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EQUIVARIANT SEMIALGEBRAIC LOCAL-TRIVIALITY

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Abstract. We prove the equivariant version of the semialgebraic local-triviality of semialgebraic maps.

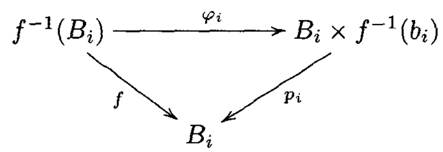
1. Introduction

In this paper we generalize the semialgebraic local-triviality of semialgebraic maps.

A *semialgebraic set* is a subset of some $\mathbb{R}^n$ defined by finite number of polynomial equations and inequalities, and a *semialgebraic map* between semialgebraic sets is a continuous map whose graph is a semialgebraic set. In this paper we only consider the semialgebraic sets in $\mathbb{R}^n$ for some n equipped with the subspace topology induced by the usual topology of $\mathbb{R}^n$, and all semialgebraic maps are continuous.

In 1980 R. M. Hardt [5] proved the semialgebraic local-triviality of semialgebraic maps as follows.

Proposition 1.1 ([5], [1, Theorem 9.3.2]). *Let M, N be two semialgebraic sets and $f\colon M \to N$ a semialgebraic map. Then there exists a finite decomposition of N into semialgebraic subsets $\{B_i\}$ such that for each B_i there exists a semialgebraic homeomorphism $\varphi_i\colon f^{-1}(B_i) \to B_i \times f^{-1}(b_i)$ such that $f|_{f^{-1}(B_i)} = p_i \circ \varphi_i$, where $b_i \in B_i$ and $p_i\colon B_i \times f^{-1}(b_i) \to B_i$ is the projection.*



The purpose of this paper is to prove the equivariant version of Proposition 1.1. For this we need some basic definitions. A semialgebraic set G in some $\mathbb{R}^m$ is called a *semialgebraic group* if it is a topological group whose multiplication and inversion are semialgebraic maps. A *semialgebraic G-set* means

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a semialgebraic set M in some $\mathbb{R}^k$ with a semialgebraic action $\theta\colon G \times M \to M$ of G. A map $f\colon M \to N$ between semialgebraic G-sets is said to be a *semialgebraic G-map* if it is a continuous G-map and a semialgebraic map between ordinary semialgebraic sets M and N, i.e., its graph is a semialgebraic subset of $M \times N$.

The main result of this paper is as follows.

Theorem 1.2. *Let G be a compact semialgebraic group. Let M, N be semialgebraic G-sets and $f\colon M \to N$ a semialgebraic G-map. Then there exists a finite decomposition of N into semialgebraic G-subsets $\{T_i\}$ such that for each T_i there exist semialgebraic G-homeomorphisms $\psi_i\colon T_i \to B_i \times G(y_i)$ and $\varphi_i\colon f^{-1}(T_i) \to B_i \times f^{-1}(G(y_i))$ such that $\psi_i \circ (f|_{f^{-1}(T_i)}) = (\mathrm{id}_{B_i} \times f|_{f^{-1}(G(y_i))}) \circ \varphi_i$, where $y_i \in T_i$ and B_i is a semialgebraic set with the trivial G-action.*

$$\begin{array}{ccc} f^{-1}(T_i) & \xrightarrow[\cong]{\varphi_i} & B_i \times f^{-1}(G(y_i)) \\ {\scriptstyle f}\downarrow & & \downarrow {\scriptstyle \mathrm{id}_{B_i} \times f} \\ T_i & \xrightarrow[\psi_i]{\cong} & B_i \times G(y_i) \end{array}$$

Note $f^{-1}(G(y_i)) = G(f^{-1}(y_i))$. In case G is trivial, Theorem 1.2 is same to Proposition 1.1 with the identification $T_i = B_i \times \{y_i\} = B_i$ by ψ_i.

To prove Theorem 1.2 we need the following result which is the equivariant semialgebraic local-triviality of a semialgebraic G-invariant map.

