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ON THE PRINCIPAL IDEAL THEOREM

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ABSTRACT. In this paper we give an example of imaginary quadratic number field k such that every ideal of k becomes principal in some proper subfields of the Hilbert class field of k.

1. Introduction

Let K be a finite extension of a number field k, cl_K and cl_k denote the ideal class groups of K and k, respectively. We say that an ideal class of k capitulates in K if it is in the kernel of the homomorphism

$$j_{K/k} : cl_k \to cl_K$$

induced by the extension of ideals from k to K. The kernel of $j_{K/k}$ is called the capitulation kernel of the extension K/k, denoted by $Cap(K/k)$. Let $\tilde{k}$ be the Hilbert class field of k. The principal ideal theorem says that cl_k always capitulates in $\tilde{k}$. However, Heider and Schmithals have found real quadratic fields k such that cl_k capitulates in a proper subfield of $\tilde{k}$ ([8]). For a prime p let $cl_k^{(p)}$ and $\tilde{k}^{(p)}$ be the p-class group of cl_k (i.e. the Sylow p-subgroup of cl_k) and the Hilbert p-class field of k, respectively. Since $cl_k^{(p)}$ capitulates in $\tilde{k}^{(p)}$ for every prime p, cl_k capitulates in a proper subfield of $\tilde{k}$ if and only if there exists a prime p such that $cl_k^{(p)}$ capitulates in a proper subfield of $\tilde{k}^{(p)}$. Iwasawa ([10]) has proved the following:

Theorem 1. *For each prime number $p \geq 2$, there exist infinitely many finite algebraic number fields k such that $cl_k^{(p)}$ capitulates in a proper subfield of $\tilde{k}^{(p)}$.*

Some examples of real quadratic fields k such that $cl_k^{(2)}$ capitulates in a proper subfield of $\tilde{k}^{(2)}$ are given in [9] and [5]. (See also [14], [3], and [6]. For more on the capitulation problem see [11].) Let $C(n)$ be a cyclic group of order n. In this paper we shall prove the following:

Theorem 2. *Let $k = \mathbb{Q}(\sqrt{-8867 \cdot 73681})$. We have $cl_k \simeq C(210) \times C(3) \times C(3) \times C(3)$.*

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(i) *For all* 40 *proper subfield* F_i*'s* $1 \le i \le 40$ *of* $\tilde{k}^{(3)}$ *with* $[F_i : k] = 27$, $cl_k^{(3)}$ *capitulates in* F_i.
(ii) *There exist at least* 81 *proper subfield* L_j*'s of* $\tilde{k}^{(3)}$ *such that* $[L_j : k] = 9$ *and* $cl_k^{(3)}$ *capitulates in* L_j.

Note that $\mathbb{Q}(\sqrt{-653329427})$ $(653329427 = 8867 \cdot 73681)$ is the first known imaginary quadratic field whose 3-rank of the ideal class group is equal to 4 (see [4]). For $l = 3$ or 5, many authors have computed the capitulation kernels in the unramified cyclic extensions of degree l of imaginary quadratic number fields k such that 3-rank of cl_k is equal to 2 or 3 when $l = 3$, or 5-rank of cl_k is equal to 2 when $l = 5$ ([15], [8], and [12]).

The paper is organized in the following way. In Section 2, we first construct the 40 unramified cyclic cubic extension K_i's $1 \le i \le 40$ of $\mathbb{Q}(\sqrt{-653329427})$ by the method given in [7]. Second, for each of these 40 extensions K_i/k we determine explicitly the capitulation kernel $Cap(K_i/k)$ using the method developed in [15] and [8]. In Section 3, we prove Theorem 2. All computations were done with the aid of PARI-GP ([1]) on a workstation (SUN Sparc II).

2. The capitulation kernels in the unramified cyclic cubic extensions of $\mathbb{Q}(\sqrt{-653329427})$

We will use the following notations. For a number field k we let E_k, d_k, I_k and cl_k be the group of units, the discriminant, the group of fractional ideals and the ideal class group of k, respectively. For a prime place $\mathfrak{p}$ of k, let $k_\mathfrak{p}$ be the completion of k at $\mathfrak{p}$ and $U_\mathfrak{p}$ the group of units of $k_\mathfrak{p}$. For a finite extension K/k, $N_{K/k}$ denotes the norm and $E_{K/k} = \{\epsilon \in E_K \mid N_{K/k}(\epsilon) = 1\}$.

