Application of Numerical Differentiation Using Differential Quadrature(DQ) to Curved Member-like Structural Analysis

곡선부재의 구조해석에서 미분구적(DQ)을 이용한 수치미분의 적용

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Key Words : Curved Member(곡선부재), Differential Quadrature(미분구적), Natural Frequency(고유진동수), Numerical Differentiation(수치미분), Structural Analysis(구조해석)

ABSTRACT

This paper deals with the application of the numerical differentiation using the differential quadrature(DQ) in the curved member-like structural analysis. Derivative values of the geometry of structure are definitely needed for analyzing the structural behavior. For verifying the numerical differentiation using DQ, free vibration problems of arch are selected. Terms of curvature composed with the derivatives of arch geometry obtained herein are agreed quite well with exact values obtained explicitly. Natural frequencies subjected to terms of curvature obtained by DQ are agreed quite well with those in the literature. The numerical differentiation using DQ can be practically utilized in the structural analysis.

요 약

이 논문은 곡선부재의 구조해석에서 수치미분의 적용에 관한 연구이다. 구조물 선형식의 미분은 구조물의 거동해석에서 반드시 필요한 수학적 계산 중의 하나이다. 구조물의 선형이 곡선인 경우에 미분식의 산출은 많은 노력과 시간을 필요로 한다. 이 연구에서는 곡선부재의 구조해석에서 미분구적(DQ)을 이용한 수치미분의 적용성을 검증하기 위하여 아치의 자유진동 문제를 택하였다. 미분구적을 이용하여 아치 곡률항의 미분값을 계산하고 이를 대수적으로 구한 정확한 값과 비교하였다. 이 연구에서 얻어진 곡률항을 이용하여 최종적으로 산출한 아치의 고유진동수는 문헌해와 매우 우수하게 근접하였다. 이러한 결과로부터 구조해석에서 미분구적을 이용한 수치미분의 적용성을 입증할 수 있었다.

1. Introduction

Since various numerical methods are definitely needed in the structural analysis, more efficient numerical methods are increasingly asked for in the analyses of structure. Therefore, efficiencies of various numerical methods were studied in the science and engineering fields(1). Historically,

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applications of the numerical integration methods were investigated by many researchers[2]. Meanwhile, works related to the applications of numerical differential method are very rare in the open literature.

This paper deals with application of the numerical differentiation(ND) using the differential quadrature(DQ) in the analyses of the curved member-like structure. Note that hereafter the term of 'numerical differentiation' is implicitly expressed as 'ND' in this study. For verifying ND using DQ, firstly, parameters of a given structure are computed by DQ; secondly, these parameters are compared with exact values obtained explicitly; and finally, structural responses obtained herein are compared with those in the open literature. Herein, free vibration problems of arch with the variable curvature are selected for applying DQ to ND in the structural analysis.

In case of the structures with the variable curvature, namely curved member-like structures such as arches, it is difficult not only to develop theories but also to get their structural responses since the terms of curvature are considered in developing theories[3]. References and their citations include the governing differential equations and solution methods in the free vibration problems of arch. Lee and Wilson[4] investigated free vibrations of the uniform arches in which both parabolic and sinusoidal arches were considered; Wilson et al.[5] reported natural frequencies of the circular arches with both prime and quadratic variable cross-sections; Wilson and Lee[6] studied in-plane free vibrations of the catenary arch with unsymmetric axis; Oh et al.[7] researched free vibration problems including both rotatory and shear deformation effects; Lee et al.[8] reported natural frequencies of the tapered cantilever arch; and Lee et al.[9] derived differential equations governing the free vibrations of arch in the rectangular co-ordinates rather than in the polar co-ordinates. In the works mentioned above, ND methods were not applied in calculating coefficients in the differential equations. In the open literature, only work related to ND methods is the paper by Lee et al.[10] in which Taylor expansion method such as the forward fifth polynomials was applied rather than DQ adapted herein.

2. Differential Quadrature

2.1 First Order Derivative

If a function is given such that $y=f(x)$ shown in Fig. 1, then its derivative dy/dx is obtained explicitly. However, the given functional equations are mathematically complicated, the analytical process obtaining its derivatives are very cumbersome and time consuming.