Theorem 1.3. *Let G be a compact semialgebraic group and M a semialgebraic G-set. Let N be a semialgebraic set and $f\colon M \to N$ a semialgebraic G-invariant map. Then there exists a finite decomposition of N into semialgebraic subsets $\{B_i\}$ such that for each B_i there exists a semialgebraic G-homeomorphism $\varphi_i\colon f^{-1}(B_i) \to B_i \times f^{-1}(b_i)$ such that $f|_{f^{-1}(B_i)} = p_i \circ \varphi_i$, where $b_i \in B_i$ and $p_i\colon B_i \times f^{-1}(b_i) \to B_i$ is the projection.*

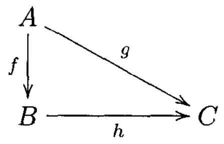
$$\begin{array}{ccc} f^{-1}(B_i) & \xrightarrow{\varphi_i} & B_i \times f^{-1}(b_i) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p_i} & \\ & B_i & \end{array}$$

This paper is organized as follows. In Section 2 we discuss some background materials on semialgebraic G-sets. In Section 3 we prove Theorem 1.3. Section 4 is devoted to the proof of Theorem 1.2.

2. Some background materials on semialgebraic G-sets

In this section we discuss some background materials on semialgebraic G-sets. It is easy to see that the composition of two semialgebraic maps is also semialgebraic. Moreover, the image and the preimage of a semialgebraic subset by a semialgebraic map are semialgebraic. See [1] for more detailed arguments on semialgebraic sets and maps. We state the following elementary proposition because it will be used several times in this paper.

Proposition 2.1 ([8, Lemma 2.4]). *Let A, B, and C be semialgebraic sets, and let $f\colon A \to B$ and $g\colon A \to C$ be semialgebraic. Assume f is surjective. If $h\colon B \to C$ is a continuous map such that $h \circ f = g$, then h is a semialgebraic map.*



If $f\colon M \to N$ is a semialgebraic map which is a homeomorphism, then Proposition 2.1 implies that the inverse f^{-1} is also semialgebraic.

H. Hironaka [6] proved the existence of semialgebraic triangulation for semialgebraic sets as follows: Let M be a semialgebraic set and $M_1, \dots, M_k$ semialgebraic subsets of M. Then there exist a finite open simplicial complex K and a semialgebraic homeomorphism $\tau\colon |K| \to M$ such that each M_j is a finite union of some of the $\tau(\sigma)$, where σ is an open simplex of K. In this case, set

$$\{B_i\} = \{\tau(\sigma) \mid \sigma \text{ is an open simplex of } K\}.$$

Then we obtain the following proposition.

Proposition 2.2. *Let M be a semialgebraic set and $M_1, \dots, M_k$ semialgebraic subsets of M. Then there exists a finite decomposition of M into semialgebraic subsets $B_1, B_2, \dots, B_n$ such that*

(1) *each M_j is a finite union of some B_i;*
(2) *$M = B_1 \cup B_2 \cup \cdots \cup B_n$;*
(3) *$B_i \cap B_{i'} = \varnothing$ if $i \neq i'$.*

In this case $\{B_i\}$ is called *compatible with* $\{M_j\}$.

Now we study some elementary theory of semialgebraic transformation groups. The following is one of the fundamental facts in the theory of semialgebraic transformation groups.

Proposition 2.3 ([3]). *Let G be a compact semialgebraic group and M a semialgebraic G-set. Then the orbit space M/G exists as a semialgebraic set such that the orbit map $\pi\colon M \to M/G$ is semialgebraic.*

As an immediate consequence of Proposition 2.3, if G is a semialgebraic group and H a compact semialgebraic subgroup of G, the homogeneous space G/H is a semialgebraic G-set. On the other hand, for a semialgebraic G-set M the orbit $G(x)$ of $x \in M$ is clearly a semialgebraic G-set. Moreover, the isotropy subgroup G_x is also a closed semialgebraic subgroup of G for all $x \in M$. When

G_x is compact, as in the theory of Lie group actions, by Proposition 2.1, we have the natural semialgebraic G-homeomorphism:

$$\alpha_x \colon G/G_x \to G(x), \quad (gG_x \mapsto gx)$$

Note that every semialgebraic group has a Lie group structure [7].

