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# CONNECTIONS ON ALMOST COMPLEX FINSLER MANIFOLDS AND KOBAYASHI HYPERBOLICITY

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Abstract. In this paper, we establish a necessary condition in terms of curvature for the Kobayashi hyperbolicity of a class of almost complex Finsler manifolds. For an almost complex Finsler manifold with the condition (R), so-called Rizza manifold, we show that there exists a unique connection compatible with the metric and the almost complex structure which has the horizontal torsion in a special form. With this connection, we define a holomorphic sectional curvature. Then we show that this holomorphic sectional curvature of an almost complex submanifold is not greater than that of the ambient manifold. This fact, in turn, implies that a Rizza manifold is hyperbolic if its holomorphic sectional curvature is bounded above by $-1$.

## 1. Introduction

It is known that if a complex manifold admits a Hermitian metric with holomorphic sectional curvature bounded above by $-1$, it is hyperbolic in the sense that its Kobayashi pseudo-distance is a distance (see, e.g., [5]). In [6], S. Kobayashi extended his concept of hyperbolicity to almost complex manifolds and obtained the hyperbolicity criterion for almost Hermitian manifolds. In [9], authors studied various connections on almost complex Finsler manifolds in pursuit of application to global properties of the manifolds. Here we further explore the usage of such connections and extend Kobayashi's results to so-called Rizza manifolds, almost complex Finsler manifolds with the compatibility condition (R) on the Finsler metric and the almost complex structure.

In §2, we set up the notations for Rizza manifolds and cook up a unitary frame bundle. For a Rizza manifold $(M, J, L)$, we consider the pull-back bundle $\tilde{\pi} : p^*TM \to \widetilde{TM}$ of the tangent bundle $\pi : TM \to M$ by the projection $p : \widetilde{TM} \to M$. Here $\widetilde{TM} = TM \setminus \{\text{zero section of } \pi : TM \to M\}$. Let $\hat{\pi} : \mathcal{F}M \to \widetilde{TM}$ be a Finsler bundle: an associated frame bundle of $\tilde{\pi} : p^*TM \to$

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$\widetilde{TM}$. We define a generalized Finsler structure $G_{ij}$ satisfying $G_{ij} = G_{pq}J_i^p J_j^q$ from $L$. Then we construct a special subbundle $\mathcal{FU}(M)$ of the Finsler bundle $\mathcal{F}M$ over $\widetilde{TM}$ : $\mathcal{FU}(M)$ is roughly the intersection of the complex frame bundle $\mathcal{FC}(M)$ defined by the almost complex structure $J$ and the orthogonal frame bundle $\mathcal{FO}(M)$ defined by the generalized metric $G_{ij}$. It turns out that $\mathcal{FU}(M) \to \widetilde{TM}$ is a principal bundle over $\widetilde{TM}$ with the structure group $\mathcal{U}(n)$.

In §3, we cook up various connections and define the torsion and the curvature. A connection on $\mathcal{FU}(M)$ induces a connection $\nabla$ on $\tilde{\pi} : p^*TM \to \widetilde{TM}$ satisfying $\nabla G = 0$ and $\nabla J = 0$. To choose the most natural connection among such connections, we impose an extra condition on the torsion of the connection $\nabla$. In order to define the torsion, we introduce a linear connection on $T(\widetilde{TM})^{\mathbb{C}}$. The complexification $p^*TM^{\mathbb{C}}$ of $p^*TM$ can be decomposed into $p^*TM^{1,0}$ and $p^*TM^{0,1}$ by the almost complex structure $J$. Now the connection $\nabla$ on $p^*TM$ can be extended complex linearly to $p^*TM^{\mathbb{C}}$. If we have a non-linear connection, i.e., $T(\widetilde{TM}) = \mathcal{H} + \mathcal{V}$, then we have $T(\widetilde{TM})^{\mathbb{C}} = \mathcal{H}^{\mathbb{C}} + \mathcal{V}^{\mathbb{C}}$. The connection $\nabla$ on $p^*TM^{\mathbb{C}}$ produces a linear connection on $T(\widetilde{TM})^{\mathbb{C}}$ and so we can define a torsion of $\nabla$ as that of a linear connection on $T(\widetilde{TM})^{\mathbb{C}}$. And hence the torsion has horizontal and vertical components. We then prove a Theorem 3.1 on the existence of a special connection. This is a generalization of the canonical connection of a almost Hermitian manifold introduced by Kobayashi [6, 7]. With the canonical connection $\nabla$, we define the holomorphic sectional curvature $\mathcal{K}_F(v)$ along $v \in T_xM$ as usual.

In §4, we prove one of the main ingredient: non-increasing property of the curvature for submanifolds. On an almost complex submanifold $M'$ of a Rizza manifold $(M, J, L)$, we will produce a canonical connection and its curvature. And then we show that the holomorphic sectional curvature of $M'$ does not exceed that of $M$ (see, e.g., [2] for the Hermitian case). This is crucial in proving the Theorem 5.2.

