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# BIJECTIVITY BETWEEN COIN-STACKS AND PERMUTATIONS AVOIDING 132-PATTERN

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ABSTRACT. We have defined a bijective map from certain set of coin-stacks onto the permutations avoiding 132-pattern and give an algorithm that finds a corresponding permutation from a given coin-stack. We also list several open problems which are similar as a CS-partition problem.

## 1. Introduction

By an *ordinary coin-stack* or, simply, a *coin-stack* we mean an arrangement of $n$ coins in rows such that the coins in the first row form a single contiguous block, and that in all higher rows each coin touches exactly two coins from the row beneath it. In addition, if first row contains exactly $k$ coins, we say that the coin-stack is of the form $(\boldsymbol{n},\boldsymbol{k})$-*stack* or $(\boldsymbol{n},\boldsymbol{k})$-*fountain* ([6, 8, 10]).

We see that there is a one-to-one correspondence between coin-stacks and certain kinds of integer-partitions given as in the following. Let $n$ and $k$ be given nonnegative integers. Then how many different integer sequences $(d_1, d_2, \ldots, d_k)$ satisfying

$$\begin{array}{ll} (i) & d_1 = 1,\ d_i \geq 1 \text{ for } i = 2, \ldots, k, \\ (ii) & n = d_1 + d_2 + \cdots + d_k, \\ (iii) & d_{i+1} - d_i \leq 1 \text{ for } i = 1, 2, \ldots, k-1 \end{array}$$

do we have?

Let us call such a sequence $(d_1, d_2, \ldots, d_k)$ a coin stack-partition, shortly, *CS-partition.*

For example, as in the Figure 1, the coin-stack can be written as a CS-partition

$$30 = 1 + 2 + 1 + 2 + 3 + 3 + 4 + 1 + 2 + 2 + 3 + 3 + 2 + 1$$

from the *right* side to the *left* side.

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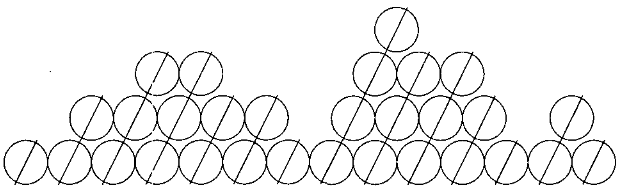

FIGURE 1. An example of (30, 14)-stack (ordinary coin-stack)

Let $[n] = \{1, 2, \ldots, n\}$ for an integer $n$ which is greater than or equal to 1, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in [p_1]^m, \beta = (\beta_1, \beta_2, \ldots, \beta_m) \in [p_2]^m$. We say that $\alpha$ and $\beta$ have same *relative order* if for all $1 \leq i < j \leq m$ one has $\alpha_i < \alpha_j$ if and only if $\beta_i < \beta_j$. For two permutations $\sigma \in S_k$ and $\pi \in S_n$ $(k < n)$, an *occurrence* of $\sigma$ in $\pi$ is a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $(\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_k})$ and $\sigma$ have same relative order; in such a context $\sigma$ is usually called the *pattern.* We say that $\pi$ is a permutation (in $S_n$) avoiding $\sigma$- pattern if there is no occurrence of $\sigma$ in $\pi$.

Let $f(n,k)$ be the number of $(n,k)$-stacks and its ordinary generating function

$$F(x,y) = \sum_{n,k \geq 0} f(n,k) x^n y^k.$$

It turned out that ([2, 6])

$$F(x,y) = \cfrac{1}{1 - \cfrac{xy}{1 - \cfrac{x^2 y}{1 - \cfrac{x^3 y}{1 - \cdots}}}}$$

and $F(1,y) = \frac{1-\sqrt{1-4y}}{2y}$, which is known as a Catalan function (see [3] for details). It is well known that the number of $n$-permutations avoiding 132-pattern is again a Catalan number $C_n = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}$. (See, for example, [1]). Note that

$$\frac{1-\sqrt{1-4y}}{2y} = \sum_{n=0}^{\infty} C_n y^n.$$

It is our purpose here to give a one-to-one correspondence between $n$-permutations avoiding 132-pattern and coin-stacks with $n$ coins in the bottom row (that is, $(*, n)$-stacks).