The differential quadrature(DQ) method is a numerical discretization technique for approximation of the derivatives. Consider a one dimensional problem over a closed interval $[x_1, x_N]$ shown in Fig. 1, in which there are N grid points with coordinates x_i for which step size $h=x_{i+1}-x_i$. Bellman et al.[11] assumed that a function $f(x)$ is sufficiently smooth over the closed interval $[x_1, x_N]$ so that its first

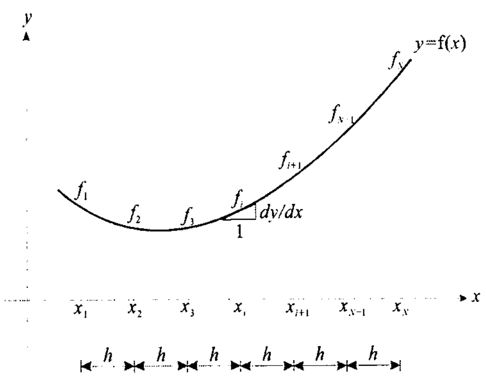

Fig. 1 First order derivative of $y=f(x)$ with step size h

order derivative $f_{x^{(1)}}(x_i)$ at the grid point x_i can be approximated by the following formulation.

$$f_{x^{(1)}}(x_i)=\sum_{j=1}^{N} a_{ij}\cdot f(x_j),\ \text{for}\ i=1,2,\cdots,N \quad (1)$$

where $f(x_j)$ represents the function value at x_j and a_{ij} are the weighting coefficients of first order derivative.

Once the weighting coefficients are determined, it is easy to use the function values to compute the derivatives. The determination of weighting coefficients a_{ij} in Eq. (1) is a key procedure in DQ approximation. There are several approaches that weighting coefficients can be efficiently computed by employing some explicit formulations. Herein, Quan and Chang's approach is chosen in which Lagrange interpolation polynomials are used as the test function. The formulations to compute the weighting coefficients a_{ij} are as follows.

$$a_{ij}=\frac{1}{x_j-x_i}\Pi_{k=1,k\neq i,j}^{N}\frac{x_i-x_k}{x_j-x_k},\ \text{for}\ i\neq j \quad (2.1)$$

$$a_{ii}=\Pi_{k=1,k\neq i}^{N}\frac{1}{x_i-x_k} \quad (2.2)$$

2.2 Second and Higher Order Derivatives

The second order derivatives are easily computed by merely substituting the first derivative values to $f(x_j)$ in Eq. (1) as the function values. Then the newly computed $f_{x^{(1)}}(x_i)$ are now the second order derivatives to the original function values.

Second order derivative $f_{x^{(2)}}(x_i)$ at the grid point x_i can be approximated by the following formulation.

$$f_{x^{(2)}}(x_i)=\sum_{j=1}^{N} b_{ij}\cdot f(x_j),\ \text{for}\ i=1,2,\cdots,N \quad (3)$$

where b_{ij} are weighting coefficients of the second order derivatives and are derived as

$$b_{ij}=\frac{2}{x_j-x_i}\left(\Pi_{k=1,k\neq i,j}^{N}\frac{x_i-x_k}{x_j-x_k}\right)\times\left(\Pi_{l=1,l\neq i,j}^{N}\frac{1}{x_i-x_l}\right)\ \text{for}\ i\neq j \quad (4.1)$$

$$b_{ii}=2\Pi_{k=1,k\neq i}^{N-1}\left[\frac{1}{x_i-x_k}\times\left(\Pi_{l=k+1,l\neq i}^{N}\frac{1}{x_i-x_l}\right)\right] \quad (4.2)$$

Similarly, weighting coefficients of the higher order derivatives can be derived but not shown in this study.

2.3 Grid Point Distribution

For applying DQ, the grid point distribution is determined. Most useful grid point distributions are: uniform grid; Chebyshev-Gauss-Lobatto grid; and grid with coordinates chosen as the roots of Chebyshev polynomials. As an example, the uniform grid with $h=x_{i+1}-x_i$ is already shown in Fig. 1. Herein, the uniform grid is selected. The coordinate of grid point x_i for the closed interval $[x_1, x_N]$ is as follows.

$$x_i=x_1+\frac{x_N-x_1}{N-1}(i-1)\ \text{for}\ i=1,2,\cdots,N \quad (5)$$

where N is the number of grid points and the term of $(x_N-x_1)/(N-1)$ is the uniform step size h.

All theories of this chapter are cited from the work by Shu(12).