In §5, we do some analysis on the holomorphic map $f : D(0,1) \to M$. The canonical connection defined in §3 enables us to do such computation. As a consequence, we generalize a theorem of Kobayashi on the hyperbolicity criterion for almost Hermitian manifolds by Kobayashi [6] to Rizza manifolds:

**Theorem 5.2** (Main Theorem). *If $(M, J, L)$ is a Rizza manifold whose holomorphic sectional curvature $\mathcal{K}_F$ is bounded above by $-1$, then $M$ is hyperbolic.*

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## 2. Preliminaries

Let $M$ be a $2n$-dimensional manifold with an almost complex structure $J$ and a Finsler metric $L$. We call the triple $(M, J, L)$ a Rizza manifold if the following condition, the Rizza condition, is satisfied:

(R) $\qquad L(x,\phi_\theta(y)) = L(x,y)$ for all $x \in M$, $y \in T_xM$ and $\theta \in \mathbb{R}$,

where $\phi_\theta(y) = (\cos\theta)y + (\sin\theta)Jy$. The Rizza condition guarantees that the tangent space at any point of a Rizza manifold is a complex Banach space.

Let $(x^1,\dots,x^{2n})$ be a local coordinate system of $M$ and $(x^1,\dots,x^{2n},y^1,\dots,y^{2n})$ be a local coordinate system of $TM$ induced by $(x^1,\dots,x^{2n})$.

Now consider the pull-back bundle $\tilde{\pi} : p^*TM \to \widetilde{TM}$ of the tangent bundle $\pi : TM \to M$ by the projection $p : \widetilde{TM} \to M$. Here $\widetilde{TM} = TM \setminus \{\text{zero section of } \pi : TM \to M\}$ is the slit tangent bundle.

$$\begin{array}{ccc} p^*TM & \xrightarrow{\tilde{p}} & TM \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \widetilde{TM} & \xrightarrow[p]{} & M \end{array}$$

Let $g_{ij} = \frac{1}{2}\frac{\partial L^2}{\partial y^i \partial y^j}$. If $g_{ij}$ satisfies $g_{ij}(x,y) = g_{pq}(x,y)J_i^p(x)J_j^q(x)$, then $g_{ij}$ is a-priori a Riemannian metric(see [3]). This is why we are interested in the condition (R) for almost complex Finsler manifolds.

Now define $G_{ij}(x,y) = \frac{1}{2}\{g_{ij}(x,y) + g_{pq}(x,y)J_i^p(x)J_j^q(x)\}$. Then $G_{ij}$ is a generalized Finsler structure and satisfies $G_{ij} = G_{pq}J_i^pJ_j^q$. This $G_{ij}$ defines a Riemannian structure $G$ on $\tilde{\pi} : p^*TM \to \widetilde{TM}$ by

$$G_{(x,y)}(U,V) = G_{ij}(x,y)u^iv^j \quad \text{for } U = u^i\frac{\partial}{\partial x^i}\Big|_{(x,y)} \text{ and } V = v^i\frac{\partial}{\partial x^i}\Big|_{(x,y)}.$$

This $G$ is an almost Hermitian structure on $p^*TM$, i.e., $G(U,V) = G(JU,JV)$.

Let $\hat{\pi} : \mathcal{F}M \to \widetilde{TM}$ be an associated frame bundle of $\tilde{\pi} : p^*TM \to \widetilde{TM}$ as in [10]. Now consider a subbundle $\mathcal{FU}(M)$ of $\mathcal{F}M$ defined in the following way. Let $\mathbb{R}^{2n}$ be equipped with a Euclidean inner product and $T_xM \simeq \tilde{\pi}^{-1}(x,y)$ with the Hermitian structure $G_{(x,y)}$. $u \in \mathcal{F}M$ with $\hat{\pi}(u) = (x,y)$ is in $\mathcal{FU}(M)$ if $u$, as a linear map $u : \mathbb{R}^{2n} \to T_xM$, is orthogonal and satisfies $J \circ u = u \circ J_o$. Here $J_o$ is the canonical complex structure of $\mathbb{R}^{2n}$.

Let $\{\epsilon_1,\dots,\epsilon_{2n}\}$ be the standard basis of $\mathbb{R}^{2n}$ and let $\{e_1,\dots,e_n,Je_1,\dots,Je_n\}$ be a local orthonormal frame field of $p^*TM \to \widetilde{TM}$ over an open set $O \subseteq \widetilde{TM}$. To each $u \in \mathcal{FU}(M)$, we assign an element $(x,y,\alpha) \in O \times \mathcal{U}(n)$, where $\hat{\pi}(u) = (x,y)$ and $\alpha = (\alpha_i^j)$ defined in the following way. If we put $u(\epsilon_a) = A_a^be_b + B_a^bJe_b$ for $a = 1,2,\dots,n$, then

$$u(\epsilon_{n+a}) = u \circ J_o(\epsilon_a) = J \circ u(\epsilon_a) = -B_a^be_b + A_a^bJe_b.$$

Furthermore, by the orthogonality of $u$, we have

$$\delta_a^c = G(u(\epsilon_a),u(\epsilon_c)) = (AA^t + BB^t)_a^c,$$
$$0 = G(u(\epsilon_a),u(\epsilon_c)) = (BA^t - AB^t)_a^c,$$

where $A = (A^b_a)$ and $B = (B^b_a)$. Now define $\alpha = A + \sqrt{-1}B$.