## 2. Some notations and main results

Let $n$ be a positive integer greater than 1, and

$$M(n) := \{(i,j) \in N \times N | 1 \leq i < j \leq n\},$$

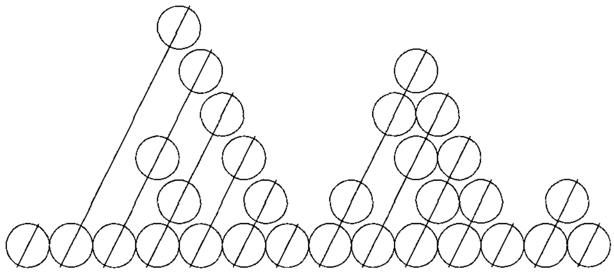

FIGURE 2. A Floated Coin-Stack (FCS) for a permutation $p = (14, 12, 10, 9, 11, 8, 13, 4, 5, 3, 6, 7, 1, 2)$

$S(n, 132)$ is the set of all $n$-permutations avoiding 132-pattern. Now, for $p = p_1p_2 \cdots p_n \in S(n, 132)$, we define $\alpha_p : M(n) \to \{0, 1\}$ by

$$\alpha_p(i,j) = \begin{cases} 1 & \text{if } p_i < p_j \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 1.** *With the notations above, the following holds:*

(a) *If* $\alpha_p(i,j) = 1$ *and* $i + 1 < j$ *then* $\alpha_p(i + 1, j) = 1$ **(Avoidance of 132-pattern)**

(b) *If* $\alpha_p(i,j) = 1 = \alpha_p(j,k)$, *then* $\alpha_p(i,k) = 1$. **(Transitivity)**

*Proof.* (a) When $\alpha_p(i,j) = 1$ and $i+1 < j$, if $\alpha_p(i+1,j) = 0$ then $p_{i+1} > p_j > p_i$. This implies that the permutation $p$ has 132-pattern.

(b) Obvious. □

**Definitions.** A *floated coin-stack* (simply, *FCS*) is a map $\alpha_p : M(n) \to \{0, 1\}$ satisfying conditions (a) and (b) in Lemma 1. The bottom row ($n$ cells) of the FCS can be regarded as permutation $p = p_1p_2 \cdots p_n$ itself. In Figure 2, we regard 14 cells in the bottom row as

$$p = (p_1, p_2, \ldots, p_{14}) = (14, 12, 10, 9, 11, 8, 13, 4, 5, 3, 6, 7, 1, 2)$$

in $S(14, 132)$.

Let $\alpha_p$ be an FCS. If $\alpha_p(i,j) = 1$, we call the position $(i,j)$ the *cell* of $p$, and denote it by $(i,j)_p$. If $\alpha_p(1,j) = \alpha_p(2,j) = \cdots = \alpha_p(i-1,j) = 0$ and $\alpha_p(i,j) = \alpha_p(i+1,j) = \cdots = \alpha_p(j-1,j) = 1$, then we call all the cells $(l,j)_p$ with $i \le l \le j-1$ the *j-branch* of $p$, and $(i,j)_p$ the *head cell* of the $j$-branch. In this case, we also call the set of all $(k,l)$-positions with $i \le k < l \le j$ the *shadow* of the $j$-branch of $p$. For example, in Figure 2 there are six branches(i.e., 5-, 7-, 9-, 11-, 12- and 14-branch). The 5-branch is a subset of the shadow of the 7-branch there.

**Lemma 2.** *With the condition above, if $i < k < j$ and $(i,j)_p$ is the head cell of the $j$-branch of $p$ in $S(n, 132)$, then either* (a) *there is no $k$-branch, or* (b) *$k$-branch is the subset of the shadow of the $j$-branch if there is.*

*Proof.* Let the head cell of the $k$-branch be outside the shadow of the $j$-branch, where $i < k < j$. Then it must be that $\alpha_p(i-1, k) = 1 = \alpha_p(k, j)$. By Lemma 1(b), $\alpha_p(i-1, j) = 1$, which contradicts that $(i,j)_p$ is the head cell of the $j$-branch. □

For two integers $k_1$ and $k_2$ with $k_1 \leq k_2$, let $[k_1, k_2]$ be the set of all integers between $k_1$ and $k_2$ including both ends.