Proposition 2.4 ([4, 9]). *Let G be a compact semialgebraic group. Then every semialgebraic G-set has only finitely many orbit types.*

Let G be a compact semialgebraic group and M a semialgebraic G-set. Then the set

$$M^G = \{x \in M \mid gx = x \ \text{ for all } \ g \in G\}$$

is a closed semialgebraic subset of M. Moreover, for a subgroup H of G, let $M_{(H)}$ denote the subspace of points on orbits of type G/H, i.e.,

$$M_{(H)} = \{x \in M \mid G_x = gHg^{-1} \ \text{ for some } g \in G\}.$$

By the same way as in the proof of Lemma 3.3 in [8], we obtain that, for any subgroup H of G, $M_{(H)}$ is a semialgebraic G-subset of M. In particular, if H is not a closed semialgebraic subgroup of G then $M_{(H)} = \varnothing$ because the isotropy subgroup G_x is a closed semialgebraic subgroup of G for each $x \in M$.

Furthermore, let H be a closed semialgebraic subgroup of a compact semialgebraic group G, then we can easily show that the normalizer $N(H)$ of H is also a closed semialgebraic subgroup of G as follows; since $N(H)$ is a closed subgroup of G, thus it remains to show that it is a semialgebraic subset of G. We define $c \colon G \times H \to G$ by $c(g, h) = ghg^{-1}$, then c is a semialgebraic map. Moreover, the set $c^{-1}(G - H)$ is a semialgebraic subset of $G \times H$. Then $N(H) = G - p(c^{-1}(G - H))$ is also semialgebraic, where $p \colon G \times H \to G$ is the projection given by $p(g, h) = g$. Therefore $N(H)$ is a closed semialgebraic subgroup of G.

We conclude this section with the following observation for semialgebraic G-sets with only one orbit type.

Proposition 2.5. *Let G be a compact semialgebraic group, and M a semialgebraic G-set with only one orbit type G/H. Then we have the following semialgebraic G-homeomorphisms:*

(1) $\alpha \colon G \times_N M^H \overset{\cong}{\to} M, \quad [g, x] \mapsto g(x)$ *where N is the normalizer of H in G.*
(2) *The map $\beta \colon M^H/N \overset{\cong}{\to} M/G$ induced from the inclusion $M^H \hookrightarrow M$.*
(3) $\gamma \colon (G/H) \times_K M^H \overset{\cong}{\to} M, \quad [gH, x] \mapsto g(x)$ *where $K = N/H$.*

Proof. These maps are well-known to be G-homeomorphisms, see e.g. [2, Chater II]. That these maps are semialgebraic follows easily from Propositions 2.1 and 2.3.

(1) The map α is a continuous homeomorphism. Thus we only need to show that it is semialgebraic. For this, we consider the following commutative

diagram;

$$\begin{array}{ccc} G\times M^H & & \\ {\scriptstyle \pi'}\downarrow & \searrow^{\theta|} & \\ G\times_N M^H & \xrightarrow[\alpha]{} & M \end{array}$$

where π' is the semialgebraic orbit map and $\theta|$ is the restriction of the semialgebraic G-action θ on M. Since π' is surjective, α is semialgebraic by Proposition 2.1.

(2) We only need to show that β is semialgebraic. For this, we consider the following commutative diagram;

$$\begin{array}{ccc} M^H & \overset{i}{\hookrightarrow} & M \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M^H/N & \xrightarrow[\beta]{} & M/G \end{array}$$

where π', π are semialgebraic orbit maps and i is the inclusion. Since π' is surjective, β is semialgebraic by Proposition 2.1.

(3) We only need to show that γ is semialgebraic. For this, we consider the following commutative diagram;

$$\begin{array}{ccc} G\times M^H & & \\ {\scriptstyle \pi'\times \mathrm{id}}\downarrow & \searrow & \\ G/H\times M^H & & {\scriptstyle \theta|} \\ {\scriptstyle \pi''}\downarrow & & \searrow \\ (G/H)\times_K M^H & \xrightarrow[\gamma]{} & M \end{array}$$

where π', π'' are semialgebraic orbit maps and $\theta|$ is the restriction of the semialgebraic G-action θ on M^H. Since $\pi''\circ(\pi'\times \mathrm{id})$ is surjective and semialgebraic, γ is semialgebraic by Proposition 2.1. □

3. Proof of Theorem 1.3

In this section we prove Theorem 1.3. For this we need the equivariant semialgebraic local-triviality of the orbit map $\pi\colon M \to M/G$ for a semialgebraic G-set.