3. Free Vibrations of Arch

3.1 Governing Equations

Free vibration problems are selected for applying DQ to ND in structural analysis. The sinusoidal arch is picked out as the objective structure. The geometry of the symmetric sinusoidal arch is defined in Fig. 2. The arch is supported by hinged or clamped ends. Its span length, rise, and shape of the middle surface

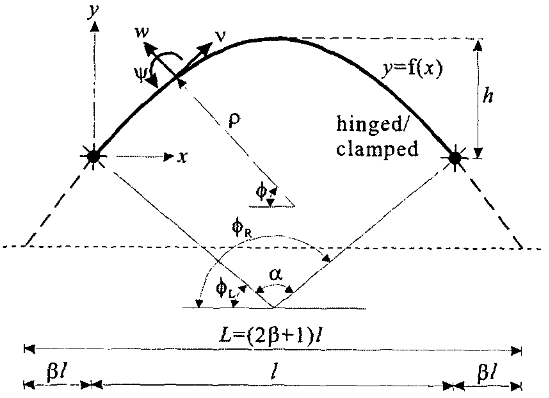

Fig. 2 Sinusoidal arch and its variables

are l, h, and $y=f(x)$, respectively. The parameter β is a non-dimensional quantity by which the chord length L is defined as $L=(2\beta+1)l$. Its radius of curvature ρ, a function of the co-ordinate x, has an inclination ϕ with the x-axis. The subtended angle of arch is α and angles representing the directions of the normal vectors at two ends are ϕ_L and ϕ_R Also in Fig. 2 are the positive directions of radial and tangential displacements, w and v, and the positive direction of rotation ψ of cross-section.

To facilitate the numerical studies, the non-dimensional system parameters of $\xi=x/l$, $\eta=y/l$, $\delta=w/l$, $\lambda=v/l$, $\zeta=\rho/l$, $f=h/l$ and $s=l/r$ are introduced. The co-ordinates, the displacements, the radius and the rise are normalized by span length l, respectively. The parameter s is the slenderness ratio in which r is the radius of gyration of cross-section.

Differential equations governing in-plane free vibrations of arch with the variable curvature were derived by many researchers[4~10]. Among these equations, the following differential equations are selected from the work by Lee and Wilson[4].

$$\delta'''' = e_1\delta''' + (e_2 + s^{-4}e_3C_k^2)\delta'' + (e_1 - s^{-4}e_4C_k^2)\delta' + (e_5 + s^{-4}e_6C_k^2)\delta + (1 - s^{-4}C_k^2)e_3\lambda' + s^{-4}e_4C_k^2\lambda \quad (6.1)$$

$$\lambda'' = e_7\delta'' + (s^{-4}C_k^2 - 1)\delta' + e_8\delta + e_9\lambda' + s^{-4}(e_3 - 1)C_k^2\lambda \quad (6.2)$$

where $(\,'\,) = d/d\phi$ and coefficients $e_1 \sim e_9$ are as follows :

$$e_1 = 5\zeta'/\zeta \quad (7.1)$$

$$e_2 = 2\zeta''/\zeta - 8(\zeta'/\zeta)^2 - 2 \quad (7.2)$$

$$e_3 = -(s\zeta)^2 \quad (7.3)$$

$$e_4 = -s^2\zeta\zeta' \quad (7.4)$$

$$e_5 = 2\zeta''/\zeta - 8(\zeta'/\zeta)^2 - (s\zeta)^2 - 1 \quad (7.5)$$

$$e_6 = (s\zeta)^4 \quad (7.6)$$

$$e_7 = \zeta'/(s^2\zeta^3) \quad (7.7)$$

$$e_8 = \zeta'[1 + 1/(s\zeta)^2]/\zeta \quad (7.8)$$

$$e_9 = \zeta'/\zeta \quad (7.9)$$

The eigenvalue, i.e. C_k, in Eq. (6.1) and (6.2) is frequency parameter defined as

$$C_k = \omega_k s^2\sqrt{m/(EA)} \quad (8)$$

where ω_k is frequency parameter, $k(=1,2,3,4,\cdots)$ is mode number, m is mass per unit length, E is Young's modulus and A is cross-sectional area, respectively.

The boundary conditions for the hinged ends($\phi=\phi_L$ and $\phi=\phi_R$) are $\lambda=0$, $\delta=0$ and $\delta''=0$. The boundary conditions of the clamped ends($\phi=\phi_L$ and $\phi=\phi_R$) are $\lambda=0$, $\delta=0$ and $\delta'=0$.