**Lemma 3.** *Let $\gamma$ be an FCS, $i < l_1 < l_2 < k$ and $i, l_1, l_2$-branches be in the shadow of $k$-branch in $\gamma$. Also let $(j_i, l_i)(i = 1, 2)$ be the head cells of $l_i$-branch, respectively. Then $j_2 \notin [j_1 + 1, l_1]$.*

*Proof.* Suppose contrary to the conclusion, that is, $j_2 \in [j_1 + 1, l_1]$. Then $\gamma(j_1, l_1) = 1 = \gamma(l_1, l_2)$. By Lemma 1(b), $\gamma(j_1, l_2) = 1$, which contradicts that head cell of $l_2$-branch is $(j_2, l_2)$ and $j_2 > j_1$. □

**Example.** Consider a coin-stack shown as in Figure 3. This is not an FCS since Lemma1(b) is not satisfied. Note that this coin-stack does not correspond to a 9-permutation avoiding 132-pattern.

*Remark.* From Lemma 3, $j_2$ is either $j_2 \leq j_1$ (the shadow of the $l_1$-branch is a subset of that of the $l_2$-branch) or $j_2 > l_1$ (the shadow of the $l_1$-branch and that of the $l_2$-branch are disjoint).

Now, let $\beta : M(n) \to \{0, 1\}$ satisfy the following:

(OCS) If $\beta(i,j) = 1$, then $j = i + 1$ or $\beta(i, j-1) = 1 = \beta(i+1, j)$.

Note that OCS stands for an ordinary coin-stack. We can see that every ordinary coin-stack can be regarded as a map $\beta : M(n) \to \{0, 1\}$ satisfying the condition (OCS). Let

$$\Psi = \{\alpha_p : M(n) \to \{0, 1\} | p \in S(n, 132)\},$$

and

$$\Phi = \{M(n) \to \{0, 1\} | \beta \text{ satisfies the condition (OCS)}\}.$$

Define $\phi : \Psi \to \Phi$ as the following: For fixed $i$, let $n_i = |\{j \in [n] | \alpha_p(i,j) = 1\}|$. Define $\beta = \phi(\alpha_p)$ (*Left Bottom-projection* or *LB-projection*) by

$$\beta(i,k) = \begin{cases} 1 & \text{for } i + 1 \leq k \leq i + n_i \\ 0 & \text{for } n_i + i < k \leq n. \end{cases}$$

Then the map $\phi$ is well-defined since if $\beta(s,t) = 1$ and $s + 1 < t$ then $t - s \leq n_s$, so $\beta(s, t-1) = 1$, and $n_s - n_{s+1} \leq 1$ (by Lemma 2), so $\beta(s+1, t) = 1$.

**Theorem 4.** (1) *$\psi : S(n, 132) \to \Psi$ defined by $\psi(p) = \alpha_p$ is a one-to-one function.*

(2) *$\phi : \Psi \to \Phi$ defined by LB-projection is a bijection.*

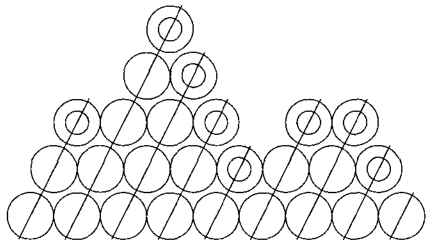

FIGURE 3. (1-9)-block

*Proof.* (1) It is obvious that $\psi$ is surjective. For injectivity, if $p \neq q$ in $S(n, 132)$, then we may assume that there are $i$ and $j$ with $p_i < p_j$ but $q_i > q_j$ and $\alpha_p(i,j) = 1 \neq 0 = \alpha_q(i,j)$. Hence $\alpha_p = \psi(p) \neq \psi(q) = \alpha_q$.

(2) We define a map $\rho : \Phi \to \Psi$ such that $\phi \circ \rho$ and $\rho \circ \phi$ are identities. Let $\beta$ be in $\Phi$. So we can find $n_i$'s $(i = 1, \ldots, n)$. Note that $n_n = 0$ and $n_k - n_{k+1} \leq 1$ $(k = 1, 2, \ldots, n-1)$.