Lemma 3.1. *Let G be a compact semialgebraic group, M a semialgebraic G-set and let $\pi\colon M \to M/G$ be the semialgebraic orbit map. Then there exists a finite decomposition of M/G into semialgebraic subsets $B_1, \dots, B_k$ such that for each B_i there exists a semialgebraic G-homeomorphism $\varphi_i\colon \pi^{-1}(B_i) \to B_i \times \pi^{-1}(b_i)$*

such that $\pi|_{\pi^{-1}(B_i)} = p_i \circ \varphi_i$, *where* $b_i \in B_i$ *and* $p_i\colon B_i \times \pi^{-1}(b_i) \to B_i$ *is the projection.*

$$\begin{array}{ccc} \pi^{-1}(B_i) & \xrightarrow{\quad\varphi_i\quad} & B_i \times \pi^{-1}(b_i) \\ & {\scriptstyle \pi}\searrow \qquad \swarrow{\scriptstyle p_i} & \\ & B_i & \end{array}$$

Proof. We first prove the case when M has only one orbit type, say G/H. By Proposition 2.5, we have semialgebraic G-homeomorphisms $\alpha\colon G\times_N M^H \to M$ and $\beta\colon (M^H)/N \to M/G$ where N is the normalizer of H. Let $\pi_*\colon M^H \to M^H/N$ be the semialgebraic orbit map. Apply Proposition 1.1 to π_*, so that there exists a finite decomposition of $(M^H)/N$ into semialgebraic subsets $\{A_1, \ldots, A_k\}$ such that for each A_i there exists a semialgebraic homeomorphism $\phi_i\colon \pi_*^{-1}(A_i) \to A_i \times N/H$ such that $\pi_*|_{\pi_*^{-1}(A_i)} = p_i \circ \phi_i$ where $p_i\colon A_i \times N/H \to A_i$ is the projection. Note $N/H \cong \pi_*^{-1}(a_i)$ for $a_i \in A_i$. Set $C_i = \phi_i^{-1}(A_i \times \{eH\}) \subset \pi_*^{-1}(A_i)$.

$$\begin{array}{ccccccc} A_i \times \{eH\} & \xleftarrow[\phi_i|]{\cong} & C_i & & & & \\ \big\downarrow & & \big\downarrow & & & & \\ A_i \times N/H & \xleftarrow[\phi_i]{\cong} & \pi_*^{-1}(A_i) & \hookrightarrow & M^H & \hookrightarrow & M \\ & {\scriptstyle p_i}\searrow \quad \swarrow{\scriptstyle \pi_*|} & & & \big\downarrow{\scriptstyle \pi_*} & & \big\downarrow{\scriptstyle \pi} \\ & A_i & & \hookrightarrow & (M^H)/N & \xrightarrow[\beta]{\cong} & M/G \end{array}$$

Then it is easy to see that $NC_i = \pi_*^{-1}(A_i)$. The subgroup N acts on $A_i \times N/H$ and $\pi_*^{-1}(A_i)$ but the homeomorphism $\phi_i\colon \pi_*^{-1}(A_i) \to A_i \times N/H$ is not necessarily N-equivariant. Therefore we need to define a new map $\gamma_i\colon N/H \times A_i \to NC_i = \pi_*^{-1}(A_i)$ by $\gamma_i(gH, x) = g\psi_i(x)$, where $\psi_i\colon A_i \to C_i$ is a semialgebraic homeomorphism defined by $\psi_i(x) = \phi_i^{-1}(x, eH)$. We claim that γ_i is a semialgebraic N-homeomorphism. Consider the following commutative diagram

$$\begin{array}{ccc} N \times A_i & \xrightarrow[\cong]{\ \mathrm{id}\times\psi_i\ } & N \times C_i \\ {\scriptstyle \pi'\times \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \theta|} \\ N/H \times A_i & \xrightarrow[\gamma_i]{\qquad} & NC_i \end{array}$$

where π' is the quotient map and $\theta|$ is the restriction of the action map $\theta\colon G \times M \to M$. Since all other maps in the above diagram are surjective and semialgebraic, γ_i is surjective and semialgebraic by Proposition 2.1. Suppose $\gamma_i(gH, x) = \gamma_i(g'H, x')$ for (gH, x), $(g'H, x') \in N/H \times A_i$. Then $g\psi_i(x) = g'\psi_i(x')$ implies that $\psi_i(x) = g^{-1}g'\psi_i(x')$. Hence $\psi_i(x)$ and $\psi_i(x')$ are contained in the same N-orbit in M^H, which implies that $x = x'$ in A_i. Therefore $\psi_i(x) = g^{-1}g'\psi_i(x)$ and thus $g^{-1}g' \in N_{\psi_i(x)} = H$. Hence $gH = g'H$