Then

$$\alpha\bar{\alpha}^t = AA^t + BB^t + \sqrt{-1}(BA^t - AB^t) = Id,$$

i.e., $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathcal{O}(2n)$. Thus $\alpha \in \mathcal{U}(n)$. And the map $u \mapsto (x, y, \alpha)$ gives a local trivialization of $\mathcal{FU}(M)$. In summary, we have

**Proposition 2.1.** *$\mathcal{FU}(M) \to \widetilde{TM}$ is a principal bundle over $\widetilde{TM}$ with the structure group $\mathcal{U}(n)$.*

## 3. Connections on the unitary frame bundles

The principal bundle $\mathcal{FU}(M)$ in Proposition 2.1 admits a connection. Its connection form $\tilde{\omega}$ is a $\mathfrak{u}(n)$-valued 1-form on $\mathcal{FU}(M)$. The connection form $\tilde{\omega}$ induces an affine connection $\nabla$ on $p^*TM \to \widetilde{TM}$ making both $G$ and $J$ parallel: $\nabla G = 0$ and $\nabla J = 0$.

Now the connection form for $\nabla$ is a locally defined $\mathfrak{u}(n)$-valued 1-form $\omega$ on $\widetilde{TM}$. Fix an orthonormal frame field $\sigma(x, y) = \{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ on $p^*TM \to \widetilde{TM}$. This $\sigma$ is a local section of $\mathcal{FU}(M) \to \widetilde{TM}$ and let $\omega = \sigma^*\tilde{\omega}$. Then $\omega$ is a $\mathfrak{u}(n)$-valued 1-form on $\widetilde{TM}$ of the form

$$(\omega^b_a) = \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix}.$$

We then have a covariant derivative on $p^*TM \to \widetilde{TM}$ defined by

$$\nabla e_a = P^b_a e_b + Q^b_a Je_b,$$
$$\nabla Je_a = -Q^b_a e_b + P^b_a Je_b.$$

Note that the complexification $(p^*TM)^{\mathbb{C}}$ of $p^*TM$ can be decomposed into $p^*TM^{1,0}$ and $p^*TM^{0,1}$ by the almost complex structure $J$, where $p^*TM^{1,0} = \{X - \sqrt{-1}JX \mid X \in p^*TM\}$ and $p^*TM^{0,1} = \{X + \sqrt{-1}JX \mid X \in p^*TM\}$. Now the connection $\nabla$ on $p^*TM$ can be extended complex linearly to $p^*TM^{\mathbb{C}}$. In terms of the local basis $\{\mathbf{e}_a = \frac{1}{2}(e_a - \sqrt{-1}Je_a)\}_{a=1}^n$ for the complex vector space of type $(1,0)$ and $\{\bar{\mathbf{e}}_a = \frac{1}{2}(e_a + \sqrt{-1}Je_a)\}_{a=1}^n$ for that of type $(0,1)$, we have

$$\nabla \mathbf{e}_a = \tfrac{1}{2}\nabla(e_a - \sqrt{-1}Je_a) = \psi^b_a \mathbf{e}_b \quad \text{and} \quad \nabla \bar{\mathbf{e}}_a = \overline{\nabla \mathbf{e}_a} = \bar{\psi}^b_a \bar{\mathbf{e}}_b,$$

where $\psi^b_a = P^b_a + \sqrt{-1}Q^b_a$.

Next we consider a non-linear connection of $T\widetilde{TM}$. Let $N^j_i$ be a fixed non-linear connection of $T\widetilde{TM}$. Then $T\widetilde{TM} = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{H}$ is the horizontal subspace of $T\widetilde{TM}$ with the basis $\left\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}\right\}_{i=1}^n$ and $\mathcal{V}$ is the vertical subspace of $T\widetilde{TM}$ with the basis $\left\{\frac{\partial}{\partial y^i}\right\}_{1=1}^n$. With the identifications $\chi^{\mathcal{H}}(\frac{\partial}{\partial x^i}) = \frac{\delta}{\delta x^i}$ from $p^*TM$ onto $\mathcal{H}$ and $\chi^{\mathcal{V}}(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$ from $p^*TM$ onto $\mathcal{V}$, we have the almost complex structure $J$ on $\mathcal{H}$ and on $\mathcal{V}$ in such a way that $J\circ\chi^{\mathcal{H}} = \chi^{\mathcal{H}}\circ J$ and $J\circ\chi^{\mathcal{V}} = \chi^{\mathcal{V}}\circ J$.