Let $k = \mathbb{Q}(\sqrt{-653329427})$. The ideal class group of k is represented by the reduced primitive quadratic forms:

$$cl_k = \langle a_1, b, c, d\rangle \simeq C(210) \times C(3) \times C(3) \times C(3),$$

where

$$a_1 = (1899, 1267, 86221), b = (11271, 8927, 16259),$$
$$c = (6157, -875, 26559), d = (1649, 457, 99081),$$

a_1 is of order 210, b, c and d are of order 3. Set $a = a_1^{70} = (689, 589, 237183)$. Then $cl_k^{(3)} = \langle a, b, c, d\rangle$. To construct the unramified cyclic cubic extensions of a quadratic number field we use the following proposition.

Proposition 1. *Let* $k = \mathbb{Q}(\sqrt{m})$ *and* $k_0 = \mathbb{Q}(\sqrt{m_0})$ *with*

$$m_0 = \begin{cases} -m/3 & \text{if } 3 \text{ divides } m, \\ -3m & \text{otherwise.} \end{cases}$$

Let α *be an algebraic integer in* k_0 *satisfying the following:*
(i) $(\alpha) = \mathfrak{a}^3$ *with* $\mathfrak{a}$ *a non-principal ideal of* k_0, *prime to* 3;
(ii) $N_{k_0/\mathbb{Q}}(\alpha) = n^3$ *for some integer* n *with* $(n, 3) = 1$; *and*

(iii) *for every prime ideal* $\mathfrak{p}$ *in* k_0 *lying above* 3, *there exists* $\xi \in k_0$ *such that* $\alpha \equiv \xi^3 \mod \mathfrak{p}^{e+1}$, *where* e *is the ramification index of* $\mathfrak{p}$.

Then the cubic polynomial

$$X^3 - 3nX - Tr_{k_0/\mathbb{Q}}(\alpha)$$

defines a cubic number field whose discriminant is the same as that of k. *Moreover, the splitting field of this cubic polynomial is an unramified cyclic cubic extension of* k.

Proof. See Theorem 1 and Lemma 2 in [7]. □

In order to determine the cubic polynomials we proceed as follows. We let $k_0 = \mathbb{Q}(\sqrt{3 \cdot 653329427})$. Then $cl_{k_0} \simeq C(6) \times C(3) \times C(3)$. First, we determine 40 triplet (x_i, y_i, n_i)'s of integers such that $x_i^2 - 3 \cdot 653329427 y_i^2 = 4n_i^3$. Second, using PARI-GP ([2] and [1]) we verify that these 40 polynomials

$$r_i(X) = X^3 - 3n_iX - x_i$$

define non-isomorphic cubic number fields of discriminant -653329427. Finally, we have used PARI-GP to simplify the coefficients of the polynomials. The 40 polynomial r_i's, $1 \le i \le 40$, are listed in Table 1. In what follows we let K_i denote the splitting field of the polynomial r_i over $\mathbb{Q}$. Then K_i's $1 \le i \le 40$ are 40 unramified cyclic cubic extensions of $k = \mathbb{Q}(\sqrt{-653329427})$.

According to Hilbert's Theorem 94 $Cap(K_i/k)$ is of order 3. Using the method developed in [15] and [8] we will explicitly determine $Cap(K_i/k)$ for $1 \le i \le 40$. Following [8], we let

$$V_k = \{x \in k^* \mid (x) = \mathfrak{a}^l \text{ for some } \mathfrak{a} \in I_k\}$$

and

$$cl_{k,l} = \{c \in cl_k \mid |c| = 1 \text{ or } l\}$$

for a prime number l. We have an exact sequence

$$1 \to E_k/E_k^l \to V_k/k^{*l} \xrightarrow{\phi_k} cl_{k,l} \to 1$$

$$(x) \mod k^{*l} \longmapsto [\mathfrak{a}]$$

of F_l-vector spaces. (Here, F_l is the finite field of integers mod l.) Let K be an unramified cyclic extension of k of degree l with Galois group $G = \langle\sigma\rangle$. Since $N_{K/k} \circ j_{K/k}(c) = c^{[K:k]}$, we have $\operatorname{Ker} j_{K/k} \subset cl_{k,l}$. We let $\mathfrak{K}(K/k) = E_K \cdot K^{*l} \cap V_k$. Then $\mathfrak{K}(K/k)/k^{*l} = \operatorname{Ker}(j_{K/k} \circ \phi_k)$. For S a set of prime places in k, let $V_k^{(S)} = \{x \in V_k | x \in k_\mathfrak{p}^{*l} \text{ for all } \mathfrak{p} \in S\}$. When $S = \{\mathfrak{p}\}$, we let $V_k^{(\mathfrak{p})} = V_k^{(S)}$. The following capitulation criterion is a generalization of the criterion that was developed by Scholz and Taussky ([15]).