which implies that γ_i is injective. This completes the proof of the claim. Clearly γ_i induces a semialgebraic N-homeomorphism $\gamma_i'\colon \pi_*^{-1}(A_i) = NC_i \to A_i \times N/H$ by $\gamma_i' = c \circ \gamma_i^{-1}$, where $c\colon N/H \times A_i \to A_i \times N/H$ is a semialgebraic map defined by $c(gH, x) = (x, gH)$. And the following diagram commutes.

$$\begin{array}{ccc} \pi_*^{-1}(A_i) = NC_i & \xrightarrow{\gamma_i'} & A_i \times N/H \\ & {\scriptstyle \pi_*} \searrow \quad \swarrow {\scriptstyle p_i} & \\ & A_i & \end{array}$$

Now let us continue our original proof. Let $B_i = \beta(A_i) \subset M/G$. Then $\{B_i\}$ is a finite semialgebraic decomposition of M/G and $\pi_*^{-1}(A_i) = (\pi^{-1}(B_i))^H$. Hence we have a semialgebraic G-homeomorphism

$$\begin{array}{lll} \varphi_i\colon & \pi^{-1}(B_i) \cong G \times_N (\pi^{-1}(B_i))^H & (\because\ \alpha^{-1}) \\ & = G \times_N \pi_*^{-1}(A_i) & \\ & \cong G \times_N (N/H \times A_i) & (\because\ \mathrm{id} \times_N \gamma_i') \\ & \cong G \times_N (N/H \times B_i) & (\because\ \mathrm{id} \times_N (\mathrm{id} \times \beta)) \\ & \cong (G \times_N N/H) \times B_i & \\ & \cong G/H \times B_i \cong B_i \times G/H & \end{array}$$

such that $\pi|_{\pi^{-1}(B_i)} = p_i \circ \varphi_i$ where $p_i\colon B_i \times G/H \to B_i$ is the projection. This completes the proof of the case when M has only one orbit type.

We now prove the general case. By Proposition 2.4, M has finite orbit types, say $G/H_1, \dots, G/H_l$. Then for each $i = 1, \dots, l$, $M_{(H_i)}$ has only one orbit type. Hence, by the previous case, the restriction $\pi|\colon M_{(H_i)} \to M_{(H_i)}/G$ has the equivariant semialgebraic local-triviality. Since M (resp. M/G) is the disjoint union of $M_{(H_i)}$ (resp. $M_{(H_i)}/G$), $\pi\colon M \to M/G$ has obviously the equivariant semialgebraic local-triviality. □

As an application of Lemma 3.1, we prove Theorem 1.3 as follows.

Proof of Theorem 1.3. By Lemma 3.1, there exists a finite decomposition of M/G into semialgebraic subsets $A_1, \dots, A_l$ such that for each A_j there exists a semialgebraic G-homeomorphism $\psi_j\colon \pi^{-1}(A_j) \to A_j \times \pi^{-1}(a_j)$ such that $\pi|_{\pi^{-1}(A_j)} = q_j \circ \psi_j$ where $a_j \in A_j$, $\pi\colon M \to M/G$ is the semialgebraic orbit map and $q_j\colon A_j \times \pi^{-1}(a_j) \to A_j$ is the projection.

On the other hand, since $f\colon M \to N$ is a semialgebraic G-invariant map, it induces a semialgebraic map $\bar{f}\colon M/G \to N$ by Proposition 2.1. Apply Proposition 1.1 to $\bar{f}$, then there exists a finite decomposition of N into semialgebraic subsets $C_1, \dots, C_m$ such that for each C_k there exists a semialgebraic homeomorphism $\phi_k\colon \bar{f}^{-1}(C_k) \to C_k \times \bar{f}^{-1}(c_k)$ such that $\bar{f}|_{\bar{f}^{-1}(C_k)} = r_k \circ \phi_k$ where $c_k \in C_k$ and $r_k\colon C_k \times \bar{f}^{-1}(c_k) \to C_k$ is the projection.

