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THE RICCI TENSOR OF REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS

JUAN DE DIOS PÉREZ AND YOUNG JIN SUH

ABSTRACT. In this paper, first we introduce the full expression of the curvature tensor of a real hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ from the equation of Gauss and derive a new formula for the Ricci tensor of M in $G_2(\mathbb{C}^{m+2})$. Next we prove that there do not exist any Hopf real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with parallel and commuting Ricci tensor. Finally we show that there do not exist any Einstein Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

Introduction

In the geometry of real hypersurfaces in complex space forms or in quaternionic space forms it can be easily checked that there do not exist any real hypersurfaces with parallel shape operator A by virtue of the equation of Codazzi.

But if we consider a real hypersurface with parallel Ricci tensor S in such space forms, the proof of its non-existence is not so easy. In the class of Hopf hypersurfaces Kimura [7] has asserted that there do not exist any real hypersurfaces in a complex projective space $\mathbb{C}P^m$ with parallel Ricci tensor, that is, $\nabla S = 0$. Moreover, he has given a classification of Hopf hypersurfaces in $\mathbb{C}P^m$ with commuting Ricci tensor, that is, $S\phi = \phi S$ (see in [8]) and showed that M is locally congruent to one of real hypersurfaces of type A_1, A_2, B, C, D and E, that is, respectively, a tube of certain radius r over a totally geodesic $\mathbb{C}P^k$, a complex quadric $\mathbb{Q}^{m-1}$, $\mathbb{C}P^1 \times \mathbb{C}P^{\frac{n-1}{2}}$, a complex two-plane Grassmannian $G_2(\mathbb{C}^5)$ and an Hermitian symmetric space $SO(10)/U(5)$.

On the other hand, in a complex hyperbolic space $\mathbb{C}H^m$ Ki and the second author [6] have given a complete classification of Hopf hypersurfaces in $\mathbb{C}H^m$ with commuting Ricci tensor and have proved that M is locally congruent to

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a horosphere, a geodesic hypersphere, a tube over a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^m$.

In a quaternionic projective space $\mathbb{Q}P^m$ the first author [9] has considered the notion of $S\phi_i = \phi_i S$, $i = 1, 2, 3$, for real hypersurfaces in $\mathbb{Q}P^m$ and classified that M is locally congruent to of A_1, A_2-type, that is, a tube over $\mathbb{Q}P^k$ with radius $0 < r < \frac{\pi}{4}$. Moreover, in also [9] he has classified real hypersurfaces in $\mathbb{Q}P^m$ with parallel Ricci tensor is an open subset of a geodesic hypersphere whose radius r satisfies $\cot^2 r = \frac{1}{2m}$.

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Then the formula concerned with the Ricci tensor mentioned above is not so simple if we consider a real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ (See [3], [4], [10], [11], [12] and [13]).

In this paper we study the analogous question related to the Ricci tensor S for real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, which has a remarkable geometrical structure. The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (See Berndt [2]).

In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperkähler manifold. So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometrical conditions for real hypersurfaces that $[\xi] = \text{Span}\ \{\xi\}$ or $\mathfrak{D}^\perp = \text{Span}\ \{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By using such kinds of geometric conditions Berndt and the second author [3] have proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

If the structure vector field ξ of M in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, M is said to be *Hopf real hypersurface.* In such a case the integral curves of the structure vector field ξ are geodesics (See Berndt and Suh [4]). Moreover, the flow generated by the integral curves of the structure vector field ξ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be *geodesic Reeb flow.* Moreover, we say that the Reeb vector field is Killing, that is, $\mathcal{L}_\xi g = 0$ for the Lie derivative along the direction of the structure vector field ξ, where g denotes the Riemannian metric induced from $G_2(\mathbb{C}^{m+2})$. Then this is equivalent to the fact that the structure tensor ϕ commutes with the shape operator A of M in $G_2(\mathbb{C}^{m+2})$.

When the Ricci tensor S of M in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor ϕ, we say that M has *commuting Ricci tensor.* Moreover, M is said to have *parallel Ricci tensor* if the Ricci tensor S of M in $G_2(\mathbb{C}^{m+2})$ has the property $\nabla S = 0$ for the induced covariant derivative ∇ of M.

In the proof of Theorem A we have proved that the one-dimensional distribution $[\xi]$ is contained in either the 3-dimensional distribution $\mathfrak{D}^{\perp}$ or in the orthogonal complement $\mathfrak{D}$ such that $T_xM = \mathfrak{D}\oplus\mathfrak{D}^{\perp}$. The case (A) in Theorem A is just the case that the one dimensional distribution $[\xi]$ is contained in $\mathfrak{D}^{\perp}$. Of course they satisfy that the Reeb vector ξ is Killing, that is, the structure tensor commutes with the shape operator. Then naturally, it satisfies that the Ricci tensor commutes with the structure tensor (see Remark 5.1 in section 5). But it can be checked easily that the Ricci tensor is not parallel.

On the other hand, it is not difficult to check that the Ricci tensor S of type (B) mentioned in Theorem A can not commute with the structure tensor ϕ and can not be parallel. Then it must be a natural problem to know whether real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with *parallel and commuting Ricci tensor* can exist or not. At least in Theorem A we know that real hypersurfaces of type (A) or of type (B) do not have parallel and commuting Ricci tensor. Of course, they are Hopf hypersurfaces.

Motivated by such a problem the main result of this paper is to prove the non-existence of all Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with *parallel and commuting Ricci tensor* as follows:

Theorem. *There do not exist any Hopf real hypersurfaces in* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, *with parallel and commuting Ricci tensor.*

On the other hand, in a complex projective space $\mathbb{C}P^m$ Cecil and Ryan [5] have proved the non-existence property for Einstein hypersurfaces. From such a view point let us define an Einstein hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ as follows:

A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be *Einstein* if the Ricci tensor S is given by $g(SX, Y) = ag(X, Y)$ for a constant function a and any vector fields X and Y on M. Then naturally we know that its Ricci tensor of M in $G_2(\mathbb{C}^{m+2})$ is parallel and commuting. So we also conclude that

Corollary. *There do not exist any Einstein Hopf real hypersurfaces in*

$$G_2(\mathbb{C}^{m+2}),\ m \geq 3.$$

In section 2 we recall Riemannian geometry of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and in section 3 we will show some fundamental properties of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. The formula for the Ricci tensor S and its covariant derivative ∇S will be shown explicitly in this section. In sections 4 and 5 we will give a complete proof of the main Theorem according to the geodesic Reeb flow satisfying $\xi\in\mathfrak{D}$ or geodesic Reeb flow satisfying $\xi\in\mathfrak{D}^{\perp}$.

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [1], [2], [3] and [4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\mathrm{tr}(JJ_1) = 0$. This fact will be used in next sections.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

Moreover, in [2] it is known that the Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned} &\bar{R}(X,Y)Z \\ &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\ &\quad + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\}, \end{aligned} \tag{1.2}$$

where $\{J_1, J_2, J_3\}$ is any canonical local basis of $\mathfrak{J}$ and X, Y and Z any vector fields on $G_2(\mathbb{C}^{m+2})$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we derive some fundamental formulas which will be used in the proof of our main theorem. Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, that is, a submanifold in $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N.

The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M. Using the above expression (1.2) for the curvature tensor $\bar{R}$, the Gauss and the Codazzi equations are respectively given by

$$\begin{aligned}
&R(X,Y)Z \\
&= g(Y,Z)X - g(X,Z)Y \\
&\quad + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\
&\quad + \sum_{\nu=1}^{3} \{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\} \\
&\quad + \sum_{\nu=1}^{3} \{g(\phi_\nu \phi Y,Z)\phi_\nu \phi X - g(\phi_\nu \phi X,Z)\phi_\nu \phi Y\} \\
&\quad - \sum_{\nu=1}^{3} \{\eta(Y)\eta_\nu(Z)\phi_\nu \phi X - \eta(X)\eta_\nu(Z)\phi_\nu \phi Y\} \\
&\quad - \sum_{\nu=1}^{3} \{\eta(X)g(\phi_\nu \phi Y,Z) - \eta(Y)g(\phi_\nu \phi X,Z)\}\xi_\nu \\
&\quad + g(AY,Z)AX - g(AX,Z)AY
\end{aligned}$$

and

$$\begin{aligned}
&(\nabla_X A)Y - (\nabla_Y A)X \\
&= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \\
&\quad + \sum_{\nu=1}^{3} \{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X,Y)\xi_\nu\} \\
&\quad + \sum_{\nu=1}^{3} \{\eta_\nu(\phi X)\phi_\nu \phi Y - \eta_\nu(\phi Y)\phi_\nu \phi X\} \\
&\quad + \sum_{\nu=1}^{3} \{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu ,
\end{aligned}$$

where R denotes the curvature tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

The following identities can be proved by a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned}
&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu \xi_{\nu+1} = \xi_{\nu+2}, \\
&\phi \xi_\nu = \phi_\nu \xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X), \\
&\phi_\nu \phi_{\nu+1} X = \phi_{\nu+2} X + \eta_{\nu+1}(X)\xi_\nu, \\
&\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2} X + \eta_\nu(X)\xi_{\nu+1}.
\end{aligned} \tag{2.1}$$