Theorem 3. *Let* K *be a finite unramified cyclic extension of degree* l *of* k *with Galois group* $G = \langle\sigma\rangle$. *Let* $\mathcal{E}$ *be a set of generators of* $E_{K/k}/E_K^{1-\sigma}$, $\mathcal{F}$ *a set of generators of* E_k/E_k^l, $\delta = \dim\ cl_{k,l}, \mu = \dim\ \operatorname{Ker} j_{K/k}$ *and* $\rho = \dim\ E_k/E_k^l$.

(a) *Let $\mathfrak{p}$ be a prime ideal of k. Assume that*
(i) *$\mathfrak{p}$ splits totally in K.*
Then $\mathfrak{K}(K/k) \subset V_k^{(\mathfrak{p})}$ if and only if $\mathfrak{p}$ satisfies the following two conditions:
(ii) *$\eta \in k_{\mathfrak{p}}^{*l}$ for all $\eta \in \mathcal{F}$.*
(iii) *$\varepsilon^{(1-\sigma)^{l-2}} \in K_{\mathfrak{P}}^{*l}$ for all $\varepsilon \in \mathcal{E}$ and for all prime divisors $\mathfrak{P}|\mathfrak{p}$ in K.*

(b) *Let $[\mathfrak{a}] \notin \operatorname{Ker} j_{K/k}$ with $\mathfrak{a}^l = (x)$. The density of the prime ideals $\mathfrak{p}$ such that $\mathfrak{p}$ satisfies the conditions* (i), (ii), (iii), *and*
(iv) *$N_{k/\mathbb{Q}}(\mathfrak{p}) = p$ and $\mathfrak{p}$ is unramified over $\mathbb{Q}$,*
(v) *$p \equiv 1 \mod l$,*
for $\mathfrak{p}$ such that $x \notin k_{\mathfrak{p}}^{*l}$, is equal to

$$\begin{cases} \frac{1}{[(k(\zeta):k]}(\frac{1}{l})^{\mu+\rho+1}(1-\frac{1}{l}) & \text{if } \zeta \notin k, \\ (\frac{1}{l})^{\mu+\rho}(1-\frac{1}{l}) & \text{if } \zeta \in k, \end{cases}$$

where ζ is a primitive l-th root of unity.

(c) *$\mathfrak{K}(K/k) = V_k^{(S)}$ for all sufficiently large sets S of prime ideals of k satisfying* (i), (ii), (iii), (iv) *and* (v). *Moreover, we can choose S such that $\#S = \delta - \mu$.*

Proof. See Lemma 1 and Theorem 2 in [8]. □

For the extensions K_i/k $1 \le i \le 40$ with $k = \mathbb{Q}(\sqrt{-653329427})$, we have $l = 3, \delta = 4, \mu = 1$ and $cl_{k,3} = cl_k^{(3)}$. From Theorem 3(c) we can explicitly determine $Cap(K_i/k)$. As we have determined the polynomials r_i defining K_i, it is easy to find the prime ideals $\mathfrak{p}$ that satisfy the conditions (i), (iv) and (v). For the conditions (ii) and (iii), we have used the technique developed in [8].

By the existence theorem of class field theory, the map $L \longmapsto N_{L/k}(cl_L)$ is a 1-to-1 correspondence between the finite unramified abelian extensions L/k and the subgroups of cl_k. In addition, the factor group $cl_k/N_{L/k}(cl_L)$ is isomorphic to the Galois group $G(L/k)$ of the extension L/k. The subgroup $N_{L/k}(cl_L)$ is called the norm group of the extension L/k. Furthermore, the prime decomposition law is helpful to determine the norm group. In fact, let $\mathfrak{p}$ be an unramified prime ideal of k. The prime decomposition law says that if the ideal class $[\mathfrak{p}]$ is contained in $N_{L/k}(cl_L)$, then $\mathfrak{p}$ splits in L into a product of n different prime ideals, where $n = [L : k]$ (see Theorem 8.4 in [13]). We can easily determine such prime ideals and the norm group $N_{K_i/k}(cl_{K_i})$ (see [8]). In Table 1, we compile the polynomials r_i, $Cap(K_i/k)$ and the norm groups $N_{K_i/k}(cl_{K_i}) \mod cl_k^3$ for all $1 \le i \le 40$.