By Proposition 2.2, there exists a finite decomposition of N into semialgebraic subsets $\{B_i\}$ which is compatible with $\{C_k\} \cup \{\bar{f}(A_j)\}$. We claim that

$\{B_i\}$ is the desired finite decomposition of N. Notice that each B_i is either $B_i \cap \bar{f}(M/G) = \varnothing$ or $B_i \subset \bar{f}(M/G) = f(M)$ by the compatibility of $\{B_i\}$.

In case $B_i \cap \bar{f}(M/G) = \varnothing$, $f^{-1}(B_i) = f^{-1}(b_i) = \varnothing$, and hence $f^{-1}(B_i) = \varnothing = B_i \times f^{-1}(b_i)$.

In case $B_i \subset \bar{f}(M/G)$, there exist $C_{k(i)}$ and $A_{j(i)}$ such that $B_i \subset C_{k(i)}$, $B_i \subset \bar{f}(A_{j(i)})$ by the compatibility of $\{B_i\}$. Thus we obtain a semialgebraic G-homeomorphism

$$\begin{aligned}\varphi_i\colon \quad f^{-1}(B_i) = \pi^{-1}(\bar{f}^{-1}(B_i)) &\underset{\psi_{j(i)}}{\overset{\cong}{\to}} \bar{f}^{-1}(B_i) \times \pi^{-1}(a_j) \\ &\underset{\phi_{k(i)}\times \mathrm{id}}{\overset{\cong}{\to}} B_i \times \bar{f}^{-1}(b_i) \times \pi^{-1}(a_j) \underset{\mathrm{id}\times h}{\overset{\cong}{\to}} B_i \times f^{-1}(b_i)\end{aligned}$$

where $b_i \in B_i$, $a_j \in \bar{f}^{-1}(b_i)$ and $h\colon \bar{f}^{-1}(b_i) \times \pi^{-1}(a_j) \to f^{-1}(b_i)$ is the semialgebraic G-homeomorphism which is the restriction of $\psi_{j(i)}^{-1}$. Note $f^{-1}(b_i) = \pi^{-1}(\bar{f}^{-1}(b_i))$. It is easy to check that the diagram

$$\begin{array}{ccc} f^{-1}(B_i) & \xrightarrow{\varphi_i} & B_i \times f^{-1}(b_i) \\ & {}_{f}\searrow \quad \swarrow_{p_i} & \\ & B_i & \end{array}$$

commutes where p_i is the projection. This completes the proof of Theorem 1.3. □

4. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2.

Let $\pi_M\colon M \to M/G$ and $\pi_N\colon N \to N/G$ denote the semialgebraic orbit maps. Apply Theorem 1.3 to π_M, then we have a finite decomposition of M/G into semialgebraic subsets $\{A_j\}$ such that for each A_j there exists a semialgebraic G-homeomorphism $\phi_j\colon \pi_M^{-1}(A_j) \to A_j \times \pi_M^{-1}(a_j)$ such that $\pi_M|_{\pi_M^{-1}(A_j)} = q_j \circ \phi_j$ where $a_j \in A_j$ and $q_j\colon A_j \times \pi_M^{-1}(a_j) \to A_j$ is the projection. Similarly, there exists a finite decomposition of N/G into semialgebraic subsets $\{C_k\}$ such that for each C_k there exists a semialgebraic G-homeomorphism $\psi'_k\colon \pi_N^{-1}(C_k) \to C_k \times \pi_N^{-1}(c_k)$ such that $\pi_N|_{\pi_N^{-1}(C_k)} = r_k \circ \psi'_k$ where $c_k \in C_k$ and $r_k\colon C_k \times \pi_N^{-1}(c_k) \to C_k$ is the projection. Moreover, since $f\colon M \to N$ is a semialgebraic G-map, it induces a semialgebraic map $\bar{f}\colon M/G \to N/G$. By Proposition 1.1, there exists a finite decomposition of N/G into semialgebraic subsets $\{D_l\}$ such that for each D_l there exists a semialgebraic homeomorphism $\chi_l\colon \bar{f}^{-1}(D_l) \to D_l \times \bar{f}^{-1}(d_l)$ such that $\bar{f}|_{\bar{f}^{-1}(D_l)} = s_l \circ \chi_l$ where $d_l \in D_l$ and $s_l\colon D_l \times \bar{f}^{-1}(d_l) \to D_l$ is the projection.