Consider the complex tangent space $\mathcal{H}^{1,0}$ of type $(1,0)$ of $\mathcal{H}^{\mathbb{C}}$ with the basis $\{e_a^{\mathcal{H}} = \frac{1}{2}\left(\chi^{\mathcal{H}}(e_a) - \sqrt{-1}J\chi^{\mathcal{H}}(e_a)\right)\}_{a=1}^{n}$ and $\mathcal{H}^{0,1}$ of type $(0,1)$ with the basis $\{\bar{e}_a^{\mathcal{H}} = \frac{1}{2}\left(\chi^{\mathcal{H}}(e_a) + \sqrt{-1}J\chi^{\mathcal{H}}(e_a)\right)\}_{a=1}^{n}$. Similarly $\mathcal{V}^{\mathbb{C}} = \mathcal{V}^{1,0}\oplus\mathcal{V}^{0,1}$, where $\mathcal{V}^{1,0}$ is spanned by $\{e_a^{\mathcal{V}} = \frac{1}{2}\left(\chi^{\mathcal{V}}(e_a) - \sqrt{-1}J\chi^{\mathcal{V}}(e_a)\right)\}_{a=1}^{n}$ and $\mathcal{V}^{0,1}$ is spanned by $\{\bar{e}_a^{\mathcal{V}} = \frac{1}{2}\left(\chi^{\mathcal{V}}(e_a) + \sqrt{-1}J\chi^{\mathcal{V}}(e_a)\right)\}_{a=1}^{n}$. Then we define a Hermitian metric on $T\widetilde{TM}^{\mathbb{C}}$ so that $\{e_a^{\mathcal{H}}, \bar{e}_a^{\mathcal{H}}, e_a^{\mathcal{V}}, \bar{e}_a^{\mathcal{V}}\}$ is orthonormal and we define the induced linear connection on $\widetilde{TM}$ by

$$\nabla e_a^{\mathcal{H}} = \psi_a^b e_b^{\mathcal{H}}, \qquad \nabla \bar{e}_a^{\mathcal{H}} = \bar{\psi}_a^b \bar{e}_b^{\mathcal{H}},$$
$$\nabla e_a^{\mathcal{V}} = \psi_a^b e_b^{\mathcal{V}}, \qquad \nabla \bar{e}_a^{\mathcal{V}} = \bar{\psi}_a^b \bar{e}_b^{\mathcal{V}}.$$

We now define the torsion of the connection $\nabla$ on $p^*TM \to \widetilde{TM}$ as that of the induced linear connection on $\widetilde{TM}$. Let $\{\theta^a, \bar{\theta}^a, \phi^a, \bar{\phi}^a\}$ be the coframe field dual to $\{e_a^{\mathcal{H}}, \bar{e}_a^{\mathcal{H}}, e_a^{\mathcal{V}}, \bar{e}_a^{\mathcal{V}}\}$. For the canonical 1-form $\eta = \theta^a\otimes e_a^{\mathcal{H}} + \phi^a\otimes e_a^{\mathcal{V}}$ on $\widetilde{TM}$, define the torsion $D\eta$ of the connection $\nabla$ by

$$\begin{aligned} D\eta =& d\theta^a\otimes e_a^{\mathcal{H}} + d\phi^a\otimes e_a^{\mathcal{V}} - \theta^a\wedge\psi_a^b e_b^{\mathcal{H}} - \phi^a\wedge\psi_a^b e_b^{\mathcal{V}} \\ =& \Theta^a\otimes e_a^{\mathcal{H}} + \Phi^a\otimes e_a^{\mathcal{V}}, \end{aligned}$$

where $\Theta^a = d\theta^a + \psi_b^a\wedge\theta^b$ and $\Phi^a = d\phi^a + \psi_b^a\wedge\phi^b$.

**Definition 3.1.** $\Theta^a = d\theta^a + \psi_b^a\wedge\theta^b$ is called the horizontal torsion of the connection $\nabla$ and $\Phi^a = d\phi^a + \psi_b^a\wedge\phi^b$ is called the vertical torsion of the connection $\nabla$.

Next we define the holomorphic sectional curvature of the connection $\nabla$. Let the curvature of the connection be

$$\Omega_b^a = d\omega_b^a + \omega_c^a\wedge\omega_b^c$$

and let a $(1,1)$-component $\Omega_b^{a\,1,1}$ of $\Omega_b^a$ be

$$R_b{}^a{}_{c\bar{d}}\theta^c\wedge\bar{\theta}^d + R_b{}^a{}_{c:\bar{d}}\theta^c\wedge\bar{\phi}^d + R_b{}^a{}_{\bar{c}:d}\bar{\theta}^c\wedge\phi^d + R_b{}^a{}_{:c\bar{d}}\phi^c\wedge\bar{\phi}^d.$$

**Definition 3.2.** The holomorphic sectional curvature $\mathcal{K}_F(v)$ of $L$ along $v\in T_xM$ is defined by

$$\begin{aligned} \mathcal{K}_F(v) &= \frac{2}{G(v)^2}\langle\Omega(\chi,\bar{\chi})\chi,\chi\rangle_v \\ &= \frac{2}{G(v)^2}G_{\sigma\bar{a}}R_b{}^\sigma{}_{c\bar{d}}v^c\bar{v}^d v^b\bar{v}^a = \frac{2}{G(v)^2}R_{b\bar{a}c\bar{d}}v^c\bar{v}^d v^b\bar{v}^a, \end{aligned}$$

where $\chi : \widetilde{TM} \to \mathcal{H}^{1,0}$ is the horizontal radial section defined by $\chi(v) = v^a e_a^{\mathcal{H}}|_v$ for $v\in\widetilde{TM}$.