Governing equations introduced above can now be solved numerically. First of all, the coefficients $e_1 \sim e_9$ in Eqs. (7.1)~(7.9) are computed before solving differential equations.

3.2 Terms of Curvature

Calculation methods of coefficients $e_1 \sim e_9$ in Eqs. (7.1)~(7.9) are now discussed. When the functional equation of the sinusoidal arch depicted in Fig. 2, i.e. $y=f(x)$, is given, these coefficients can be calculated explicitly. The non-dimensional functional equation of sinusoidal arch $y=f(x)$ with l, h and β is expressed as

$$\eta = b_1 \sin(b_2\xi + b_3) + f - b_1, \quad 0 \le \xi \le 1 \tag{9.1}$$

where

$$b_1 = f/(1-\sin b_3) \tag{9.2}$$

$$b_2 = \pi/(1+2\beta) \tag{9.3}$$

$$b_3 = \pi\beta/(1+2\beta) \tag{9.4}$$

Using Eqs. (9.1)~(9.4) gives the functional relationship between the polar co-ordinate ϕ and the rectangular coordinate ξ. See Fig. 2. The result is

$$\phi = \pi/2 - \tan^{-1}[b_1 b_2 \cos(b_2\xi + b_3)], \quad \phi_L \le \phi \le \phi_R \tag{10.1}$$

or

$$\xi = b_2^{-1}\cos^{-1}[b_1^{-1} b_2^{-1}\tan(\pi/2-\phi)] - b_3 b_2^{-1}, \quad 0 \le \xi \le 1 \tag{10.2}$$

Note that ϕ_L and ϕ_R in Eq. (10.1) are computed by merely substituting $\xi=0$ and $\xi=1$ into Eq. (10.1), respectively, so that subtended angle $\alpha=\phi_R-\phi_L$.

Now, radius of curvature ζ at any co-ordinate ξ is obtained by using the well-known equation, or

$$\zeta = [1+(\eta^{i})^2]^{3/2}/\eta^{ii} \tag{11}$$

where $(\ )^{i} = d/d\xi$.

Using Eqs. (9)~(11), terms of the curvature ζ, ζ^{i} and ζ^{ii} are obtained mathematically as follows:

$$\zeta = (1+b_1^2 b_2^2 b_5^2)^{3/2}/(b_1 b_2^2 b_4) \tag{12.1}$$

$$\zeta^{i} = -b_5(1+b_1^2 b_2^2 b_5^2)^{3/2}(1+b_1^2 b_2^2 b_5^2 + 3b_1^2 b_2^2 b_4^2)/(b_1^2 b_2^3 b_4) \tag{12.2}$$

$$\begin{aligned}\zeta^{ii} = (1+b_1^2 b_2^2 b_5^2)^{3/2}[&(1+b_1^2 b_2^2 b_5^2) \\ &\times(1+b_1^2 b_2^2 b_5^2 + 3b_1^2 b_2^2 b_4^2) \\ &+3b_1^2 b_2^2 b_5^2(1+b_1^2 b_2^2 b_5^2 + 3b_1^2 b_2^2 b_4^2) \\ &-4b_1^2 b_2^2 b_5^2(1+b_1^2 b_2^2 b_5^2) + 3b_5^2/b_4^2 \\ &\times(1+b_1^2 b_2^2 b_5^2)(1+b_1^2 b_2^2 b_5^2 \\ &+3b_1^2 b_2^2 b_4^2)]/(b_1^3 b_2^4 b_4^3)\end{aligned} \tag{12.3}$$

where

$$b_4 = \sin(b_2\xi + b_3) \tag{12.4}$$

$$b_5 = \cos(b_2\xi + b_3) \tag{12.5}$$

As discussed above, the mathematical procedure for calculating terms of the curvature are complicated and cumbersome, and also takes very much time. It is expected that obtaining more higher derivatives is more complicated and takes more time, and there should be some possibility to be led to wrong results. Furthermore, derivatives ζ^{i} and ζ^{ii} of Eqs. (12.2) and (12.3) can not be used yet directly for calculating coefficients $e_1 \sim e_9$ because derivatives ζ' and ζ'' are needed rather than derivatives ζ^{i} and ζ^{ii}. This means that one more step is needed in order to transfer derivatives with respect to ξ to those with respect to ϕ by using Eq. (10.1) or (10.2), which is also sometimes complicated and cumbersome.