By Proposition 2.2, there exists a finite decomposition of N/G into semialgebraic subsets $\{B_i\}$ which is compatible with $\{\bar{f}(A_j)\} \cup \{C_k\} \cup \{D_l\}$. Notice that each B_i is either $B_i \cap \bar{f}(M/G) = \varnothing$ or $B_i \subset \bar{f}(M/G) = \pi_N(f(M))$ by the compatibility of $\{B_i\}$.

In case $B_i \cap \bar{f}(M/G) = \varnothing$, $\pi_N^{-1}(b_i) = \varnothing$ for all $b_i \in B_i$. Set $T_i = \pi_N^{-1}(B_i)$, then $f^{-1}(T_i) = \varnothing$ and $f^{-1}(G(y_i)) = \varnothing$ for all $y_i \in T_i$. Hence $f^{-1}(T_i) = \varnothing = B_i \times f^{-1}(G(y_i))$.

In case $B_i \subset \bar{f}(M/G)$, there exist $A_{j(i)}$, $C_{k(i)}$ and $D_{l(i)}$ such that $B_i \subset \bar{f}(A_{j(i)})$, $B_i \subset C_{k(i)}$ and $B_i \subset D_{l(i)}$ by the compatibility of $\{B_i\}$. Put $T_i = \pi_N^{-1}(B_i)$, then T_i is a semialgebraic G-subset of N which is semialgebraically G-homeomorphic to $B_i \times \pi_N^{-1}(b_i)$ by $\psi'_{k(i)}$. Put

$$\psi_i = \psi'_{k(i)}|\colon T_i = \pi_N^{-1}(B_i) \xrightarrow{\cong} B_i \times \pi_N^{-1}(b_i)$$

where $\psi'_{k(i)}|$ denotes the restriction of $\psi'_{k(i)}$.

On the other hand, $f^{-1}(T_i) = \pi_M^{-1}(\bar{f}^{-1}(B_i))$ is semialgebraically G-homeomorphic to $\bar{f}^{-1}(B_i) \times \pi_M^{-1}(a_{j(i)})$ by $\phi_{j(i)}$ where $a_{j(i)} \in \bar{f}^{-1}(b_i) \subset A_{j(i)}$. Thus we have a semialgebraic G-homeomorphism

$$\begin{aligned}\varphi_i\colon\quad & f^{-1}(T_i) = \pi_M^{-1}(\bar{f}^{-1}(B_i)) \underset{\phi_{j(i)}|}{\xrightarrow{\cong}} \bar{f}^{-1}(B_i) \times \pi_M^{-1}(a_{j(i)}) \\ & \underset{\chi_{l(i)}|\times \mathrm{id}}{\xrightarrow{\cong}} B_i \times \bar{f}^{-1}(b_i) \times \pi_M^{-1}(a_{j(i)}) \underset{\mathrm{id}_{B_i}\times h}{\xrightarrow{\cong}} B_i \times f^{-1}(\pi_N^{-1}(b_i))\end{aligned}$$

where $h\colon \bar{f}^{-1}(b_i) \times \pi_M^{-1}(a_{j(i)}) \to \pi_M^{-1}(\bar{f}^{-1}(b_i)) = f^{-1}(\pi_N^{-1}(b_i))$ is a semialgebraic G-homeomorphism which is the restriction of $\phi_{j(i)}^{-1}$.

It is easy to check that the diagram

$$\begin{array}{ccc} f^{-1}(T_i) & \underset{\cong}{\xrightarrow{\varphi_i}} & B_i \times f^{-1}(\pi_N^{-1}(b_i)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{B_i}\times f|} \\ T_i & \underset{\psi_i}{\xrightarrow{\cong}} & B_i \times \pi_N^{-1}(b_i) \end{array}$$

commutes where $f|\colon f^{-1}(\pi_N^{-1}(b_i)) \to \pi_N^{-1}(b_i)$ is the restriction of f. Note $\pi_N^{-1}(b_i) = G(y_i)$ for all $y_i \in \pi_N^{-1}(b_i) \subset T_i$. This completes the proof of Theorem 1.2.

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