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{2.2}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.3}$$

$$\begin{aligned}(\nabla_X\phi_\nu)Y = &- q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\ &- g(AX,Y)\xi_\nu.\end{aligned} \tag{2.4}$$

Summing up these formulas, we find the following

$$\begin{aligned}\nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu) \\ &= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu) \\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX \\ &\quad - g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned} \tag{2.5}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{2.6}$$

3. Proof of main theorem

Now let us contract Y and Z in the equation of Gauss in section 2. Then the Ricci tensor S of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is given by

$$\begin{aligned}SX &= \sum_{i=1}^{4m-1} R(X,e_i)e_i \\ &= (4m+10)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &\quad + \sum_{\nu=1}^{3}\{(\mathrm{Tr}\phi_\nu\phi)\phi_\nu\phi X - (\phi_\nu\phi)^2X\} \\ &\quad - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(X)\phi_\nu\phi\xi_\nu\} \\ &\quad - \sum_{\nu=1}^{3}\{(\mathrm{Tr}\ \phi_\nu\phi)\eta(X) - \eta(\phi_\nu\phi X)\}\xi_\nu + hAX - A^2X,\end{aligned} \tag{3.1}$$

where h denotes the trace of the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. From the formula $JJ_\nu = J_\nu J$, $\mathrm{Tr}\ JJ_\nu = 0$, $\nu = 1,2,3$ we calculate the following for any basis $\{e_1, \ldots, e_{4m-1}, N\}$ of the tangent space of $G_2(\mathbb{C}^{m+2})$

$$\begin{aligned}0 &= \mathrm{Tr}\ JJ_\nu \\ &= \sum_{k=1}^{4m-1} g(JJ_\nu e_k, e_k) + g(JJ_\nu N, N) \\ &= \mathrm{Tr}\ \phi\phi_\nu - \eta_\nu(\xi) - g(J_\nu N, JN) \\ &= \mathrm{Tr}\ \phi\phi_\nu - 2\eta_\nu(\xi)\end{aligned} \tag{3.2}$$

and

$$(3.3)\qquad \begin{aligned}(\phi_\nu\phi)^2X =&\phi_\nu\phi(\phi\phi_\nu X - \eta_\nu(X)\xi + \eta(X)\xi_\nu)\\ =&\phi_\nu(-\phi_\nu X + \eta(\phi_\nu X)\xi) + \eta(X)\phi_\nu{}^2\xi\\ =&X - \eta_\nu(X)\xi_\nu + \eta(\phi_\nu X)\phi_\nu\xi + \eta(X)\{-\xi + \eta_\nu(\xi)\xi\}.\end{aligned}$$

Substituting (3.2) and (3.3) into (3.1), we have

$$(3.4)\qquad \begin{aligned}SX =&(4m+10)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu\\ &+\sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi\}\\ &+hAX - A^2X\\ =&(4m+7)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu\\ &+\sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\}\\ &+hAX - A^2X.\end{aligned}$$

Now the covariant derivative, bearing in mind that S is parallel, of (3.4) becomes

$$(3.5)\qquad \begin{aligned}&(\nabla_Y S)X\\ =&-3(\nabla_Y\eta)(X)\xi - 3\eta(X)\nabla_Y\xi\\ &-3\sum_{\nu=1}^{3}(\nabla_Y\eta_\nu)(X)\xi_\nu - 3\sum_{\nu=1}^{3}\eta_\nu(X)\nabla_Y\xi_\nu\\ &+\sum_{\nu=1}^{3}\Big\{Y(\eta_\nu(\xi))\phi_\nu\phi X + \eta_\nu(\xi)(\nabla_Y\phi_\nu)\phi X\\ &+\eta_\nu(\xi)\phi_\nu(\nabla_Y\phi)X - (\nabla_Y\eta)(\phi_\nu X)\phi_\nu\xi\\ &-\eta((\nabla_Y\phi_\nu)X)\phi_\nu\xi - \eta(\phi_\nu X)\nabla_Y(\phi_\nu\xi)\\ &-(\nabla_Y\eta)(X)\eta_\nu(\xi)\xi_\nu - \eta(X)\nabla_Y(\eta_\nu(\xi))\xi_\nu - \eta(X)\eta_\nu(\xi)\nabla_Y\xi_\nu\Big\}\\ &+(Yh)AX + h(\nabla_Y A)X - (\nabla_Y A^2)X\\ =&\,0.\end{aligned}$$

Then from (3.5), together with the formulas in section 2, we have

$$(3.6)\qquad \begin{aligned}&(\nabla_Y S)X\\ =&-3g(\phi AY, X)\xi - 3\eta(X)\phi AY\\ &-3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(X) - q_{\nu+1}(Y)\eta_{\nu+2}(X)\\ &+g(\phi_\nu AY, X)\}\xi_\nu\\ &-3\sum_{\nu=1}^{3}\eta_\nu(X)\{q_{\nu+2}(Y)\xi_{\nu+1} - q_{\nu+1}(Y)\xi_{\nu+2} + \phi_\nu AY\}\\ &+\sum_{\nu=1}^{3}\Big[Y(\eta_\nu(\xi))\phi_\nu\phi X + \eta_\nu(\xi)\{-q_{\nu+1}(Y)\phi_{\nu+2}\phi X\end{aligned}$$

$$
\begin{aligned}
&+ q_{\nu+2}(Y)\phi_{\nu+1}\phi X + \eta_\nu(\phi X)AY - g(AY, \phi X)\xi_\nu\} \\
&+ \eta_\nu(\xi)\{\eta(X)\phi_\nu AY - g(AY, X)\phi_\nu\xi\} - g(\phi AY, \phi_\nu X)\phi_\nu\xi \\
&+ \{q_{\nu+1}(Y)\eta(\phi_{\nu+2}X) - q_{\nu+2}(Y)\eta(\phi_{\nu+1}X) - \eta_\nu(X)\eta(AY) \\
&+ \eta(\xi_\nu)g(AY, X)\}\phi_\nu\xi \\
&- \eta(\phi_\nu X)\{q_{\nu+2}(Y)\phi_{\nu+1}\xi - q_{\nu+1}(Y)\phi_{\nu+2}\xi + \phi_\nu\phi AY \\
&- \eta(AY)\xi_\nu + \eta(\xi_\nu)AY\} \\
&- g(\phi AY, X)\eta_\nu(\xi)\xi_\nu - \eta(X)Y(\eta_\nu(\xi))\xi_\nu - \eta(X)\eta_\nu(\xi)\nabla_Y\xi_\nu\Big] \\
&+ (Yh)AX + h(\nabla_Y A)X - (\nabla_Y A^2)X \\
=\,& 0.
\end{aligned}
$$

Putting $X = \xi$ in (3.6), we have

$$
\begin{aligned}
0 =& - 3\phi AY \\
&- 3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\xi) - q_{\nu+1}(Y)\eta_{\nu+2}(\xi) + g(\phi_\nu AY, \xi)\}\xi_\nu \\
&- 3\sum_{\nu=1}^{3}\eta_\nu(\xi)\{q_{\nu+2}(Y)\xi_{\nu+1} - q_{\nu+1}(Y)\xi_{\nu+2} + \phi_\nu AY\} \\
&+ \sum_{\nu=1}^{3}\Big[\eta_\nu(\xi)\{\phi_\nu AY - \eta(AY)\phi_\nu\xi\} - g(\phi AY, \phi_\nu\xi)\phi_\nu\xi \\
&- Y(\eta_\nu(\xi))\xi_\nu - \eta_\nu(\xi)\{q_{\nu+2}(Y)\xi_{\nu+1} - q_{\nu+1}(Y)\xi_{\nu+2} + \phi_\nu AY\}\Big] \\
&+ (Yh)A\xi + h(\nabla_Y A)\xi - (\nabla_Y A^2)\xi.
\end{aligned}
\tag{3.7}
$$

Now if we suppose that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, then (3.7) together with $A\xi = \alpha\xi$ implies

$$
\begin{aligned}
0 =&(h\alpha - \alpha^2 - 3)\phi AY + Y(\alpha h)\xi - hA\phi AY - (Y\alpha^2)\xi + A^2\phi AY \\
&- 4\sum_{\nu=1}^{3}\Big\{q_{\nu+2}(Y)\eta_{\nu+1}(\xi) - q_{\nu+1}(Y)\eta_{\nu+2}(\xi)\Big\}\xi_\nu \\
&- 5\sum_{\nu=1}^{3}g(\phi_\nu AY, \xi)\xi_\nu - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)q_{\nu+2}(Y)\xi_{\nu+1} \\
&+ 4\sum_{\nu=1}^{3}\eta_\nu(\xi)q_{\nu+1}(Y)\xi_{\nu+2} - 3\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu AY \\
&- \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\eta(AY) + g(\phi AY, \phi_\nu\xi)\}\phi_\nu\xi.
\end{aligned}
\tag{3.8}
$$

Now we should verify that $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.