Finally we are ready to define a special connection on a Rizza manifold $(M,J,L)$. We call this uniquely defined connection the canonical connection of the Rizza manifold $(M,J,L)$.

**Theorem 3.1.** *For a Rizza manifold $(M,J,L)$, there exists a unique connection $\nabla$ on $p^*TM \to \widetilde{TM}$ such that $\nabla G=0$, $\nabla J=0$ and*

$$\Theta^a(\mathcal{H},\bar{\mathcal{H}})=0 \ \text{and}\ \Theta^a(\mathcal{H},\bar{\mathcal{V}})=0,$$

*where $\Theta^a$ is the horizontal torsion of $\nabla$.*

*Proof.* Let $\psi_o$ be the connection form on $p^*TM$ induced by a connection on $\mathcal{FU}(M)$. Then $\psi_o$ is a $\mathfrak{u}(n)$-valued 1-form. And hence $\nabla G=0$ and $\nabla J=0$, where $\nabla$ is the connection on $p^*TM$ whose connection form is $\psi_o$. Let the $(1,1)$-component of the horizontal torsion $\Theta_o^a$ be

$$(\Theta_o^a)^{1,1}=A^a_{b\bar c}\theta^b\wedge\bar\theta^c+B^a_{b\bar c}\theta^b\wedge\bar\phi^c+C^a_{\bar b c}\bar\theta^b\wedge\phi^c+D^a_{b\bar c}\phi^b\wedge\bar\phi^c.$$

Now define a new $\mathfrak{u}(n)$-valued connection $\psi$ by

$$\psi^a_b=\psi_{o}{}^a_b+A^a_{b\bar c}\bar\theta^c+B^a_{b\bar c}\bar\phi^c-\bar A^b_{a\bar c}\theta^c-\bar B^b_{a\bar c}\phi^c.$$

Clearly $G$ and $J$ are parallel with respect to the new connection $\psi$. The $(1,1)$-component of its horizontal torsion $\Theta^a$ is

$$\begin{aligned}(\Theta^a)^{1,1}&=(\Theta_o^a)^{1,1}-A^a_{b\bar c}\theta^b\wedge\bar\theta^c-B^a_{b\bar c}\theta^b\wedge\bar\phi^c\\&=C^a_{\bar b c}\bar\theta^b\wedge\phi^c+D^a_{b\bar c}\phi^b\wedge\bar\phi^c.\end{aligned}$$

Then $\Theta^a(\mathcal{H},\bar{\mathcal{H}})=0$ and $\Theta^a(\mathcal{H},\bar{\mathcal{V}})=0$. Thus $\psi$ is the desired connection.

Now for the uniqueness of such connections, assume that there exist two such connections $\psi_1$ and $\psi_2$ with the corresponding horizontal torsions $\Theta_1$ and $\Theta_2$ respectively. Since $\psi_1$ and $\psi_2$ are skew-Hermitian, we may put

$$\psi_1{}^a_b-\psi_2{}^a_b=E^a_{bc}\theta^c-\bar E^b_{ac}\bar\theta^c+F^a_{bc}\phi^c-\bar F^b_{ac}\bar\phi^c.$$

Then

$$\begin{aligned}(\Theta_1{}^a-\Theta_2{}^a)^{1,1}&=[(\psi_1{}^a_b-\psi_2{}^a_b)\wedge\theta^b]^{1,1}\\&=-\bar E^b_{ac}\bar\theta^c\wedge\theta^b-\bar F^b_{ac}\bar\phi^c\wedge\theta^b.\end{aligned}$$

Since $\Theta_i^a(\mathcal{H},\bar{\mathcal{H}})=0$ and $\Theta_i^a(\mathcal{H},\bar{\mathcal{V}})=0$ for $i=1,2$, $\bar E^b_{ac}=\bar F^b_{ac}=0$ which means $\psi_1{}^a_b=\psi_2{}^a_b$. □

Modifying the proof of Theorem 3.1, we further prescribe the horizontal component of the torsion:

**Corollary 3.2.** *For a Rizza manifold $(M,J,L)$, and given $P^a_{b\bar c}$ and $Q^a_{b\bar c}$ there exists a unique connection $\nabla$ on $p^*TM\to\widetilde{TM}$ such that $\nabla G=0$, $\nabla J=0$ and*

$$\Theta^a(e^{\mathcal{H}}_b,\bar e^{\mathcal{H}}_c)=P^a_{b\bar c}\ \text{and}\ \Theta^a(e^{\mathcal{H}}_b,\bar e^{\mathcal{V}}_c)=Q^a_{b\bar c},$$

*where $\Theta^a$ is the horizontal torsion of $\nabla$.*

We have the following analogue of Theorem 3.1 by imposing the conditions on the vertical torsion.