Table 1. $Cap(K_i/k)$ and the norm groups of K_i/k

i	r_i	$Cap(K_i/k)$	$N_{K_i/k}(cl_{K_i}) \mod cl_k^3$
1	$r_1(x) = x^3 + 506 * x - 2237$	a^2c	$\langle a, bc, d\rangle$
2	$r_2(x) = x^3 - x^2 + 463 * x + 2932$	bcd	$\langle a, b^2c, b^2d\rangle$
3	$r_3(x) = x^3 - x^2 + 367 * x + 3988$	d	$\langle ab, bc^2, d\rangle$
4	$r_4(x) = x^3 + 236 * x - 4717$	$ab^2c^2d^2$	$\langle ab^2, bd, c\rangle$
5	$r_5(x) = x^3 + 110 * x - 4899$	a^2d^2	$\langle ab^2, d, bc^2\rangle$
6	$r_6(x) = x^3 - x^2 + 9 * x - 4922$	$a^2b^2cd^2$	$\langle ac, b^2, cd^2\rangle$
7	$r_7(x) = x^3 - x^2 - 153 * x - 4922$	$ab^2c^2d^2$	$\langle ab, c, d\rangle$
8	$r_8(x) = x^3 - 250 * x - 5149$	bc^2	$\langle a, bc^2, d\rangle$
9	$r_9(x) = x^3 - x^2 - 763 * x + 9746$	$a^2b^2cd^2$	$\langle a, b, cd\rangle$
10	$r_{10}(x) = x^3 - 1114 * x - 15133$	$a^2c^2d^2$	$\langle a, bd, cd\rangle$
11	$r_{11}(x) = x^3 - x^2 - 1597 * x + 25592$	bc	$\langle ad^2, b, c\rangle$
12	$r_{12}(x) = x^3 - x^2 + 1581 * x + 23980$	d	$\langle ab, c, b^2d\rangle$
13	$r_{13}(x) = x^3 + 1493 * x - 32490$	a^2bcd^2	$\langle ad^2, b^2d, bc^2\rangle$
14	$r_{14}(x) = x^3 - x^2 + 1296 * x + 34588$	bc^2d^2	$\langle ad, b, c\rangle$
15	$r_{15}(x) = x^3 - x^2 + 576 * x + 38800$	$ab^2c^2d^2$	$\langle a, bc, bd\rangle$
16	$r_{16}(x) = x^3 - x^2 - 1440 * x - 44144$	abc	$\langle ac, b^2d, cd\rangle$
17	$r_{17}(x) = x^3 - x^2 - 627 * x + 44892$	ab^2	$\langle a, b, c\rangle$
18	$r_{18}(x) = x^3 - x^2 + 1435 * x - 71238$	bd^2	$\langle a, bd, c\rangle$
19	$r_{19}(x) = x^3 - x^2 - 1483 * x + 183824$	bc^2	$\langle a, bd^2, bc\rangle$
20	$r_{20}(x) = x^3 - 139 * x - 314822$	bc	$\langle b, cd, ac\rangle$
21	$r_{21}(x) = x^3 + 2519 * x - 301076$	$a^2c^2d^2$	$\langle a^2d, c, bd^2\rangle$
22	$r_{22}(x) = x^3 - x^2 - 4373 * x - 220344$	ab^2c^2	$\langle ab, c^2d, bc\rangle$
23	$r_{23}(x) = x^3 - x^2 - 3205 * x - 148994$	$a^2bc^2d^2$	$\langle a, c, d\rangle$
24	$r_{24}(x) = x^3 - x^2 + 4917 * x + 4014$	b^2cd^2	$\langle cd, b^2d, ad\rangle$
25	$r_{25}(x) = x^3 - 4264 * x - 107283$	ab	$\langle a, c, bd^2\rangle$
26	$r_{26}(x) = x^3 - x^2 - 4127 * x - 100802$	a^2bcd^2	$\langle a, b, d\rangle$
27	$r_{27}(x) = x^3 - x^2 - 5639 * x - 208380$	a^2c	$\langle a^2c, bc, d\rangle$
28	$r_{28}(x) = x^3 - x^2 - 3511 * x + 311738$	b^2cd^2	$\langle ab, cd, bd\rangle$
29	$r_{29}(x) = x^3 - x^2 + 3779 * x - 277372$	a	$\langle ac, bc, d\rangle$
30	$r_{30}(x) = x^3 - x^2 + 6526 * x - 41100$	ab^2d	$\langle a^2c, b^2c, cd\rangle$
31	$r_{31}(x) = x^3 - x^2 - 5029 * x + 270560$	b^2d^2	$\langle ac, cd^2, bc^2\rangle$
32	$r_{32}(x) = x^3 - x^2 + 6289 * x - 257810$	cd^2	$\langle a, b, cd^2\rangle$
33	$r_{33}(x) = x^3 + 3614 * x - 301$	d	$\langle ac, b, d\rangle$
34	$r_{34}(x) = x^3 - x^2 + 4101 * x + 84798$	abc	$\langle ac^2, b, cd\rangle$
35	$r_{35}(x) = x^3 - x^2 + 2611 * x + 121620$	ab^2	$\langle ad^2, b, c^2d\rangle$
36	$r_{36}(x) = x^3 - x^2 + 3193 * x + 61498$	bd^2	$\langle ab, bd, c\rangle$
37	$r_{37}(x) = x^3 - 8371 * x - 297406$	b	$\langle ab^2, c, d\rangle$
38	$r_{38}(x) = x^3 - x^2 - 3041 * x + 65758$	bcd^2	$\langle b, c, d\rangle$
39	$r_{39}(x) = x^3 - 3718 * x - 127865$	$a^2c^2d^2$	$\langle ac, bd, cd^2\rangle$
40	$r_{40}(x) = x^3 - 4600 * x - 152169$	bc^2d^2	$\langle ac^2, b, d\rangle$