On the other hand, by differentiating $A\xi = \alpha\xi$ and using the equation of Codazzi in section 2, we have the following

$$
\begin{aligned}
&- 2g(\phi X, Y) + 2\sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) \\
&\quad - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\} \\
&= g((\nabla_X A)Y - (\nabla_Y A)X, \xi)
\end{aligned}
$$

$$
\begin{aligned}
&= g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\
&= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) \\
&\quad - 2g(A\phi AX, Y).
\end{aligned}
$$

Putting $X = \xi$ gives $Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y)$. From this, substituting into the above equation, we have the following

$$
\begin{aligned}
A\phi AY = &\frac{\alpha}{2}(A\phi + \phi A)Y + \phi Y \\
&+ \sum_{\nu=1}^{3}\{\eta_\nu(Y)\phi\xi_\nu + \eta_\nu(\phi Y)\xi_\nu + \eta_\nu(\xi)\phi_\nu Y \\
&- 2\eta(Y)\eta_\nu(\xi)\phi\xi_\nu - 2\eta_\nu(\xi)\eta_\nu(\phi Y)\xi\}.
\end{aligned}
\tag{3.9}
$$

Then substituting (3.9) into (3.8), we have

$$
\begin{aligned}
0 =&\{h\alpha - \alpha^2 - 3\}\phi AY + Y(h\alpha)\xi - hA\phi AY - (Y\alpha^2)\xi \\
&+ \frac{\alpha}{2}A^2\phi Y + A\phi Y \\
&+ \sum_{\nu=1}^{3}\{\eta_\nu(Y)A\phi\xi_\nu + \eta_\nu(\phi Y)A\xi_\nu \\
&+ \eta_\nu(\xi)A\phi_\nu Y - 2\eta(Y)\eta_\nu(\xi)A\phi\xi_\nu - 2\alpha\eta_\nu(\xi)\eta_\nu(\phi Y)\xi\} \\
&+ \frac{\alpha^2}{4}(A\phi + \phi A)Y + \frac{\alpha}{2}\phi Y \\
&+ \frac{\alpha}{2}\sum_{\nu=1}^{3}\{\eta_\nu(Y)\phi\xi_\nu + \eta_\nu(\phi Y)\xi_\nu \\
&+ \eta_\nu(\xi)\phi_\nu Y - 2\eta(Y)\eta_\nu(\xi)\phi\xi_\nu - 2\eta_\nu(\xi)\eta_\nu(\phi Y)\xi\} \\
&- 4\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\xi) - q_{\nu+1}(Y)\eta_{\nu+2}(\xi)\}\xi_\nu \\
&- 5g(\phi_\nu AY, \xi)\xi_\nu - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)q_{\nu+2}(Y)\xi_{\nu+1} \\
&+ 4\sum_{\nu=1}^{3}\eta_\nu(\xi)q_{\nu+1}(Y)\xi_{\nu+2} - 3\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu AY \\
&- \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\eta(AY) + g(\phi AY, \phi_\nu\xi)\}\phi_\nu\xi,
\end{aligned}
\tag{3.10}
$$

where we have used $A\xi = \alpha\xi$ in the fourth line. From this let us verify that $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. In order to do this we suppose that $\xi = X_1 + X_2$ for some $X_1 \in \mathfrak{D}$ and $X_2 \in \mathfrak{D}^\perp$. Now putting $Y = \xi$ in (3.10), we have

$$
\begin{aligned}
&\xi(h\alpha)\xi - (\xi\alpha^2)\xi \\
&+ \sum_{\nu=1}^{3}\{\eta_\nu(X_2)A\phi\xi_\nu + \eta_\nu(X_2)A\phi_\nu\xi - 2\eta_\nu(X_2)A\phi_\nu\xi\} \\
&+ \frac{\alpha}{2}\sum_{\nu=1}^{3}\{\eta_\nu(X_2)\phi\xi_\nu + \eta_\nu(X_2)\phi_\nu\xi - 2\eta_\nu(X_2)\phi_\nu\xi\} \\
&- 4\sum_{\nu=1}^{3}\{q_{\nu+2}(\xi)\eta_{\nu+1}(\xi) - q_{\nu+1}(\xi)\eta_{\nu+2}(\xi)\}\xi_\nu
\end{aligned}
$$

$$
\begin{aligned}
&- 4\sum_{\nu=1}^{3}\eta_\nu(\xi)q_{\nu+2}(\xi)\xi_{\nu+1} + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)q_{\nu+1}(\xi)\xi_{\nu+2} \\
&- 4\alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi \\
= {}& 0.
\end{aligned}
$$

Then by comparing $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components in above equation, we have the following

$$
\{\xi(h\alpha) - \xi(\alpha^2)\}X_1 - 4\alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu X_1 = 0. \tag{3.11}
$$

From this formula we assert the following

Lemma 3.1. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with parallel Ricci tensor. If M has a geodesic Reeb flow ξ, then either $\xi\in\mathfrak{D}$ or $\xi\in\mathfrak{D}^\perp$.*

Proof. We proceed by showing that for each $x\in M$, either $\xi\in\mathfrak{D}$ at x or $\xi\in\mathfrak{D}^\perp$ at x. The result then follows by continuity since ξ is a unit vector. First, consider any point x the open subset $\mathfrak{U} = \{x\in M|\alpha(x)\neq 0\}$. If $X_1(x) = 0$, then $\xi\in\mathfrak{D}^\perp$ at x. If $X_1(x)\neq 0$, taking its inner product with (3.11), we have

$$
\xi(h\alpha) - \xi(\alpha^2) = 0.
$$

From this, together with (3.11) again, we have

$$
4\alpha(x)\sum_{\nu=1}^{3}\eta_\nu(X_2(x))\phi_\nu X_1(x) = 0. \tag{3.12}
$$

Then it is not difficult to verify that a nontrivial linear combination of ϕ_1, ϕ_2 and ϕ_3 can not be singular. Thus $\eta_\nu(X_2) = 0$ for $\nu = 1, 2, 3$. This gives $X_2(x) = 0$ and that $\xi\in\mathfrak{D}$ at x.

Now consider a point x where $\alpha = 0$. Suppose that α vanishes in a neighborhood of the point x. Then by differentiating $A\xi = 0$ and using the same method as in Berndt and the second author [3], we have the following for any tangent vector field Y

$$
Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y). \tag{3.13}
$$

This gives

$$
\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y) = 0. \tag{3.14}
$$

From this, replacing Y by ϕY for any $Y\in\mathfrak{D}$, we have

$$
\sum_{\nu=1}^{3}\eta_\nu(\xi)^2\eta(Y) = 0. \tag{3.15}
$$

On the other hand, replacing Y by ϕY into (3.14), we have

$$
\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(Y) = \sum_{\nu=1}^{3}\eta_\nu(\xi)^2\eta(Y). \tag{3.16}
$$

Then by putting $Y = \xi_\mu$, $\mu = 1, 2, 3$, respectively, into (3.16), we have

$$
\{1 - \sum_{\nu=1}^{3}\eta_\nu(\xi)^2\}\eta_\mu(\xi) = 0
$$

for $\mu = 1, 2, 3$. From this we consider the following two cases :

$$\sum_{\nu=1}^{3} \eta_\nu(\xi)^2 = 1 \text{ or } \eta_\mu(\xi) = 0 \text{ for } \mu = 1, 2, 3. \tag{3.17}$$

At a point satisfying the first case of (3.17) we have $\xi \in \mathfrak{D}^\perp$ of (3.17) we have $\xi \in \mathfrak{D}^\perp$(since the $\mathfrak{D}^\perp$-component of ξ has length 1). For points satisfying the second case of (3.17), we know $\xi \in \mathfrak{D}$.

Finally, we consider a point x such that $\alpha(x) = 0$ but the point x is the limit of a sequence of points where $\alpha \neq 0$. Such a sequence will have an infinite subsequence where $X_2 = 0$(in which case $\xi \in \mathfrak{D}^\perp$ at x, by continuity) or an infinite subsequence where $X_1 = 0$(in which case $\xi \in \mathfrak{D}^\perp$ at x). This completes the proof. □

By virtue of Lemma 3.1, for the proof of our Main Theorem, in section 4 we will consider the first case where M has a geodesic Reeb flow with $\xi \in \mathfrak{D}$. In section 5 in order to complete the proof of our Theorem we will discuss the remaining case where M has a geodesic Reeb flow with $\xi \in \mathfrak{D}^\perp$.