**Theorem 3.3.** *For a Rizza manifold $(M, J, L)$, there exists a unique connection $\nabla$ on $p^*TM \to \widetilde{TM}$ such that $\nabla G = 0$, $\nabla J = 0$ and*

$$\Phi^a(\mathcal{V}, \bar{\mathcal{H}}) = 0 \text{ and } \Phi^a(\mathcal{V}, \bar{\mathcal{V}}) = 0,$$

*where $\Phi^a$ is the vertical torsion of $\nabla$.*

**Corollary 3.4.** *For a Rizza manifold $(M, J, L)$, and given $P^a_{b\bar{c}}$ and $Q^a_{b\bar{c}}$ there exists a unique connection $\nabla$ on $p^*TM \to \widetilde{TM}$ such that $\nabla G = 0$, $\nabla J = 0$ and*

$$\Theta^a(e^{\mathcal{V}}_b, \bar{e}^{\mathcal{H}}_c) = P^a_{b\bar{c}} \text{ and } \Theta^a(e^{\mathcal{V}}_b, \bar{e}^{\mathcal{V}}_c) = Q^a_{b\bar{c}},$$

*where $\Theta^a$ is the horizontal torsion of $\nabla$.*

*Remark.* For a complex Finsler manifold, the Chern-Finsler connection has a torsion whose horizontal $(1,1)$ torsion vanishes and whose vertical torsion has no $\bar{\theta}^a \wedge \phi^b$ and $\phi^a \wedge \bar{\phi}^b$ terms. So both connections in Theorem 3.1 and Theorem 3.3 are generalizations of the Chern-Finsler connection for complex Finsler manifolds. As we will see in §4, the connection in Theorem 3.1 is much more useful than the one in Theorem 3.3. This is why we call the connection in Theorem 3.1 the canonical connection.

## 4. Curvature of almost complex submanifolds of Rizza manifolds

Let $(M, J, L)$ be a Rizza manifold of dimension $2n$ and let $M'$ be a $2m$-dimensional almost complex submanifold of $M$, i.e., at each point of $x \in M'$, the tangent space $T_xM'$ is invariant under $J$. A Rizza metric $L$ on $M$ induces a Rizza metric on $M'$. Here we show that the holomorphic sectional curvature of $M'$ is not greater than that of $M$. This is a generalization of the well-known property in Hermitian geometry(see, e.g., [2]) and is essential in the proof of Lemma 5.1. Hereafter, we denote by $R'_{b\bar{a}c\bar{d}}$, $\mathcal{K}'_F$, etc., the corresponding quantities $R_{b\bar{a}c\bar{d}}$, $\mathcal{K}_F$, on $M'$.

Let $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ be a local orthonormal frame field on $p^*TM$ such that $\{e_1, \dots, e_m, Je_1, \dots, Je_m\}$ are tangent to $M'$. If $\{\theta^a, \bar{\theta}^a\}$ is dual to $\{e^{\mathcal{H}}_a, \bar{e}^{\mathcal{H}}_a\}$, then $\theta^{m+1} = \cdots = \theta^n = 0$ on $M'$. Let $(\psi^a_b)$ be the canonical connection on $M$ and $\Theta^a = d\theta^a + \psi^a_b \wedge \theta^b$ be its horizontal torsion.

If we restrict $\Theta^r = d\theta^r + \sum_{b=1}^n \psi^r_b \wedge \theta^b$ to $\widetilde{TM'}$ for $r = m+1, \dots, n$, then we have $\Theta^r = \sum_{b=1}^m \psi^r_b \wedge \theta^b$. Since $\Theta^a(\mathcal{H}, \bar{\mathcal{H}}) = 0$ and $\Theta^a(\mathcal{H}, \bar{\mathcal{V}}) = 0$, $\psi^r_b$ $(1 \le b \le m,\ m+1 \le r \le n)$ restricted to $\widetilde{TM'}$ are of type $(1,0)$. And hence we may put $\psi^r_b = h^r_{bc}\theta^c + h^r_{b:c}\phi^c$ on $\widetilde{TM'}$.

Note that $\psi' = (\psi^a_b)_{a,b=1,\dots,m}$ defines a canonical connection on $M'$. Let $\Omega'$ be the curvature of $\psi'$. Then

$$\begin{aligned}\Omega'^a_b &= d\psi'^a_b + \sum_{c=1}^{m} \psi'^a_c \wedge \psi'^c_b = \Omega^a_b - \sum_{r=m+1}^{n} \psi^a_r \wedge \psi^r_b = \Omega^a_b - \sum_{r=m+1}^{n} \psi^r_b \wedge \bar{\psi}^r_a \\ &= \Omega^a_b - \sum h^r_{bc}\overline{h^r_{ad}}\theta^c \wedge \bar{\theta}^d - \text{ terms containing no } \theta^c \wedge \bar{\theta}^d\end{aligned}$$

and so $R'_{b\bar{a}c\bar{d}} = R_{b\bar{a}c\bar{d}} - \sum_{r=m+1}^{n} h^r_{bc}\bar{h}^r_{ad}$.