3. Proof of Theorem 2

We keep the notations in Section 2.

i) The group $cl_k^{(3)}$ has 40 subgroups of order 3: $\langle \alpha_i \rangle$ $1 \le i \le 40$. Let F_i be the corresponding subfield of $\tilde{k}^{(3)}$, i.e.

$$N_{F_i/k}(cl_{F_i}) \mod cl_k^3 = \langle \alpha_i \mod cl_k^3 \rangle \text{ and } [F_i : k] = 3^3.$$

Each field F_i has 13 subfield K_{i_t}'s $t = 1, \dots, 13$ such that $[K_{i_t} : k] = 3$. If H and K are subgroups of cl_k, the subgroup generated by H and K is called the join of H and K and is denoted $H \bigvee K$. Note that for every $1 \le t \le 13$, the subgroup $Cap(K_{i_t}/k)$ is contained in $Cap(F_i/k)$. We claim that for every $1 \le i \le 40$,

$$(*) \qquad \bigvee_{t=1}^{13} Cap(K_{i_t}/k) = cl_k^{(3)}.$$

In fact, we write $Cap(K_{i_t}/k) = \langle a^{t_1} b^{t_2} c^{t_3} d^{t_4} \rangle$, $t_1, t_2, t_3, t_4 \in \mathbb{Z}/3\mathbb{Z}$ and consider the 13 by 4 matrix over $\mathbb{Z}/3\mathbb{Z}$, M_i, such that t-th row of M_i is (t_1, t_2, t_3, t_4). In order to prove $(*)$ it is sufficient to verify that the rank of M_i is equal to 4. We have verified on a computer that the rank of M_i is equal to 4 for all 40 subfield F_i's of $\tilde{k}^{(3)}$ with $[F_i : k] = 27$. In Table 2 we give some extracts of our computational results: ten arbitrary chosen fields among 40 field F_i's. For each F_i, we give 13 subfields K_{i_t} of F_i and $Cap(K_{i_t}/k)$ with $1 \le t \le 13$. In the column of F_i we give α_i a generator of $N_{F_i/k}(cl_{F_i}) \mod cl_k^3$.