Meanwhile, the radius ζ is easily computed in this study, since ζ expressed in Eq. (11) composed with η^{i} and η^{ii} is calculated by DQ. Once ζ is obtained, both ζ' and ζ'' are obtained by DQ. Subsequently, coefficients $e_1 \sim e_9$ in Eqs. (6.1) and (6.2) are computed.

Once again, it is accentuated that structural parameters such as the coefficients $e_1 \sim e_9$ are computed by only DQ omitting complicated and cumbersome procedures discussed above for obtaining derivatives such as Eqs. (12.1)~(12.3).

3.3 Computing Procedure

In order to calculate the coefficients $e_1 \sim e_9$, terms of curvature ζ, ζ' and ζ'' are calculated approximately by DQ. The calculating procedure is summarized as follows:

(1) Arch parameters f, β and s are input. Number N in DQ is input.

(2) Set of grid points ϕ is calculated by Eqs. (5) and (10.1).

(3) Polar co-ordinate ϕ_i is selected from the set of ϕ.

(4) Rectangular co-ordinate ξ_i corresponding ϕ_i is calculated by Eq. (10.2).

(5) Set of grid points ξ is calculated by Eq. (5) for which set of ξ has to include ξ_i obtained in step 4).

(6) Set of the co-ordinates η corresponding ξ obtained in step 5) is calculated by Eq. (9.1).

(7) Derivatives η^i and η^{ii} at ξ_i are calculated by DQ. See Eqs. (1) and (3).

(8) Radius ζ_i at ξ_i, namely at ϕ_i, is calculated by Eq. (11).

(9) After calculating all ζ_i, derivatives ζ_i' and ζ_i'' at each grid point ϕ_i are computed by DQ.

(10) Coefficients $e_1 \sim e_9$ at ϕ_i are computed by Eqs. (7.1)~(7.9).

Once coefficients are computed, differential Eqs. (6.1) and (6.2) subjected to the boundary conditions can now be solved numerically and frequency parameters C_k are obtained. In this study, the Runge-Kutta and determinant search methods[4] are used. For understanding the solution methods in detail, readers can refer the work by Lee and Wilson[4].

Table 1 Convergence analysis

N	Terms of curvature		
	ζ	ζ'	ζ''
11	+0.972843	−3.05686	+1.84053
21	+0.972843	−3.03080	+1.76550
41	+0.972843	−3.03074	+1.76547
51	+0.972843	−3.03074	+1.76547
101	+0.972843	−3.03074	+1.76547

* $f=0.3$, $\beta=0.5$, $\phi_i=0.891$

4. Numerical Examples and Discussion

4.1 Convergence Analysis

Accuracies of ND are absolutely affected by the step size h which can be obtained by the number of grid points N as shown in Fig. 1. It is general that the appropriate h is estimated by the convergence analysis in this kind of the approximation problems.

Prior to executing numerical examples, convergence analysis on terms of curvature is done. Shown in Table 1 are the terms of curvature ζ, ζ' and ζ'' calculated by varying numbers N=11, 21, 41, 51 and 101. Then the step size h is obtained by the Eq. (5). System parameters used in calculation are f=0.3, β =0.5 and $\phi_i = \phi_L + \alpha/10 = 0.891$.

The results with the six digit figures are presented. In case of ζ calculation, using N =11 gives accuracy of six significant figures and calculations of ζ' and ζ'' need N=41 for the accuracy of six significant figures. More greater number N should be needed for achieving same accuracy in calculating higher derivatives.

Consequently, it is very important to select the suitable number N after convergence analysis is executed. Hereafter, N=51 is chosen

Table 2 Comparison of ζ, ζ' and ζ''

Terms	DQ	Exact
ζ	+0.972843	+0.972843
ζ'	−3.03074	−3.03075
ζ''	+1.76547	+1.76549

* $f=0.3$, $\beta=0.5$, $\phi_i=0.891$

for numerical examples.

4.2 Comparison of Terms of Curvature

Terms of curvature ζ, ζ' and ζ'' are computed by DQ and these values are compared with those computed by Eqs. (12.1)~(12.3) explicitly where it is recalled that latter ones are exact. The results are shown in Table 2 in which system parameters are f=0.3, β=0.5 and ϕ_i=0.891. Two results are agreed closely within at least five significant figures. It is cleared that DQ is practical in the real engineering systems.