4. Real hypersurfaces with geodesic Reeb flow satisfying $\xi \in \mathfrak{D}$

Let us consider a Hopf real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with parallel and commuting Ricci tensor. Now in this section we show that the distribution $\mathfrak{D}$ of M in $G_2(\mathbb{C}^{m+2})$ satisfies $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. Then by differentiating $S\phi = \phi S$ and using $\nabla S = 0$ we have

$$\eta(SY)AX - g(AX, SY)\xi = \eta(Y)SAX - g(AX, Y)S\xi. \tag{4.1}$$

Moreover, by (3.4) and $\xi \in \mathfrak{D}$ we have

$$\begin{aligned} S\xi &= 4(m+1)\xi + hA\xi - A^2\xi, \\ SY &= (4m+7)Y - 3\eta(Y)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(Y)\xi_\nu \\ &\quad - \sum_{\nu=1}^{3}\eta(\phi_\nu Y)\phi_\nu\xi + hAY - A^2Y, \\ \eta(SY) &= g(\xi, SY) = \{4(m+1) + \alpha h - \alpha^2\}\eta(Y). \end{aligned}$$

Substituting these formulas into (4.1), we have

$$\begin{aligned} &\{4(m+1) + \alpha h - \alpha^2\}\eta(Y)AX \\ &- \Big\{(4m+7)g(AX, Y) - 3\eta(Y)\eta(AX) - 3\sum_{\nu=1}^{3}\eta_\nu(Y)\eta_\nu(AX) \\ &- \sum_{\nu=1}^{3}\eta_\nu(\phi Y)g(\phi_\nu\xi, AX) + hg(AX, AY) - g(AX, A^2Y)\Big\}\xi \\ = {}&\eta(Y)\Big\{(4m+7)AX - 3\eta(AX)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(AX)\xi_\nu \\ &- \sum_{\nu=1}^{3}\eta(\phi_\nu AX)\phi_\nu\xi + hA^2X - A^3X\Big\} \\ &- g(AX, Y)\{4(m+1)\xi + hA\xi - A^2\xi\} \end{aligned} \tag{4.2}$$

for any vector fields X and Y on M. Putting $X = \xi_\mu$ in (4.2) and using $\xi \in \mathfrak{D}$, we have

$$A^3\xi_\mu = hA^2\xi_\mu + (\alpha^2 - \alpha h)A\xi_\mu, \tag{4.3}$$

where we have used $\sum_{\nu=1}^3 \eta_\nu(\xi_\mu)\eta_\nu(AX) = g(AX, \xi_\mu)$ and $g(\phi\xi_\mu, \xi_\nu) = 0$. Now putting $Y = \xi$ in (4.2), we have

$$\begin{aligned} &\{4(m+1) + \alpha h - \alpha^2\}AX - \{(4m+7)\eta(AX) - 3\eta(AX) \\ &\quad + \alpha h\eta(AX) - \alpha^2\eta(AX)\}\xi \\ &= \Big[(4m+7)AX - 3\alpha\eta(X)\xi - 3\sum_{\nu=1}^3 \eta_\nu(AX)\xi_\nu - \sum_{\nu=1}^3 \eta(\phi_\nu AX)\phi_\nu\xi \\ &\quad + hA^2X - A^3X\Big] - \eta(AX)\{4(m+1)\xi + (\alpha h - \alpha^2)\xi\}. \end{aligned} \tag{4.4}$$

Then it follows that

$$\begin{aligned} &(3 - \alpha h + \alpha^2)AX + hA^2X - A^3X - 3\alpha\eta(X)\xi \\ &- 3\sum_{\nu=1}^3 \eta_\nu(AX)\xi_\nu - \sum_{\nu=1}^3 \eta(\phi_\nu AX)\phi_\nu\xi = 0. \end{aligned}$$

From this, putting $X = \xi_\mu$, we have

$$\begin{aligned} A^3\xi_\mu =& hA^2\xi_\mu + (\alpha^2 - \alpha h)A\xi_\mu + 3A\xi_\mu \\ &- 3\sum_{\nu=1}^3 \eta_\nu(A\xi_\mu)\xi_\nu - \sum_{\nu=1}^3 \eta(\phi_\nu A\xi_\mu)\phi_\nu\xi. \end{aligned} \tag{4.5}$$

By comparing (4.3) and (4.5) we have

$$3A\xi_\mu = 3\sum_{\nu=1}^3 \eta_\nu(A\xi_\mu)\xi_\nu + \sum_{\nu=1}^3 \eta(\phi_\nu A\xi_\mu)\phi_\nu\xi. \tag{4.6}$$

From this, if we take an inner product with $\phi_\lambda\xi$, we have

$$3g(A\xi_\mu, \phi_\lambda\xi) = \eta(\phi_\lambda A\xi_\mu) = -g(A\xi_\mu, \phi_\lambda\xi).$$

So we have $g(A\xi_\mu, \phi_\lambda\xi) = 0$. From this, together with (4.6), it follows that $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. Then by Theorem A we know that M is locally congruent to a real hypersurface of type (B), because $\xi \in \mathfrak{D}$.

Now it remains to check whether the Ricci tensor of real hypersurfaces of type (B) is parallel or not. So let us suppose that the Ricci tensor S of type (B) is parallel. That is, $(\nabla_Y S)X = 0$. Then in such a case $\xi \in \mathfrak{D}$, if we put $X = \xi$ in (3.5), the parallel Ricci tensor implies

$$\begin{aligned} 0 =& (\nabla_Y S)\xi \\ =& -3\nabla_Y\xi - 3\sum_{\nu=1}^3 (\nabla_Y \eta_\nu)(\xi)\xi_\nu \\ &+ \sum_{\nu=1}^3 \{-(\nabla_Y\eta)(\phi_\nu\xi)\phi_\nu\xi - \eta((\nabla_Y\phi_\nu)\xi)\phi_\nu\xi\} \\ &+ h(\nabla_Y A)\xi - (\nabla_Y A^2)\xi. \end{aligned}$$

Since we have assumed that M is a Hopf hypersurface, it follows that

$$0 = -3\phi AY + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AY)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(AY)\phi_\nu\xi + \alpha h\phi AY - hA\phi AY - \alpha^2\phi AY + A^2\phi AY. \tag{4.7}$$

Now let us introduce Proposition B due to Berndt and the second author [3] as follows:

Proposition B. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r)\ ,\ \beta = 2\cot(2r)\ ,\ \gamma = 0\ ,\ \lambda = \cot(r)\ ,\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1\ ,\ m(\beta) = 3 = m(\gamma)\ ,\ m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi\ ,\ T_\beta = \mathfrak{J}J\xi\ ,\ T_\gamma = \mathfrak{J}\xi\ ,\ T_\lambda\ ,\ T_\mu\ ,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp\ ,\ \mathfrak{J}T_\lambda = T_\lambda\ ,\ \mathfrak{J}T_\mu = T_\mu\ ,\ JT_\lambda = T_\mu\ .$$

Putting $Y = \xi_1 \in T_\beta$ in (4.7), then by Proposition B we have

$$(\alpha h - \alpha^2 - 4)\beta = 0. \tag{4.8}$$

On the other hand, the trace h of type (B) is given by

$$\begin{aligned} h =& \alpha + 6\cot 2r + (4n-4)(\cot r - \tan r) \\ =& \alpha + (4n-1)(\cot r - \tan r). \end{aligned}$$

Then substituting this into (4.8), we have $0 = -16n$, which makes a contradiction.

5. Real hypersurfaces with geodesic Reeb flow satisfying $\xi \in \mathfrak{D}^\perp$

Now let us consider a Hopf real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with parallel and commuting Ricci tensor. In this section we discuss geodesic Reeb flow satisfying $\xi \in \mathfrak{D}^\perp$. Since we have assumed that $\xi \in \mathfrak{D}^\perp = \text{Span}\ \{\xi_1, \xi_2, \xi_3\}$, there exists an Hermitian structure $J_1 \in \mathfrak{J}$ such that $JN = J_1N$, that is, $\xi = \xi_1$.

Now differentiating $S\phi = \phi S$ implies the following

$$(\nabla_Y S)\phi X + S(\nabla_Y \phi)X = (\nabla_Y \phi)SX + \phi(\nabla_Y S)X.$$

Now let us calculate term by term in above formula as follows :

$$\begin{aligned}
&(\nabla_Y S)\phi X \\
&= -3g(\phi AY, \phi X)\xi - 3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\phi X) - q_{\nu+1}(Y)\eta_{\nu+2}(\phi X) \\
&\quad + g(\phi_\nu AY, \phi X)\}\xi_\nu \\
&\quad - 3\sum_{\nu=1}^{3}\eta_\nu(\phi X)\{q_{\nu+2}(Y)\xi_{\nu+1} - q_{\nu+1}(Y)\xi_{\nu+2} + \phi_\nu A\phi X\} \\
&\quad + \sum_{\nu=1}^{3}\Big[Y(\eta_\nu(\xi))\phi_\nu\phi^2 X + \eta_\nu(\xi)\{-q_{\nu+1}(Y)\phi_{\nu+2}\phi^2 X \\
&\quad + q_{\nu+2}(Y)\phi_{\nu+1}\phi^2 X + \eta_\nu(\phi^2 X)AY - g(AY, \phi^2 X)\xi_\nu\} \\
&\quad - \eta_\nu(\xi)g(AY, \phi X)\phi_\nu\xi - g(\phi AY, \phi_\nu\phi X)\phi_\nu\xi \\
&\quad + \{q_{\nu+1}(Y)\eta(\phi_{\nu+2}\phi X) - q_{\nu+2}(Y)\eta(\phi_{\nu+1}\phi X) \\
&\quad - \eta_\nu(\phi X)\eta(AY) + \eta(\xi_\nu)g(AY, \phi X)\}\phi_\nu\xi \\
&\quad - \eta(\phi_\nu\phi X)\{q_{\nu+2}(Y)\phi_{\nu+1}\xi - q_{\nu+1}(Y)\phi_{\nu+2}\xi \\
&\quad + \phi_\nu\phi AY - \eta(AY)\xi_\nu + \eta(\xi_\nu)AY\} - g(\phi AY, \phi X)\eta_\nu(\xi)\xi_\nu\Big] \\
&\quad + (Yh)A\phi X + h(\nabla_Y A)\phi X - (\nabla_Y A^2)\phi X,
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
&S(\nabla_Y\phi)X \\
&= \eta(X)S(AY) - g(AY, X)S\xi \\
&= \eta(X)\Big[(4m+7)AY - 3\eta(AY)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(AY)\xi_\nu \\
&\quad + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi AY - \eta(\phi_\nu AY)\phi_\nu\xi \\
&\quad - \eta(AY)\eta_\nu(\xi)\xi_\nu\} + hA^2 Y - A^3 Y\Big] \\
&\quad - g(AY, X)\Big[4(m+1)\xi - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu + hA\xi - A^2\xi\Big],
\end{aligned} \tag{5.2}$$