Here we introduce some terminologies. For $\beta \in \Phi$, let $(\mathbf{i}-\mathbf{j})$-*block* be a restricted map (or its image) of coin-stack $\beta$ to the set $L(i,j)$, where

$$L(i,j) = \{(k,l) | \quad i \leq k < l \leq j, \beta(m, m+1) = 1 \text{ for } i \leq m \leq j-1, \\ \beta(i-1, i) = 0 = \beta(j, j+1)\}.$$

In the OCS given as in Figure 1 there are three blocks. These are $(2-7)$-block, $(8-12)$-block, and $(13-14)$-block.

We will define $\gamma = \rho(\beta) : M(n) \to \{0, 1\}$.

We also define the *Right Top-projection* or *RT-projection*. This is, in fact, an inverse map to the LB-projection.

First of all, we restrict our concern to each of blocks, say $(i-j)$-block $L(i,j)$. Consider the restriction

$$\gamma|_{L(i,j)} : L(i,j) \to \{0,1\} \text{ and } n^{(0)} = (n_i, n_{i+1}, \ldots, n_{j-1} = 1, n_j = 0).$$

Define $\gamma(l, j) := 1$ for $i \leq l \leq j-1$, which is a $j$-branch of $\gamma$. Let $n^{(1)} = (n_i - 1, n_{i+1} - 1, \ldots, n_{j-2} - 1, 0, 0)$. $n^{(1)}$ has sub-blocks, and on each sub-block we construct next branch of $\gamma$ in the same manner. Continue this work until there are no incomplete blocks left.

**Example.** (1) Consider a $(1-9)$-block as shown in Figure 3. Choose right-top cells on each line to make a 9-branch.

(2) Make a 9-branch and choose next right-top cells as in Figure 4.

(3) Make a 5-branch and a 8-branch. No incomplete blocks are left(3- and 4-branches completed too).

(4) Figure 6 is an RT-projection of Figure 3.

| i \ j | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $p_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | | $p_2$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | | | $p_3$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | | | | $p_4$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | | | | | $p_5$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | | | | | | $p_6$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | | | | | | | $p_7$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | | | | | | | | $p_8$ | 1 | 1 | 1 | 0 | 0 | 0 |
| 9 | | | | | | | | | $p_9$ | 1 | 1 | 0 | 0 | 0 |
| 10 | | | | | | | | | | $p_{10}$ | 1 | 1 | 0 | 0 |
| 11 | | | | | | | | | | | $p_{11}$ | 1 | 0 | 0 |
| 12 | | | | | | | | | | | | $p_{12}$ | 0 | 0 |
| 13 | | | | | | | | | | | | | $p_{13}$ | 1 |
| 14 | | | | | | | | | | | | | | $p_{14}$ |

| i \ j | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $p_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | | $p_2$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | | | $p_3$ | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | | | | $p_4$ | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | | | | | $p_5$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | | | | | | $p_6$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | | | | | | | $p_7$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | | | | | | | | $p_8$ | 1 | 0 | 1 | 1 | 0 | 0 |
| 9 | | | | | | | | | $p_9$ | 0 | 1 | 1 | 0 | 0 |
| 10 | | | | | | | | | | $p_{10}$ | 1 | 1 | 0 | 0 |
| 11 | | | | | | | | | | | $p_{11}$ | 1 | 0 | 0 |
| 12 | | | | | | | | | | | | $p_{12}$ | 0 | 0 |
| 13 | | | | | | | | | | | | | $p_{13}$ | 1 |
| 14 | | | | | | | | | | | | | | $p_{14}$ |

TABLE 1. 0-1-code tables(OCS of $\beta = \phi(\alpha_p)$ : top, FCS of $\alpha_p$ : bottom) for the given permutation $p = (14, 12, 10, 9, 11, 8, 13, 4, 5, 3, 6, 7, 1, 2)$ in $S_{14}(132)$

The number of cells is finite in $\beta$. Hence, this work will be completed after finite number of times. Obviously this $\gamma$ satisfies conditions (a) and (b) of

