ϕ_1') of the Dolgachev surfaces X_1 and X_2 which is known to admit a Kähler structure.

Let $X_1\sharp_{\phi_1,\phi_2}X_2$ be the smooth, closed, oriented 4-manifold obtained from

$$X_1\sharp_{\phi_1}X_2 - (F_1' \amalg F_2')$$

by using ϕ_2 to identify $N(F_1') - F_1'$ and $N(F_2') - F_2'$.

Then $X_1\sharp_{\phi_1,\phi_2}X_2$ is an elliptic surface over T^2 with the first Betti number $b_1 = 2$. At the level of smooth manifolds, this method is easily recognized as the technique producing new elliptic surfaces from old ones.

Since ϕ_1 and ϕ_2 are fiber preserving maps, there is an anti-holomorphic involution τ on the elliptic surface $X_1\sharp_{\phi_1,\phi_2}X_2$ such that

$$\tau = \begin{cases} \sigma_i \quad X_i - (N(F_i) \amalg N(F_i')) \subset X_1\sharp_{\phi_1,\phi_2}X_2, i = 1,2, \\ \sigma_1(x') = \sigma_2(\phi_2(x')) \qquad (N(F_1) - F_1)\sharp_{\phi_1}(N(F_2) - F_2), \\ \sigma_1(x) = \sigma_2(\phi_1(x)) \quad (N(F_1') - F_1')\sharp_{\phi_2}(N(F_2') - F_2'), \end{cases}$$

for all $x \in N(F_1) - F_1$ and $x' \in N(F_1') - F_1'$.

For the complex structure J on $X_1\sharp_{\phi_1,\phi_2}X_2$, $\tau_* \circ J = -J \circ \tau_*$, that is, τ is anti-holomorphic. However, for the anti-symplectic involution σ in Proposition 2.5, since there is no complex structure on $X_1\sharp_{\phi_1',\phi_2'}X_2$, σ is not an anti-holomorphic involution.

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