$$(\nabla_Y\phi)SX = \eta(SX)AY - g(AY, SX)\xi, \tag{5.3}$$

$$\begin{aligned}
&\phi(\nabla_Y S)X \\
&= -3\eta(X)\phi^2 AY \\
&\quad - 3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(X) - q_{\nu+1}(Y)\eta_{\nu+2}(X) + g(\phi_\nu AY, X)\}\phi\xi_\nu \\
&\quad - 3\sum_{\nu=1}^{3}\eta_\nu(X)\{q_{\nu+2}(Y)\phi\xi_{\nu+1} - q_{\nu+1}(Y)\phi\xi_{\nu+2} + \phi\phi_\nu AY\} \\
&\quad + \sum_{\nu=1}^{3}\Big[Y(\eta_\nu(\xi))\phi\phi_\nu\phi X + \eta_\nu(\xi)\{-q_{\nu+1}(Y)\phi\phi_{\nu+2}\phi X \\
&\quad + q_{\nu+2}(Y)\phi\phi_{\nu+1}\phi X + \eta_\nu(\phi X)\phi AY - g(AY, \phi X)\phi\xi_\nu\} \\
&\quad + \eta_\nu(\xi)\{\eta(X)\phi\phi_\nu AY - g(AY, X)\phi\phi_\nu\xi\} - g(\phi AY, \phi_\nu X)\phi\phi_\nu\xi \\
&\quad + \{q_{\nu+1}(Y)\eta(\phi_{\nu+2}X) - q_{\nu+2}(Y)\eta(\phi_{\nu+1}X)
\end{aligned}$$

$$
\begin{aligned}
&- \eta_\nu(X)\eta(AY) + \eta(\xi_\nu)g(AY,X)\}\phi\phi_\nu\xi \\
&- \eta(\phi_\nu X)\{q_{\nu+2}(Y)\phi\phi_{\nu+1}\xi - q_{\nu+1}(Y)\phi\phi_{\nu+2}\xi + \phi\phi_\nu\phi AY \\
&- \eta(AY)\phi\eta_\nu + \eta(\xi_\nu)\phi AY\} \\
&- g(\phi AY,X)\eta_\nu(\xi)\phi\xi_\nu - \eta(X)Y(\eta_\nu(\xi))\phi\xi_\nu - \eta(X)\eta_\nu(\xi)\phi\nabla_Y\xi_\nu\Big] \\
&+ (Yh)\phi AX + h\phi(\nabla_Y A)X - \phi(\nabla_Y A^2)X,
\end{aligned}
\tag{5.4}
$$

where we have used the formulas related to the Ricci tensor S given by

$$
\begin{aligned}
S\xi &= (4m+7)\xi - 3\xi - 3\sum\nolimits_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu - \sum\nolimits_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu + hA\xi - A^2\xi \\
&= 4(m+1)\xi - 4\sum\nolimits_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu + (h\alpha - \alpha^2)\xi,
\end{aligned}
$$

$$
\begin{aligned}
g(S\xi, AY) &= 4(m+1)\eta(AY) - 4\sum\nolimits_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(AY) + (h\alpha - \alpha^2)\eta(AY) \\
&= \{4(m+1)\alpha + (h\alpha - \alpha^2)\}\alpha\eta(Y) - 4\sum\nolimits_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(AY),
\end{aligned}
$$

and

$$
\begin{aligned}
\eta(SX) &= 4(m+1)\eta(X) - 3\sum\nolimits_{\nu=1}^{3}\eta_\nu(X)\eta(\xi_\nu) \\
&\quad + \sum\nolimits_{\nu=1}^{3}\{\eta_\nu(\xi)\eta_\nu(\phi^2X) - \eta(X)\eta_\nu(\xi)^2\} \\
&\quad + h\eta(AX) - \eta(A^2X) \\
&= 4(m+1)\eta(X) - 3\eta(X) - \eta(X) + (\alpha h - \alpha^2)\eta(X) \\
&= \{4m + \alpha(h-\alpha)\}\eta(X).
\end{aligned}
$$

Now let us consider the case where $\xi \in \mathfrak{D}^\perp, \xi = \xi_1$ for a Hopf hypersurface in $G_2(\mathbb{C}^{m+1})$. Then it was known that

$$\eta_1(\phi X) = 0,\ \eta_2(\phi X) = -g(\phi\xi_2, X) = \eta_3(X),$$

and

$$\eta_3(\phi X) = -g(\phi\xi_3, X) = -\eta_2(X).$$

By using these formulas to (5.1), (5.2), (5.3) and (5.4) respectively, we have

$$
\begin{aligned}
&(\nabla_Y S)\phi X \\
=& -3\{g(AX,Y) - \alpha\eta(X)\eta(Y)\}\xi - 3\sum_\nu g(\phi_\nu AY, \phi X)\xi_\nu \\
&- 3\eta_3(X)\phi_2 A\phi Y + 3\eta_2(X)\phi_3 A\phi Y \\
&+ \Big[- q_2(Y)\{-\phi_3 X + \eta(X)\xi_2\} + q_3(Y)\{-\phi_2 X - \eta(X)\xi_3\} \\
&- g(AY, \phi^2 X)\xi\Big] + \{g(AY,\phi_2 X) - \eta_3(X)\eta(AY) + \eta(X)\eta_3(AY)\}\xi_3 \\
&- \{g(AY,\phi_3 X) + \eta_2(X)\eta(AY) - \eta(X)\eta_2(AY)\}\xi_2 + \eta_2(X)\phi_2\phi AY \\
&+ \eta_3(X)\phi_3\phi AY + (Yh)A\phi X + h(\nabla_Y A)\phi X - (\nabla_Y A^2)\phi X,
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
&S(\nabla_Y\phi)X\\
=&\eta(X)S(AY)-g(AY,X)S\xi\\
=&\eta(X)\Big[(4m+7)AY-3\alpha\eta(Y)\xi-3\sum\nolimits_{\nu=1}^{3}\eta_\nu(AY)\xi_\nu\\
&\quad+\phi_1\phi AY-\eta(\phi_2 AY)\phi_2\xi-\eta(\phi_3 AY)\phi_3\xi-\eta(AY)\xi\\
&\quad+hA^2Y-A^3Y\Big]-g(AY,X)\{4m\xi+hA\xi-A^2\xi\}\\
=&\eta(X)\Big[(4m+7)AY-7\alpha\eta(Y)-2\eta_2(AY)\xi_2-2\eta_3(AY)\xi_3\\
&\quad+\phi_1\phi AY+hA^2Y-A^3Y\Big]-g(AX,Y)\{4m+(h-\alpha)\alpha\}\xi
\end{aligned}
\tag{5.6}
$$

and the term in the right side becomes respectively

$$
\begin{aligned}
(\nabla_Y\phi)SX&=\eta(SX)AY-g(AY,SX)\xi\\
&=\{4m+\alpha(h-\alpha)\}\eta(X)AY-g(AY,SX)\xi,
\end{aligned}
\tag{5.7}
$$

and

$$
\begin{aligned}
&\phi(\nabla_Y S)X\\
=&-4\eta(X)\phi^2AY\\
&+3g(\phi_2AY,X)\xi_3-3g(\phi_3AY,X)\xi_2-3\eta_1(X)\phi\phi_1AY\\
&-3\eta_2(X)\phi\phi_2AY-3\eta_3(X)\phi\phi_3AY\\
&+\{-q_2(Y)\phi\phi_3\phi X+q_3(Y)\phi\phi_2\phi X\}+\eta(X)\phi\phi_1AY\\
&+g(\phi AY,\phi_2X)\xi_2+g(\phi AY,\phi_3X)\xi_3-\eta_3(X)\phi\phi_2\phi AY\\
&+\eta_2(X)\phi\phi_3\phi AY+(Yh)\phi AX+h\phi(\nabla_YA)X-\phi(\nabla_YA^2)X.
\end{aligned}
\tag{5.8}
$$