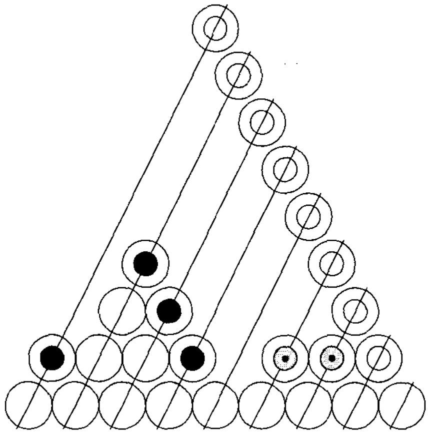

FIGURE 4. 9-branch completed

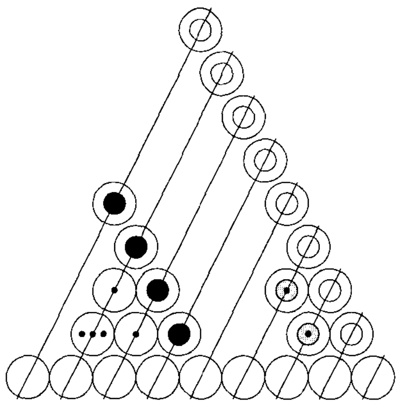

FIGURE 5. 5- and 8-branches completed

Lemma 1. Therefore $\gamma = \rho(\beta)$ is a floated coin-stack and $\rho$ is a bijection (since it is an inverse map of $\phi$). This completes the proof of (2). □

## 3. Algorithm for finding $p$ in $S(n, 132)$ from a floated coin-stack

Let $\gamma$ be a floated coin-stack. We seek $p$ in $S(n, 132)$ such that $\psi(p) = \gamma$. We define a *maxbranch* by the branch which is not in the shadow of other larger branch, and a *maxblock* by the block made by the maxbranch using LB-projection.

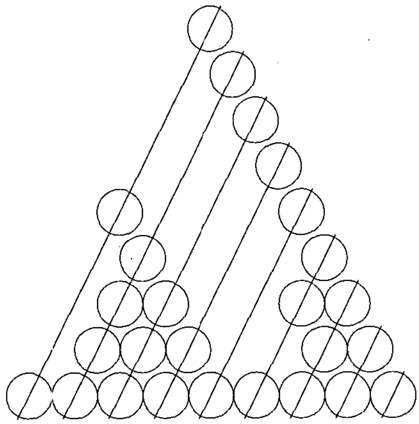

FIGURE 6. RT-projection of Figure 3.

**begin Algorithm**

Step 1: Arrange maxblocks in reverse order.
Step 2: For each maxblock, we do the following.
For an $(i-j)$-maxblock,
Step 2-1: Take out $p_j$ in $(p_i, p_{i+1}, \ldots, p_{j-1}, p_j)$, and adjoin it to the right side of the resulting sequence, producing $(p_i, p_{i+1}, \ldots, p_{j-1})p_j$.
Step 2-2: If there is no $j-1$-branch, take out $p_{j-1}$ in $(p_i, p_{i+1}, \ldots, p_{j-1})$, and adjoin it to the left side of the resulting sequence, producing $p_{j-1}(p_i, p_{i+1}, \ldots, p_{j-2})$, and return to the Step 1 with the block $(p_i, p_{i+1}, \ldots, p_{j-2})$.
Otherwise, return to the Step 1 with the block $(p_i, p_{i+1}, \ldots, p_{j-1})$.
Step 3: Put together the results obtained in Step 2, which corresponds to $123\cdots n$.
Step 4: Find permutation $p$.

**end Algorithm**

**Example.** Begin with $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}, p_{13}, p_{14})$ of the FCS given in Figure 2.