For a unit vector $v = \sum_{a=1}^{m} v^a e_a$ in $T_xM'$, we have

$$\begin{aligned}\mathcal{K}'_F(v) &= 2R'_{b\bar{a}c\bar{d}}\bar{v}^a v^b v^c \bar{v}^d = \mathcal{K}_F(v) - 2\sum_{r=m+1}^{n} h^r_{bc}\bar{h}^r_{ad}\bar{v}^a v^b v^c \bar{v}^d \\ &= \mathcal{K}_F(v) - 2\sum_{r=m+1}^{n}\left|\sum_{b,c=1}^{n} h^r_{bc}v^b v^c\right|^2 \le \mathcal{K}_F(v).\end{aligned}$$

In summary, we have

**Theorem 4.1.** *Let $M'$ be an almost complex submanifold of a Rizza manifold $(M, J, L)$. Then*

$$\mathcal{K}'_F(v) \le \mathcal{K}_F(v) \ \text{ for all } v \in T_xM'.$$

*for the canonical connection on $TM'$.*

## 5. Kobayashi Hyperbolicity of Rizza manifolds

In this section, we establish a necessary condition for the Kobayashi hyperbolicity of Rizza manifolds. Recall that a mapping $f : D(0,1) \to M$ from a unit disk $D(0,1)$ into an almost complex manifold $M$ is called holomorphic if $J \circ f_* = f_* \circ \tilde{J}$, where $\tilde{J}$ is the complex structure on $D(0,1)$. The concepts of Kobayashi pseudo-distance and hyperbolicity for complex manifolds can be extended to our setting as usual. For a description of intrinsic metrics and hyperbolicity in detail, we refer to [5].

Now we do some analysis on the holomorphic maps.

**Lemma 5.1.** *Let $w \in T_pM$, $w \neq 0$. For a holomorphic map $f : D(0,1) \to M$ with $f_*(a\frac{\partial}{\partial z}|_0) = w$,*

$$\frac{L(w)}{\sqrt{2}} \le |a| \quad \text{if } \mathcal{K}_F \le -1.$$

*Proof.* Let $S = \{z \in D(0,1) : f_* \text{ is singular at } z\}$ and $D' = D(0,1) \setminus S$ and $M' = f(D')$. Since $f$ is holomorphic and $S$ is discrete, $M'$ is an almost complex submanifold of $M$. Now we have a basis $\{f_u, f_v\}$ of $T_{f(z)}M'$ for $z \in D'$, where $f_u = f_*(\frac{\partial}{\partial u})$ and $f_v = f_*(\frac{\partial}{\partial v})$. Note that $f_v = J \circ f_u$. Let $(u, v, y^1, y^2)$ be a local coordinate system of $TM'$ given by $x = f(u,v)$ and $y = y^1 f_u + y^2 f_v$.

For a local orthonormal frame field of $p^*TM' \to \widetilde{TM'}$, consider

$$e(x,y) = \frac{f_u}{\sqrt{G_{(x,y)}(f_u, f_u)}} \quad \text{and} \quad Je(x,y) = \frac{f_v}{\sqrt{G_{(x,y)}(f_u, f_u)}}.$$

Now instead of a rectangular coordinate system $(y^1, y^2)$, let us use a polar coordinate system $(r, \theta)$. Then for $y \in \widetilde{TM'}$, the Rizza condition (R)

$$L^2(x,y) = L^2(x, r\cos\theta f_u + r\sin\theta f_v) = r^2 L^2(x, f_u)$$

implies that $\frac{\partial^3 L^2}{\partial r^3} = 0$ and $\frac{\partial L^2}{\partial \theta} = 0$. Thus

$$\frac{\partial}{\partial y^k} g_{ij}(x,y) = \frac{\partial^3}{\partial y^k \partial y^i \partial y^j}\left(\frac{L^2}{2}\right) = \sum_{s+t=3} \sum_{t=0}^{3} a_{s,t} \frac{\partial^3}{\partial r^s \partial \theta^t}\left(\frac{L^2}{2}\right) = 0,$$

i.e., $g_{ij}(x,y)$ is independent of $y$ and so is $G_{ij}(x,y)$. Therefore

$$\sqrt{G_{(x,y)}(f_u, f_u)} = \sqrt{G_{(x,f_u)}(f_u, f_u)} = L(f_u)$$

is a function of $x \in f(D(0,r))$, say $h(x)$. Then

$$e(x,y) = \frac{f_u}{h(x)} \qquad \text{and} \qquad Je(x,y) = \frac{f_v}{h(x)}.$$

Following the notations in §2, we put

$$\mathbf{e} = \tfrac{1}{2}(e - \sqrt{-1}Je) = \tfrac{1}{2h}(f_u - \sqrt{-1}f_v),$$

$$e^{\mathcal{H}} = \tfrac{1}{2h}\left(\tfrac{\delta}{\delta u} - \sqrt{-1}\tfrac{\delta}{\delta v}\right), \qquad e^{\mathcal{V}} = \tfrac{1}{2h}\left(\tfrac{\partial}{\partial y^1} - \sqrt{-1}\tfrac{\partial}{\partial y^2}\right),$$

$$\theta = h(du + \sqrt{-1}dv), \qquad \phi = h(\delta y^1 + \sqrt{-1}\delta y^2).$$