Now summing up (5.5) with (5.6) in L and (5.7) with (5.8) in R, we have the following respectively

$$
\begin{aligned}
L=&-3\{g(AX,Y)-\alpha\eta(X)\eta(Y)\}\xi-3\sum_\nu g(\phi_\nu AY,\phi X)\xi_\nu\\
&-3\eta_3(X)\phi_2A\phi Y+3\eta_2(X)\phi_3A\phi Y\\
&+\Big[q_2(Y)\{\phi_3X-\eta(X)\xi_2\}-q_3(Y)\{\phi_2X+\eta(X)\xi_3\}\Big]\\
&+\{g(AY,\phi_2X)-\eta_3(X)\eta(AY)+\eta(X)\eta_3(AY)\}\xi_3\\
&-\{g(AY,\phi_3X)+\eta_2(X)\eta(Y)-\eta(X)\eta_2(AY)\}\xi_2+\eta_2(X)\phi_2\phi AY\\
&+\eta_3(X)\phi_3\phi AY+(Yh)A\phi X+h(\nabla_YA)\phi X-(\nabla_YA^2)\phi X\\
&+\eta(X)\Big[(4m+7)AY-7\alpha\eta(Y)\xi-2\eta_2(AY)\xi_2-2\eta_3(AY)\xi_3\\
&+\phi_1\phi AY+hA^2Y-A^3Y\Big]-g(AX,Y)\{4m+(h-\alpha)\alpha\}\xi,
\end{aligned}
$$

and

$$\begin{aligned}R =&\{4m+\alpha(h-\alpha)\}\eta(X)AY - g(AY, SX)\xi \\ &- 4\eta(X)\phi^2 AY + 4g(\phi_2 AY, X)\xi_3 - 4g(\phi_3 AY, X)\xi_2 \\ &- 3\eta_1(X)\phi\phi_1 AY - 3\eta_2(X)\phi\phi_2 AY - 3\eta_3(X)\phi\phi_3 AY \\ &+ \{-q_2(Y)\phi\phi_3\phi X + q_3(Y)\phi\phi_2\phi X\} + \eta(X)\phi\phi_1 AY \\ &+ g(\phi AY, \phi_2 X)\xi_2 + g(\phi AY, \phi_3 X)\xi_3 - \eta_3(X)\phi\phi_2\phi AY \\ &+ \eta_2(X)\phi\phi_3\phi AY + (Yh)\phi AX + h\phi(\nabla_Y A)X - \phi(\nabla_Y A^2)X.\end{aligned}$$

Then from $L = R$ it follows that

$$\begin{aligned}&- 3\{g(AX, Y) - \alpha\eta(X)\eta(Y)\}\xi - 3\sum_\nu g(\phi AY, \phi_\nu X)\xi_\nu \\ &+ 3\sum_\nu \eta_\nu(AY)\eta(X)\xi_\nu - 3\sum_\nu \eta(AY)\eta_\nu(X)\xi_\nu \\ &- 3\eta_3(X)\phi_2 A\phi Y + 3\eta_2(X)\phi_3 A\phi Y \\ &- \{\alpha\eta_3(X)\eta(Y) - \eta(X)\eta_3(AY)\}\xi_3 \\ &- \{\alpha\eta_2(X)\eta(Y) - \eta(X)\eta_2(AY)\}\xi_2 \\ &+ \eta(X)\Big[(4m+7)AY - 7\alpha\eta(Y)\xi - 2\eta_2(AY)\xi_2 - 2\eta_3(AY)\xi_3 \\ &+ \phi_1\phi AY + hA^2Y - A^3Y\Big] - g(AX, Y)\{4m + (h-\alpha)\alpha\}\xi \\ =\ & \{4(m+1) + \alpha(h-\alpha)\}\eta(X)AY - g(AY, SX)\xi - 4\alpha\eta(X)\eta(Y)\xi \\ &+ 4g(\phi_2 AY, X)\xi_3 - 4g(\phi_3 AY, X)\xi_2 - 3\eta_1(X)\phi\phi_1 AY \\ &- 3\eta_2(X)\phi\phi_2 AY - 3\eta_3(X)\phi\phi_3 AY \\ &+ \{q_2(Y)\eta_2(X) + q_3(Y)\eta_3(X)\}\xi \\ &+ \eta(X)\phi\phi_1 AY + 4g(\phi AY, \phi_2 X)\xi_2 + 4g(\phi AY, \phi_3 X)\xi_3 \\ &+ \eta_3(X)\phi_2 AY - \eta_3(X)\eta_3(AY)\xi + \alpha\eta_3(X)\eta(Y)\xi_3 \\ &- \eta_2(X)\phi_3 AY - \eta_2(X)\eta_2(AY)\xi + \alpha\eta_2(X)\eta(Y)\xi_2 \\ &+ (Yh)\phi AX + h\phi(\nabla_Y A)X - \phi(\nabla_Y A^2)X.\end{aligned} \tag{5.9}$$

Putting $Y = \xi$ in the formula (5.9), we have

$$\begin{aligned}&- 3\alpha\eta(X)\xi + 3\alpha\sum_\nu \eta_\nu(\xi)\eta(X)\xi_\nu - \alpha\eta_3(X)\xi_3 \\ &- \alpha\eta_2(X)\xi_2 - 3\alpha\sum_\nu \eta_\nu(X)\xi_\nu \\ &+ \alpha\eta(X)\{4m + 7 + (h-\alpha)\alpha\}\xi - \alpha\eta(X)\{4m + (h-\alpha)\alpha\}\xi \\ =\ & \alpha\{4(m+1) + \alpha(h-\alpha)\}\eta(X)\xi - \alpha g(\xi, SX)\xi + 4\alpha g(\phi_2\xi, X)\xi_3\end{aligned}$$

$$\begin{aligned}
&- 4\alpha g(\phi_3\xi, X)\xi_2 - 3\alpha\eta_2(X)\phi\phi_2\xi - 3\alpha\eta_3(X)\phi\phi_3\xi \\
&+ \{q_2(\xi)\eta_2(X) + q_3(\xi)\eta_3(X)\}\xi + \alpha\eta_3(X)\phi_2\xi + \alpha\eta_3(X)\xi_3 \\
&- \alpha\eta_2(X)\phi_3\xi + \alpha\eta_2(X)\xi_2 + (\xi h)\phi AX + h\phi(\nabla_\xi A)X - \phi(\nabla_\xi A^2)X .
\end{aligned} \tag{5.10}$$

Putting $X = \xi$ in (5.9), it can be arranged as follows:

$$\begin{aligned}
&3\sum_\nu \eta_\nu(AY)\xi_\nu - 3\eta(AY)\xi + \eta_3(AY)\xi_3 + \eta_2(AY)\xi_2 \\
&+ \Big[(4m+7)AY - 7\alpha\eta(Y)\xi - 2\eta_2(AY)\xi_2 - 2\eta_3(AY)\xi_3 \\
&+ hA^2Y - A^3Y\Big] - \alpha\eta(Y)\big\{4m + (h-\alpha)\alpha\big\}\xi \\
&= \big\{4(m+1) + \alpha(h-\alpha)\big\}AY \\
&- \Big[\{4(m+1) + (h-\alpha)\alpha\}\alpha\eta(Y) - 4\sum_\nu \eta_\nu(\xi)\eta_\nu(AY)\Big]\xi \\
&- 4\alpha\eta(Y)\xi + 8g(\phi_2 AY, \xi)\xi_3 - 8g(\phi_3 AY, \xi)\xi_2 \\
&- 3\phi\phi_1 AY + h\phi\{(Y\alpha)\xi + \alpha\phi AY - A\phi AY\} \\
&- \phi\{(Y\alpha^2)\xi + \alpha^2\phi AY - A^2\phi AY\} .
\end{aligned} \tag{5.11}$$

On the other hand, we know that $\xi \in \mathfrak{D}^\perp, \xi = \xi_1$. By the assumption $S\phi = \phi S$ and $\nabla S = 0$, we have $S(\nabla_Y\phi)X = (\nabla_Y\phi)SX$. Then each side can be calculated as follows:

$$\begin{aligned}
S(\nabla_Y\phi)X =&\eta(X)S(AY) - g(AY, X)S\xi \\
=& \eta(X)\Big[(4m+7)AY - 3\eta(AY)\xi - 3\sum_\nu \eta_\nu(AY)\xi_\nu \\
&+ \sum_\nu \{\eta_\nu(\xi)\phi_\nu\phi AY - \eta(\phi_\nu AY)\phi_\nu\xi - \eta(AY)\eta_\nu(\xi)\xi_\nu\} \\
&+ hA^2Y - A^3Y\Big] \\
&- g(AY, X)\Big[4(m+1)\xi - 4\sum_\nu \eta_\nu(\xi)\xi_\nu + hA\xi - A^2\xi\Big],
\end{aligned}$$

and respectively

$$\begin{aligned}
(\nabla_Y\phi)SX &= \eta(SX)AY - g(AY, SX)\xi \\
&= \{4m + \alpha(h-\alpha)\}\eta(X)AY - g(AY, SX)\xi.
\end{aligned}$$

Then using (2.1) to both sides of $S(\nabla_Y\phi)X = (\nabla_Y\phi)SX$, we have the following

$$\begin{aligned}
\eta(X)\Big[&(4m+7)AY - 4\alpha\eta(Y)\xi - 3\sum_\nu \eta_\nu(AY)\xi_\nu + \phi_1\phi AY \\
&- \eta(\phi_2 AY)\phi_2\xi - \eta(\phi_3 AY)\phi_3\xi + hA^2Y - A^3Y\Big]
\end{aligned}$$

$$
\begin{aligned}
&- g(AX,Y)\big\{4m+(h-\alpha)\alpha\big\}\xi \\
&=\big\{4m+\alpha(h-\alpha)\big\}\eta(X)AY - g(AY,SX)\xi.
\end{aligned} \tag{5.12}
$$