ii) The group $cl_k^{(3)} \simeq C(3) \times C(3) \times C(3) \times C(3)$ has 130 subgroups of index 9. We denote by L_j for $1 \le j \le 130$ the subfields of $\tilde{k}^{(3)}$ such that $[L_j : k] = 9$. Every subfield L_j contains four subfields denoted by K_{j_1}, K_{j_2}, K_{j_3} and K_{j_4}. For all $1 \le j \le 130$, we will determine whether $\bigvee_{u=1}^{4} Cap(K_{j_u}/k) = cl_k^{(3)}$ or not. For this purpose we build the 4 by 4 matrix over $\mathbb{Z}/3\mathbb{Z}$, N_j, such that u-th row of N_j is (l_1, l_2, l_3, l_4) if $Cap(K_{j_u}/k) = \langle a^{l_1} b^{l_2} c^{l_3} d^{l_4} \rangle$. The fact that $\bigvee_{u=1}^{4} Cap(K_{j_u}/k) = cl_k^{(3)}$ is equivalent to the fact that det $N_j \neq 0$ in $\mathbb{Z}/3\mathbb{Z}$. We have verified on a computer that exactly 81 field L_j's among 130 fields satisfy det $N_j \neq 0$ in $\mathbb{Z}/3\mathbb{Z}$. In Table 3, we give some extracts of our computational results: 18 arbitrary chosen fields among 130 field L_j's. For each subfield L_j, we give four subfields K_{j_u} $1 \le u \le 4$, $Cap(K_{j_u}/k)$ and det N_j. In the column of L_j we give two generators of $N_{L_j/k}(cl_{L_j}) \mod cl_k^3$.

This completes the proof of Theorem 2.

Table 2

F_i	subfields K_{i_t}	$Cap(K_{i_t}/k)$	F_i	subfields K_{i_t}	$Cap(K_{i_t}/k)$
d	K_1	$a^2b^0c^1d^0$	ac^2d	K_2	$a^0b^1c^1d^1$
	K_3	$a^0b^0c^0d^1$		K_4	$a^1b^2c^2d^2$
	K_5	$a^2b^0c^0d^2$		K_5	$a^2b^0c^0d^2$
	K_7	$a^1b^2c^2d^2$		K_{12}	$a^0b^0c^0d^1$
	K_8	$a^0b^1c^2d^0$		K_{14}	$a^0b^1c^2d^2$
	K_{23}	$a^2b^1c^2d^2$		K_{15}	$a^1b^2c^2d^2$
	K_{26}	$a^2b^1c^1d^2$		K_{16}	$a^1b^1c^1d^0$
	K_{27}	$a^2b^0c^1d^0$		K_{20}	$a^0b^1c^1d^0$
	K_{29}	$a^1b^0c^0d^0$		K_{23}	$a^2b^1c^2d^2$
	K_{33}	$a^0b^0c^0d^1$		K_{27}	$a^2b^0c^1d^0$
	K_{37}	$a^0b^1c^0d^0$		K_{28}	$a^0b^2c^1d^2$
	K_{38}	$a^0b^1c^1d^2$		K_{32}	$a^0b^0c^1d^2$
	K_{40}	$a^0b^1c^2d^2$		K_{40}	$a^0b^1c^2d^2$
c	K_4	$a^1b^2c^2d^2$	ac^2d^2	K_5	$a^2b^0c^0d^2$
	K_7	$a^1b^2c^2d^2$		K_6	$a^2b^2c^1d^2$
	K_{11}	$a^0b^1c^1d^0$		K_9	$a^2b^2c^1d^2$
	K_{12}	$a^0b^0c^0d^1$		K_{10}	$a^2b^0c^2d^2$
	K_{14}	$a^0b^1c^2d^2$		K_{11}	$a^0b^1c^1d^0$
	K_{17}	$a^1b^2c^0d^0$		K_{19}	$a^0b^1c^2d^0$
	K_{18}	$a^0b^1c^0d^2$		K_{21}	$a^2b^0c^2d^2$
	K_{21}	$a^2b^0c^2d^2$		K_{23}	$a^2b^1c^2d^2$
	K_{23}	$a^2b^1c^2d^2$		K_{27}	$a^2b^0c^1d^0$
c	K_{25}	$a^1b^1c^0d^0$	ac^2d^2	K_{31}	$a^0b^2c^0d^2$
	K_{36}	$a^0b^1c^0d^2$		K_{36}	$a^0b^1c^0d^2$
	K_{37}	$a^0b^1c^0d^0$		K_{39}	$a^2b^0c^2d^2$
	K_{38}	$a^0b^1c^1d^2$		K_{40}	$a^0b^1c^2d^2$
bd^2	K_2	$a^0b^1c^1d^1$	$abcd$	K_1	$a^2b^0c^1d^0$
	K_{12}	$a^0b^0c^0d^1$		K_2	$a^0b^1c^1d^1$
	K_{13}	$a^2b^1c^1d^2$		K_5	$a^2b^0c^0d^2$
	K_{16}	$a^1b^1c^1d^0$		K_7	$a^1b^2c^2d^2$
	K_{19}	$a^0b^1c^2d^0$		K_9	$a^2b^2c^1d^2$
	K_{21}	$a^2b^0c^2d^2$		K_{14}	$a^0b^1c^2d^2$
	K_{24}	$a^0b^2c^1d^2$		K_{18}	$a^0b^1c^0d^2$
	K_{25}	$a^1b^1c^0d^0$		K_{21}	$a^2b^0c^2d^2$
	K_{26}	$a^2b^1c^1d^2$		K_{24}	$a^0b^2c^1d^2$
	K_{31}	$a^0b^2c^0d^2$		K_{28}	$a^0b^2c^1d^2$
	K_{33}	$a^0b^0c^0d^1$		K_{33}	$a^0b^0c^0d^1$
	K_{38}	$a^0b^1c^1d^2$		K_{35}	$a^1b^2c^0d^0$
	K_{40}	$a^0b^1c^2d^2$		K_{39}	$a^2b^0c^2d^2$
bc	K_1	$a^2b^0c^1d^0$	$abcd^2$	K_1	$a^2b^0c^1d^0$
	K_{11}	$a^0b^1c^1d^0$		K_4	$a^1b^2c^2d^2$