Step 1: $(p_{13}, p_{14})(p_8, p_9, p_{10}, p_{11}, p_{12})(p_2, p_3, p_4, p_5, p_6, p_7)(p_1)$

Step 2: (1) $(p_{13}, p_{14}) \longrightarrow (p_{13})p_{14} \longrightarrow p_{13}p_{14}$

(2) $(p_8, p_9, p_{10}, p_{11}, p_{12}) \longrightarrow (p_8, p_9, p_{10}, p_{11})p_{12} \longrightarrow$
$(p_8, p_9, p_{10})p_{11}p_{12} \longrightarrow p_{10}(p_8, p_9)p_{11}p_{12} \longrightarrow$
$p_{10}(p_8)p_9p_{11}p_{12} \longrightarrow p_{10}p_8p_9p_{11}p_{12}$

(3) $(p_2, p_3, p_4, p_5, p_6, p_7) \longrightarrow (p_2, p_3, p_4, p_5, p_6)p_7 \longrightarrow$
$p_6(p_2, p_3, p_4, p_5)p_7 \longrightarrow p_6(p_3, p_4, p_5)p_2p_7 \longrightarrow$
$p_6(p_3, p_4)p_5p_6p_7 \longrightarrow p_6p_4(p_3)p_5p_2p_7 \longrightarrow p_6p_4p_3p_5p_2p_7$

(4) $p_1$

Step 3: $p_{13}p_{14}p_{10}p_8p_9p_{11}p_{12}p_6p_4p_3p_5p_2p_7p_1 = 123\cdots(13)(14)$
Step 4: $p = (14, 12, 10, 9, 11, 8, 13, 4, 5, 3, 6, 7, 1, 2)$.

## 4. Further problems

First of all, we can interpret a particular pattern avoidance of the type $S_n(132) \cap S_n(\tau)$ for some $\tau \in S_k$ into a certain type of the ordinary coin-stacks. References [2] and [3] has dealt with and mentioned about this a little. Recently, first author of this article discovered that certain types of ordinary coin-stacks appeared in [3] are related with Chebyshev polynomials of the second kind.

Next, there are several partition problems similar to CS-partitions. One of them is so-called restricted growth functions, which is defined as follows:

$$\begin{array}{ll} (i) & d_1 = 1,\ d_i \geq 1 \text{ for } i = 2, \ldots, k, \\ (ii) & n = d_1 + d_2 + \cdots + d_k, \\ (iii) & d_{i+1} - \max\{d_1, d_2, \ldots, d_i\} \leq 1 \text{ for } i = 1, 2, \ldots, k-1 \end{array}$$

These were studied by Milne ([5]), and Stanton and White ([9]) introduced List Algorithm, Rank Algorithm and Unrank Algorithm. Restricted growth functions are related with the number of set-partitions, which is again can be expressed in terms of the Bell number or the Stirling number of the second kind.

Another partition is the $(r, p)$-histogram, which is defined as follows:

$$\begin{array}{ll} (i) & d_1 = r,\ d_i \geq 1 \text{ for } i = 2, \ldots, k, \\ (ii) & n = d_1 + d_2 + \cdots + d_k, \\ (iii) & d_{i+1} - d_i \leq p \text{ for } i = 1, 2, \ldots, k-1 \end{array}$$

These were studied by Merlini and Sprugnoli ([4]). Generating function of $(1, p)$-histogram has certain relations with Schur polynomials. (See [7] for details.)

Studying the following partition problems seems to be curious and interesting:

$$\begin{array}{ll} (i) & d_1 = r,\ d_i \geq 1 \text{ for } i = 2, \ldots, k, \\ (ii) & n = d_1 + d_2 + \cdots + d_k, \\ (iii) & d_{i+1} - \max\{d_1, d_2, \ldots, d_i\} \leq p \text{ for } i = 1, 2, \ldots, k-1 \end{array}$$

or

$$\begin{array}{ll} (i) & d_1 = r,\ d_i \geq 1 \text{ for } i = 2, \ldots, k, \\ (ii) & n = d_1 + d_2 + \cdots + d_k, \\ (iii) & d_{i+1} - \max\{d_{i-m}, d_{i-m+1}, \ldots, d_i\} \leq p \text{ for } i = 1, 2, \ldots, k-1. \end{array}$$

Are there any combinatorial meaning for various $(r, p, m)$, like coin-stacking in CS-partitions or set-partitions in restricted growth functions? Even for the simple case where $r = p = m = 1$ in the last problem, the answer is unknown so far.

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