From this, if we put $X=\xi$, we have

$$
\begin{aligned}
&\{7+(\alpha-h)\alpha\}AY + hA^2Y - A^3Y + \phi_1\phi AY \\
&- 7\alpha\eta(Y)\xi - 2\eta_2(AY)\xi_2 - 2\eta_3(AY)\xi_3 \\
&- \big\{4m+(h-\alpha)\alpha\big\}\alpha\eta(Y)\xi + g(AY,S\xi)\xi \\
&=0
\end{aligned} \tag{$*$}
$$

where the last term is given by

$$
\begin{aligned}
g(AY,S\xi) &= \{4(m+1)\alpha + h\alpha^2 - \alpha^3\}\eta(Y) - 4\sum_\nu \eta_\nu(\xi)\eta_\nu(AY) \\
&= \{4m+h\alpha-\alpha^2\}\alpha\eta(Y).
\end{aligned}
$$

Thus the formula $(*)$ implies the following for Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+1})$ satisfying $\xi\in\mathfrak{D}^\perp$

$$
\begin{aligned}
&A^3Y - hA^2Y - \{7+(\alpha-h)\alpha\}AY - \phi_1\phi AY + 7\alpha\eta(Y)\xi \\
&\quad + 2\eta_2(AY)\xi_2 + 2\eta_3(AY)\xi_3 = 0.
\end{aligned} \tag{5.13}
$$

Then from (5.7) and (5.13) we have

$$
\begin{aligned}
&3\sum_\nu \eta_\nu(AY)\xi_\nu - 3\eta(AY)\xi + \eta_3(AY)\xi_3 + \eta_2(AY)\xi_2 \\
&+ \Big[(4m+7)AY - \{7+(\alpha-h)\alpha\}AY - \phi_1\phi AY\Big] \\
&- \alpha\eta(Y)\{4m+(h-\alpha)\alpha\}\xi \\
&= \{4(m+1)+\alpha(h-\alpha)\}AY \\
&- \Big[\{4(m+1)\alpha + h\alpha^2 - \alpha^3\}\eta(Y) - 4\sum_\nu \eta_\nu(\xi)\eta_\nu(AY)\Big]\xi \\
&- 4\alpha\eta(Y)\xi + 8g(\phi_2AY,\xi)\xi_3 - 8g(\phi_3AY,\xi)\xi_2 \\
&- 3\phi\phi_1AY + h\phi\{(Y\alpha)\xi + \alpha\phi AY - A\phi AY\} \\
&- \phi\{(Y\alpha^2)\xi + \alpha^2\phi AY - A^2\phi AY\}.
\end{aligned} \tag{5.14}
$$

Then (5.14) can be rewritten as follows:

$$
\begin{aligned}
(4-\alpha h+\alpha^2)AY &= 2\phi_1\phi AY + \alpha\{4-\alpha(h-\alpha)\}\eta(Y)\xi - 2\eta_3(AY)\xi_3 \\
&\quad - 2\eta_2(AY)\xi_2 + h\phi A\phi AY - \phi A^2\phi AY.
\end{aligned} \tag{5.15}
$$

On the other hand, from $S\phi = \phi S$ it follows that

$$
\begin{aligned}
&h\phi AX - \phi A^2 X \\
&= hA\phi X - A^2\phi X - 3\sum_\nu \eta_\nu(\phi X)\xi_\nu + \phi_1\phi^2 X - \sum_\nu \eta(\phi_\nu\phi X)\phi_\nu\xi \\
&\quad + 3\sum_\nu \eta_\nu(X)\phi\xi_\nu - \phi\phi_1\phi X + \sum_\nu \eta(\phi_\nu X)\phi^2\xi_\nu \\
&= hA\phi X - A^2\phi X - 3\eta_2(\phi X)\xi_2 - 3\eta_3(\phi X)\xi_3 + \eta_2(X)\phi_2\xi \\
&\quad + \eta_3(X)\phi_3\xi + 3\eta_2(X)\phi\xi_2 + 3\eta_3(X)\phi\xi_3 - \sum_\nu \eta(\phi_\nu X)\xi_\nu \\
&= hA\phi X - A^2\phi X.
\end{aligned}
\tag{5.16}
$$

By virtue of this formula, (5.15) can be calculated as follows:

$$3\phi_1\phi AY + 3AY = \alpha\{3 + \alpha(h-\alpha)\}\eta(Y)\xi + 4\eta_2(AY)\xi_2 + 4\eta_3(AY)\xi_3. \tag{5.17}$$

Then by putting $Y = \xi$ in (5.17), we have $\alpha^2(h-\alpha) = 0$, which implies $\alpha = 0$ or $\alpha = h$. Moreover, if we take $Y \in \mathfrak{D}$ and make an inner product (5.17) with ξ_2, then we have

$$
\begin{aligned}
4\eta_2(AY) &= 3g(\phi_1\phi AY, \xi_2) + 3g(AY, \xi_2) \\
&= 3g(AY, \phi\xi_3) + 3g(AY, \xi_2) \\
&= 6\eta_2(AY),
\end{aligned}
\tag{5.18}
$$

where we have used $\phi\phi_1\xi_2 = \phi\xi_3 = \phi_3\xi = \phi\xi_1 = \xi_2$. From this it follows that $\eta_2(AY) = 0$. Similarly, we have $\eta_3(AY) = 0$. Then we can assert that $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. Moreover, we know that $\xi \in \mathfrak{D}^\perp$ in this section. Then by virtue of theorem due to Berndt and Suh [3], M is locally congruent to a real hypersurfaces of type (A), that is, M is a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Related to hypersurfaces of type (A) in Theorem A we introduce another Proposition due to Berndt and the second author [3] as follows:

Proposition C. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \pi/2$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r)\ ,\ \beta = \sqrt{2}\cot(\sqrt{2}r)\ ,\ \lambda = -\sqrt{2}\tan(\sqrt{2}r),\ \mu = 0$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1\ ,\ m(\beta) = 2\ ,\ m(\lambda) = 2m-2 = m(\mu),$$

and the corresponding eigenspaces we have

$$
\begin{aligned}
T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN,\\
T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N,\\
T_\lambda &= \{X | X \perp \mathbb{H}\xi, JX = J_1X\},\\
T_\mu &= \{X | X \perp \mathbb{H}\xi, JX = -J_1X\}.
\end{aligned}
$$

First, let us check whether real hypersurfaces of type (A) in $G_2(\mathbb{C}^{m+2})$ satisfy that its Ricci tensor is parallel or not when the geodesic Reeb flow of $\xi \in \mathfrak{D}$ satisfies $h = \alpha$. Then we have

$$\alpha = h = \alpha + 2\beta + (2m-2)\lambda = \alpha + 2\beta - 2(m-1)\sqrt{2}\tan\sqrt{2}r.$$

Thus we have $\sqrt{2}\cot\sqrt{2}r = \beta = \sqrt{2}(m-1)\tan\sqrt{2}r$. So, we obtain

$$\tan^2\sqrt{2}r = \frac{1}{m-1}. \tag{5.19}$$

Then from $\nabla S = 0$, by (3.7) we have

(5.20)

$$
\begin{aligned}
0 = &- 3\phi AY - 3\{q_3(Y)\eta_2(\xi) - q_2(Y)\eta_3(\xi) + g(\phi_1 AY, \xi)\}\xi\\
&- 4\{q_1(Y)\eta_3(\xi) - q_3(Y)\eta_1(\xi) + g(\phi_2 AY, \xi)\}\xi_2\\
&- 4\{q_2(Y)\eta_1(\xi) - q_1(Y)\eta_2(\xi) + g(\phi_3 AY, \xi)\}\xi_3\\
&- 3\{q_3(Y)\xi_2 - q_2(Y)\xi_3 + \phi_1 AY\}\\
&+ \Big[\{\phi_1 AY - \eta(AY)\phi_1\xi\} - g(\phi AY, \phi_1\xi)\phi_1\xi - g(\phi AY, \phi_2\xi)\phi_2\xi\\
&- g(\phi AY \phi_3\xi)\phi_3\xi - \{q_3(Y)\xi_2 - q_2(Y)\xi_3 + \phi_1 AY\}\Big]\\
&+ (Yh)A\xi + h(\nabla_Y A)\xi - (\nabla_Y A^2)\xi\\
= &- 3\phi AY - 3\phi_1 AY - 4\eta_3(AY)\xi_2 + 4\eta_2(AY)\xi_3 - q_3(Y)\xi_2 + q_2(Y)\xi_3\\
&+ h\{(Y\alpha)\xi + \alpha\phi AY - A\phi AY\} - \{(Y\alpha^2)\xi + \alpha^2\phi AY - A^2\phi AY\}\\
= &- 3\phi AY - 3\phi_1 AY - 6\eta_3(AY)\xi_2 + 6\eta_2(AY)\xi_3\\
&+ h\{\alpha\phi AY - A\phi AY\} - \{\alpha^2\phi AY - A^2\phi AY\},
\end{aligned}
$$

where we have used $q_3(X) = 2\eta_3(AX)$ and $q_2(X) = 2\eta_2(AX)$ derived from the equation $\nabla_X\xi_1 = \nabla_X\xi = \phi AX$ and (2.3). From this, substituting these formulas $A\xi = \alpha\xi$, $A\xi_2 = \beta\xi_2$, $A\xi_3 = \beta\xi_3$ and putting $Y = \xi_2$, we have

$$0 = -3\phi A\xi_2 - 3\phi_1 A\xi_2 + 6\beta\xi_3 + h(\alpha\phi A\xi_2 - A\phi A\xi_2) - (\alpha^2\phi A\xi_2 - A^2\phi A\xi_2).$$