Table 2 (cont.)

F_i	subfields K_{i_t}	$Cap(K_{i_t}/k)$	F_i	subfields K_{i_t}	$Cap(K_{i_t}/k)$
	K_{14}	$a^0b^1c^2d^2$		K_5	$a^2b^0c^0d^2$
	K_{15}	$a^1b^2c^2d^2$		K_7	$a^1b^2c^2d^2$
	K_{16}	$a^1b^1c^1d^0$		K_{10}	$a^2b^0c^2d^2$
	K_{17}	$a^1b^2c^0d^0$		K_{11}	$a^0b^1c^1d^0$
	K_{19}	$a^0b^1c^2d^0$		K_{16}	$a^1b^1c^1d^0$
	K_{22}	$a^1b^2c^2d^0$		K_{22}	$a^1b^2c^2d^0$
	K_{24}	$a^0b^2c^1d^2$		K_{25}	$a^1b^1c^0d^0$
	K_{27}	$a^2b^0c^1d^0$		K_{31}	$a^0b^2c^0d^2$
	K_{29}	$a^1b^0c^0d^0$		K_{32}	$a^0b^0c^1d^2$
	K_{38}	$a^0b^1c^1d^2$		K_{33}	$a^0b^0c^0d^1$
	K_{39}	$a^2b^0c^2d^2$		K_{34}	$a^1b^1c^1d^0$
bcd	K_1	$a^2b^0c^1d^0$	abc^2	K_2	$a^0b^1c^1d^1$
	K_2	$a^0b^1c^1d^1$		K_7	$a^1b^2c^2d^2$
	K_4	$a^1b^2c^2d^2$		K_8	$a^0b^1c^2d^0$
	K_9	$a^2b^2c^1d^2$		K_{10}	$a^2b^0c^2d^2$
	K_{13}	$a^2b^1c^1d^2$		K_{12}	$a^0b^0c^0d^1$
	K_{18}	$a^0b^1c^0d^2$		K_{16}	$a^1b^1c^1d^0$
	K_{20}	$a^0b^1c^1d^0$		K_{17}	$a^1b^2c^0d^0$
	K_{27}	$a^2b^0c^1d^0$		K_{29}	$a^1b^0c^0d^0$
	K_{29}	$a^1b^0c^0d^0$		K_{34}	$a^1b^1c^1d^0$
bcd	K_{31}	$a^0b^2c^0d^2$	abc^2	K_{35}	$a^1b^2c^0d^0$
	K_{34}	$a^1b^1c^1d^0$		K_{36}	$a^0b^1c^0d^2$
	K_{36}	$a^0b^1c^0d^2$		K_{39}	$a^2b^0c^2d^2$
	K_{38}	$a^0b^1c^1d^2$		K_{40}	$a^0b^1c^2d^2$