Notice that the canonical connection $\psi$ of $M'$ is of the form $\alpha\theta - \bar{\alpha}\bar{\theta} + \beta\phi - \bar{\beta}\bar{\phi}$. And its horizontal torsion is

$$\Theta = d\theta + \psi \wedge \theta = d(\log h) \wedge \theta + \psi \wedge \theta.$$

Then, by direct computation, $\Theta(e^{\mathcal{H}}, \bar{e}^{\mathcal{H}}) = 0$ implies $\alpha = d(\log h)(\bar{e}^{\mathcal{H}}) = \frac{1}{2h^2}(h_u - \sqrt{-1}h_v)$ and $\Theta(e^{\mathcal{H}}, \bar{e}^{\mathcal{V}}) = 0$ implies $\beta = 0$.

Next its curvature is $\Omega = d\alpha \wedge \theta - d\bar{\alpha} \wedge \bar{\theta} + \alpha \wedge d\theta - \bar{\alpha} \wedge d\bar{\theta}$. Since

$$d\alpha(\bar{e}^{\mathcal{H}}) = d\bar{\alpha}(e^{\mathcal{H}}) = \frac{\Delta \log h}{4h^2} - \frac{|\nabla h|^2}{4h^4},$$

$$\alpha d\theta(e^{\mathcal{H}}, \bar{e}^{\mathcal{H}}) = -\bar{\alpha} d\bar{\theta}(e^{\mathcal{H}}, \bar{e}^{\mathcal{H}}) = -\frac{|\nabla h|^2}{4h^4},$$

we have

$$\Omega(e^{\mathcal{H}}, \bar{e}^{\mathcal{H}}) = -\frac{\Delta \log h}{2h^2}.$$

Thus the holomorphic sectional curvature along $e \in T_x M'$ is

$$\mathcal{K}_F(e) = 2R_{1\bar{1}1\bar{1}} = -\frac{\Delta \log h}{h^2},$$

which is $\leq -1$ by Theorem 4.1.

Put $0 < r < 1$. Let $D(0,r)$ be the disk with center 0 and radius $r$. Consider $g(z) = \frac{h(z)}{\rho_r(z)}$ on $D' \cap D(0,r)$, where $\rho_r(z) = \frac{2r}{r^2-|z|^2}$. Note that $d\sigma^2 = \rho_r(z)dzd\bar{z}$ is the Poincaré metric on $D(0,r)$ and its (Gaussian) curvature is $\mathcal{K} = -\frac{\Delta \log \rho_r}{\rho_r{}^2} = -1$. And $g(z)$ is continuous and non-negative. Since $h(z)$ is bounded above on $\overline{D(0,r)}$ and $\rho_r(z) \to \infty$ as $|z| \to r$, $\lim_{|z|\to r} g(z) = 0$. Thus $g(z)$ attains its maximum at some $\tau_0 \in D' \cap D(0,r)$. And hence

$$
\begin{aligned}
0 \geq \Delta \log g(\tau_0) &= \Delta \log h - \Delta \log \rho_r \\
&= -\mathcal{K}_F(e) \cdot h^2(\tau_0) - \rho_r{}^2(\tau_0) \\
&\geq h^2(\tau_0) - \rho_r{}^2(\tau_0)\,,
\end{aligned}
$$

i.e., $g(\tau_0) = \frac{h(\tau_0)}{\rho_r(\tau_0)} \leq 1$ which means $\frac{h(0)\cdot r}{2} = g(0) \leq g(\tau_0) =\leq 1$, for $0 < r < 1$. Letting $r \to 1^-$, $h(0) \leq 2$. Thus

$$L(w) = |a| L\left(\tfrac{1}{2}(f_u - \sqrt{-1} f_v)\right) = \frac{|a| h(0)}{\sqrt{2}} \leq \sqrt{2}|a|.$$

□

Recall that the infinitesimal Kobayashi metric $K$ of $w \in T_pM$, $w \neq 0$, is defined by $K(w) = \inf\{|a|\}$, where the infimum is taken over the holomorphic maps $f : D(0,1) \to M$ satisfying $f(0) = p$ and $f_*\left(a\frac{\partial}{\partial z}|_0\right) = w$. In case, the Kobayashi pseudo-distance $d_K$ induced by $K$ is a distance, we call $M$ hyperbolic.

By Lemma 5.1, we have

$$\mathcal{K}_F(w) \geq \frac{L(w)}{\sqrt{2}} > 0 \quad \text{for all } 0 \neq w \in TM\,,$$

which means that $d_K(p,q) > 0$ for $p \neq q \in M$.

In summary, we have the following theorem, which is a generalization of the theorem of Kobayashi [6] on the hyperbolicity criterion for almost Hermitian manifolds to Rizza manifolds.

**Theorem 5.2** (Main Theorem). *If $(M, J, L)$ is a Rizza manifold whose holomorphic sectional curvature $\mathcal{K}_F$ is bounded above by $-1$, then $M$ is hyperbolic.*

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