So we have

$$0 = 6\beta - h\beta(\alpha - \beta) + \beta(\alpha^2 - \beta^2) = \beta\{6 - h(\alpha - \beta) + \alpha^2 - \beta^2\}.$$

Since $\beta \neq 0$, it follows that

$$6 = h(\alpha - \beta) - (\alpha^2 - \beta^2) = (\alpha - \beta)\{h - (\alpha + \beta)\}.$$

On the other hand, its trace of the tube of type (A) in Theorem A becomes

$$\begin{aligned} h &= \alpha + 2\beta + (2m-2)\lambda + (2m-2)\mu \\ &= \alpha + 2\beta + (2m-2)(-\sqrt{2}\tan\sqrt{2}r), \end{aligned}$$

$$\begin{aligned} \alpha - \beta &= \sqrt{8}\cot\sqrt{8}r - \sqrt{2}\cot\sqrt{2}r \\ &= 2\sqrt{2}\cot 2\sqrt{2}r - \sqrt{2}\cot\sqrt{2}r \\ &= \sqrt{2}(\cot\sqrt{2}r - \tan\sqrt{2}r) - \sqrt{2}\cot\sqrt{2}r \\ &= -\sqrt{2}\tan\sqrt{2}r, \end{aligned}$$

and

$$\begin{aligned} h - (\alpha+\beta) &= \beta - (2m-2)\sqrt{2}\tan\sqrt{2}r \\ &= \sqrt{2}\cot\sqrt{2}r - (2m-2)\sqrt{2}\tan\sqrt{2}r. \end{aligned}$$

Then it follows that

$$\begin{aligned} 6 &= -\sqrt{2}\tan\sqrt{2}r\{\sqrt{2}\cot\sqrt{2}r - (2m-2)\sqrt{2}\tan\sqrt{2}r\} \\ &= -2 + 4(m-1)\tan^2\sqrt{2}r. \end{aligned}$$

$$4 = 2(m-1)\tan^2\sqrt{2}r. \tag{5.21}$$

So we have

$$\tan^2\sqrt{2}r = \frac{2}{m-1}.$$

So by comparing (5.8) with (5.19) and (5.21), we make a contradiction. So we conclude that M in $G_2(\mathbb{C}^{m+2})$ with parallel and commuting Ricci tensor can not exist.

Next let us check whether real hypersurfaces of type (A), that is, tubes over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ satisfy $\nabla S = 0$ or not. So we assume that M satisfies $\nabla S = 0$ with vanishing geodesic Reeb flow, that is, $\alpha = 0$. Then by putting $X = \xi = \xi_1$ in (3.5), we have

$$\begin{aligned} 0 =&(\nabla_Y S)\xi \\ =& -3\mathring{\nabla}_Y\xi - 3\sum\nolimits_{\nu=1}^{3}(\nabla_Y\eta_\nu)(\xi)\xi_\nu - 3\sum\nolimits_{\nu=1}^{3}\eta_\nu(\xi)\nabla_Y\xi_\nu \\ &+ \Big\{\phi_1(\nabla_Y\phi)\xi - (\nabla_Y\eta)(\phi_2\xi)\phi_2\xi - (\nabla_Y\eta)(\phi_3\xi)\phi_3\xi \\ &- \eta((\nabla_Y\phi_2)\xi)\phi_2\xi - \eta((\nabla_Y\phi_3)\xi)\phi_3\xi - (\nabla_Y\eta)(\xi)\xi_1 - \nabla_Y\xi_1\Big\} \\ &+ h(\nabla_Y A)\xi - (\nabla_Y A^2)\xi, \end{aligned} \tag{5.22}$$

where we have used the fact that the mean curvature of type (A) mentioned in Proposition C is constant.

On the other hand, from (2.2),(2.3) and (2.4) we have the following formulas

$$\phi_1(\nabla_Y\phi)\xi = \phi_1 AY,$$

$$\begin{aligned}(\nabla_Y\eta)\phi_2\xi =&g(\nabla_Y\xi,\phi_2\xi) = -g(\phi AY,\xi_3)\\ =&g(AY,\phi_3\xi_1) = \eta_2(AY),\\ (\nabla_Y\eta)\phi_3\xi =&g(\nabla_Y\xi,\phi_3\xi) = g(\phi AY,\xi_2)\\ =&-g(AY,\phi_3\xi_1) = \eta_3(AY),\\ (\nabla_Y\phi_2)\xi =& q_1(Y)\phi_3\xi - g(AY,\xi)\xi_2,\\ (\nabla_Y\phi_3)\xi =& -q_1(Y)\phi_2\xi - g(AY,\xi)\xi_3,\end{aligned}$$

and

$$(\nabla_Y\eta)\xi = g(\nabla_Y\xi,\xi) = g(\phi A\xi,\xi) = 0.$$

Substituting these formulas into (5.22), and bearing in mind that $\xi=\xi_1$ and $\alpha=0$, we have

$$\begin{aligned}0 =&(\nabla_Y S)\xi\\ =&-3\nabla_Y\xi - 3(\nabla_Y\eta_2)(\xi)\xi_2 - 3(\nabla_Y\eta_3)(\xi)\xi_3 - 3\nabla_Y\xi\\ &+\{\phi_1 AY - \eta_2(AY)\phi_2\xi - \eta_3(AY)\phi_3\xi - \phi AY\}\\ &- hA\phi AY + A^2\phi AY\\ =&-6\phi AY - 3\Big[\{-q_3(Y)+\eta_3(AY)\}\xi_2 + \{q_2(Y)-\eta_2(AY)\}\xi_3\Big]\\ &+\{\phi_1 AY + \eta_2(AY)\xi_3 - \eta_3(AY)\xi_2 - \phi AY\}\\ &- hA\phi AY + A^2\phi AY.\end{aligned}\tag{5.23}$$

For a tube of radius $r=\frac{\pi}{4\sqrt{2}}$, by Proposition C we know $\alpha=\sqrt{8}\cot\frac{\pi}{2}=0$,$\beta=\sqrt{2}\cot\frac{\pi}{4}=\sqrt{2}$, and $\lambda=-\sqrt{2}\tan\frac{\pi}{4}=-\sqrt{2}$. Then its trace of the tube becomes

$$h=\alpha+2\beta-(2m-2)\sqrt{2}=-2(m-2)\sqrt{2}.$$

Then for any $Y\in T_{-\sqrt{2}}$ such that $AY=-\sqrt{2}Y$ and $\phi Y=\phi_1 Y$ we know that

$$q_2(Y)=q_3(Y)=0$$

from the formula $\nabla_Y\xi=\nabla_Y\xi_1$.

Now if we put $Y\in T_{-\sqrt{2}}$ in (5.7) and use $\phi Y=\phi_1 Y$ given in Proposition C, we have

$$\begin{aligned}0=&-6\phi AY - hA\phi AY + A^2\phi AY\\ =&-\sqrt{2}\{-6\phi Y - hA\phi Y + A^2\phi Y\}\\ =&\sqrt{2}(4-\sqrt{2}h)\phi Y\end{aligned}\tag{5.24}$$

where in the third equality we have used the fact that the eigen space $T_{-\sqrt{2}}$ is invariant by the structure tensor ϕ, that is, $\phi T_{-\sqrt{2}}\subset T_{-\sqrt{2}}$.

In fact, in Lemma 12 due to Berndt and the second author [4] we have the following

$$A\phi_1 Y = \frac{4}{2\lambda}\phi_1 Y = \frac{4}{-2\sqrt{2}}\phi Y = -\sqrt{2}\phi_1 Y,$$

that is, $\phi Y=\phi_1 Y\in T_{-\sqrt{2}}$.

Then by (5.24) we have

$$0 = 4 - \sqrt{2}h = 4 - \sqrt{2}\{-2(m-2)\sqrt{2}\} = 4(m-1),$$

which gives a contradiction.

Remark 5.1. It can be easily verified that a real hypersurface of type (A) in Theorem A , a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, has the property that the Ricci tensor S commutes with the structure tensor ϕ.

Remark 5.2. In the paper due to the second author [11] we have proved that there do not exist any real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with parallel shape operator, that is $\nabla A = 0$. Such a geometric condition is stronger than the parallelness of the Ricci tensor in this paper. In the paper [12] we also have proved the non-existence property of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with commuting shape operator, that is, $A\phi_i = \phi_i A$, $i = 1, 2, 3$.

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Juan de Dios Pérez
Departamento de Geometría y Topología
Facultad de Ciencias
Universidad de Granada
18071-Granada, Spain
E-mail address: jdperez@ugr.es

Young Jin Suh
Kyungpook National University
Department of Mathematics
Taegu 702-701, Korea
E-mail address: yjsuh@mail.knu.ac.kr