4.3 Comparison of Coefficients

Governing differential Eqs. (6.1) and (6.2) have the coefficients $e_1 \sim e_9$ with which terms of curvature ζ, ζ' and ζ'' are composed. It is natural that frequency parameters can be calculated accurately only if these coefficients are obtained accurately. For validating that coefficients obtained by DQ are sufficiently exact, both coefficients by DQ and exact ones are compared in Table 3 in which system parameters are f=0.3, β=0.5, s=100 and ϕ_i =0.891. Two results are agreed quite well within at least four significant figures.

Table 3 Comparison of coefficients

Coeff.	DQ	Exact
e_1	−15.5768	−15.5768
e_2	−43.3479	−43.3482
e_3	−10566.1	−10566.1
e_4	+29484.5	+29484.5
e_5	−9506.59	−9506.59
e_6	+111642000.	+111642000.
e_7	−0.000329172	−0.000329172
e_8	−3.11536	−3.11536
e_9	−3.11536	−3.11536

* f=0.3, β=0.5, s=100, ϕ_i=0.891

4.4 Comparison of Frequency Parameters

For verifying application of DQ to ND, frequency parameters, C_k(k=1,2), computed by both DQ and functional method are computed. Firstly, coefficients $e_1 \sim e_9$ in the differential Eqs. (6.1) and (6.2) are calculated by both DQ and functional method separately. Secondly, differential equations with coefficients by DQ and functional method are solved separately. Finally, two results of C_k are compared. In calculating C_k, system parameters β=0.5 and s=100 with varying f=0.1, 0.2, 0.3, 0.4 and 0.5 are applied. The boundary conditions of hinged-hinged and clamped-clamped ends are considered. The results are presented in Table 4 which shows that C_k values by DQ agree quite well with those by the functional method. It is clearly fact that ND using DQ leads accurate results in calculating frequencies of the curved member-like structure such as arch.

Table 4 Comparison of C_k(k=1, 2) between DQ and Exact(β=0.5, s=100)

· Hinged-hinged ends

f	C_1		C_2	
	DQ	Exact	DQ	Exact
0.1	38.16	38.42	60.07	60.23
0.2	30.91	30.84	71.99	71.85
0.3	26.00	26.15	63.70	63.68
0.4	21.63	21.85	55.11	55.02
0.5	15.42	15.36	40.93	41.07

· Clamped-clamped ends

f	C_1		C_2	
	DQ	Exact	DQ	Exact
0.1	56.14	56.98	66.99	66.79
0.2	46.54	46.71	93.19	93.04
0.3	27.83	27.98	70.85	70.79
0.4	26.58	26.72	64.34	64.59
0.5	18.98	19.07	48.89	48.76

Table 5 Comparison of C_k between this study and reference(f=0.25, s=200)

·Hinged-hinged ends

Data source	Frequency parameter, C_k			
	$k=1$	$k=2$	$k=3$	$k=4$
This study	26.7	64.7	118.	184.
Reference (4)	26.6	64.6	117.	183.

·Clamped-clamped ends

Data source	Frequency parameter, C_k			
	$k=1$	$k=2$	$k=3$	$k=4$
This study	40.9	85.2	147.	217.
Reference (4)	40.8	85.0	147.	216.

4.5 Comparison of Frequency Parameters with Reference Value

For comparing frequency parameters C_k calculated in this study with those of reference[4], the lowest four $C_k (k=1,2,3,4)$ of the parabolic arch are presented in Table 5, whose system parameters are f=0.25 and s=200. It is noted that the parameter β is unneeded since the chord length L is not available for the parabolic arches.

Two results are agreed very well with each other. From this table and others discussed above, it is finally concluded that ND using DQ can be practically utilized in the analyses of structure especially built with the curved member-like structure such as arch.

5. Concluding Remarks

Structural parameters composed with the derivatives of functional equation of curved member play an important role in the structural analysis. For the curved member-like structures with the variable curvature, it is complicated and also time consuming to obtain derivatives of functional equation explicitly in itself. Application of numerical differentiation using the differential quadrature(DQ) is investigated in this paper. Herein, free vibration problems of the sinusoidal and parabolic arches are chosen for numerical examples. Structural parameters such as curvatures, derivatives of curvature and coefficients of governing equations are computed numerically by DQ. All results obtained herein are agreed quite well with those of exact solutions. Also frequency parameters of this study are agreed closely with those of reference values. It is concluded that numerical differentiation using DQ can be practically utilized to analyses of the curved member-like structures without the functional procedures.

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