Table 3

L_j	K_{j_u}	$Cap(K_{j_u}/k)$	det N_j	L_j	K_{j_u}	$Cap(K_{j_u}/k)$	det N_j
d	K_7	$a^1b^2c^2d^2$		bc^2	K_8	$a^0b^1c^2d^0$	
c	K_{23}	$a^2b^1c^2d^2$		ad	K_{14}	$a^0b^1c^2d^2$	
	K_{37}	$a^0b^1c^0d^0$			K_{30}	$a^1b^2c^0d^1$	
	K_{38}	$a^0b^1c^1d^2$	2		K_{31}	$a^0b^2c^0d^2$	2
cd	K_{19}	$a^0b^1c^2d^0$		bc^2d	K_8	$a^0b^1c^2d^0$	
ab^2d	K_{28}	$a^0b^2c^1d^2$		ad^2	K_{16}	$a^1b^1c^1d^0$	
	K_{34}	$a^1b^1c^1d^0$			K_{35}	$a^1b^2c^0d^0$	
	K_{37}	$a^0b^1c^0d^0$	1		K_{36}	$a^0b^1c^0d^2$	0
cd	K_{10}	$a^2b^0c^2d^2$		bc^2	K_3	$a^0b^0c^0d^1$	
ab^2d^2	K_{20}	$a^0b^1c^1d^0$		acd	K_{10}	$a^2b^0c^2d^2$	
	K_{24}	$a^0b^2c^1d^2$			K_{13}	$a^2b^1c^1d^2$	
	K_{37}	$a^0b^1c^0d^0$	1		K_{14}	$a^0b^1c^2d^2$	0

Table 3 (cont.)

L_j	K_{j_u}	$Cap(K_{j_u}/k)$	det N_j	L_j	K_{j_u}	$Cap(K_{j_u}/k)$	det N_j
cd^2	K_{15}	$a^1b^2c^2d^2$		bc^2	K_2	$a^0b^1c^1d^1$	
bd	K_{22}	$a^1b^2c^2d^0$		acd^2	K_3	$a^0b^0c^0d^1$	
	K_{38}	$a^0b^1c^1d^2$			K_{11}	$a^0b^1c^1d^0$	
	K_{39}	$a^2b^0c^2d^2$	-1		K_{30}	$a^1b^2c^0d^1$	0
cd^2	K_2	$a^0b^1c^1d^1$		bcd^2	K_1	$a^2b^0c^1d^0$	
bd^2	K_{13}	$a^2b^1c^1d^2$		a	K_{10}	$a^2b^0c^2d^2$	
	K_{31}	$a^0b^2c^0d^2$			K_{25}	$a^1b^1c^0d^0$	
	K_{38}	$a^0b^1c^1d^2$	1		K_{32}	$a^0b^0c^1d^2$	0
cd^2	K_2	$a^0b^1c^1d^1$		bcd^2	K_6	$a^2b^2c^1d^2$	
a	K_{15}	$a^1b^2c^2d^2$		ac^2d^2	K_{10}	$a^2b^0c^2d^2$	
	K_{23}	$a^2b^1c^2d^2$			K_{21}	$a^2b^0c^2d^2$	
	K_{32}	$a^0b^0c^1d^2$	-1		K_{27}	$a^2b^0c^1d^0$	0
cd^2	K_6	$a^2b^2c^1d^2$		bc^2	K_2	$a^0b^1c^1d^1$	
ad	K_{23}	$a^2b^1c^2d^2$		a	K_8	$a^0b^1c^2d^0$	
	K_{31}	$a^0b^2c^0d^2$			K_{10}	$a^2b^0c^2d^2$	
	K_{39}	$a^2b^0c^2d^2$	-1		K_{17}	$a^1b^2c^0d^0$	0
bcd^2	K_{25}	$a^1b^1c^0d^0$		d	K_3	$a^0b^0c^0d^1$	
ac^2	K_{27}	$a^2b^0c^1d^0$		ac	K_{23}	$a^2b^1c^2d^2$	
	K_{30}	$a^1b^2c^0d^1$			K_{29}	$a^1b^0c^0d^0$	
	K_{35}	$a^1b^2c^0d^0$	1		K_{33}	$a^0b^0c^0d^1$	0
bcd^2	K_{12}	$a^0b^0c^0d^1$		d	K_1	$a^2b^0c^1d^0$	
ac^2d	K_{27}	$a^2b^0c^1d^0$		bc	K_{27}	$a^2b^0c^1d^0$	
	K_{28}	$a^0b^2c^1d^2$			K_{29}	$a^1b^0c^0d^0$	
	K_{32}	$a^0b^0c^1d^2$	-1		K_{38}	$a^0b^1c^1d^2$	0

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