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IDEMPOTENT MATRIX PRESERVERS OVER BOOLEAN ALGEBRAS

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ABSTRACT. We consider the set of $n \times n$ idempotent matrices and we characterize the linear operators that preserve idempotent matrices over Boolean algebras. We also obtain characterizations of linear operators that preserve idempotent matrices over a chain semiring, the nonnegative integers and the nonnegative reals.

1. Introduction

One of the most active and fertile subjects in matrix theory during the past one hundred years is the linear preserver problem, which concerns the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. Although the linear preservers concerned are mostly linear operators on matrix spaces over some fields or rings, the same problem has been extended to matrices over various semirings ([1]-[10]).

Chan et al. [6] showed that if T is a linear operator on the real (or complex) $n \times n$ matrices that preserve idempotent matrices and fixes the identity matrix, then there exists an $n \times n$ invertible real (or complex) matrix U such that either

$$T(X) = UXU^{-1} \tag{1.1}$$

or

$$T(X) = UX^tU^{-1} \tag{1.2}$$

for all $n \times n$ real (or complex) matrix X, where X^t denotes the transpose of X. Beasley and Pullman [3] extended the results of Chan et al. [6] by removing the condition that the operator fixes the identity matrix.

In this paper, we will characterize the linear operators that preserve idempotent matrices over Boolean algebras and related semirings. In Section 2, we list some fundamental concepts and preliminary Lemmas. In Section 3, we characterize the linear operators that preserve idempotent matrices over the binary Boolean algebra and related semiring (see Theorems 3.1, 3.3 and

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Corollary 3.2). In Section 4, we give a linear operator preserving idempotent matrices which is neither a form (1.1) nor (1.2) (see Example 4.4). Also we have a characterization of linear operators that preserve idempotent matrices over general Boolean algebras (see Theorem 4.5).

2. Preliminaries

A *semiring* $\mathbb{S}$ consists of a set $\mathbb{S}$ and two binary operations, addition $(+)$ and multiplication $(\cdot)$, such that:

(a) $(\mathbb{S}, +)$ is a commutative monoid with identity element 0;
(b) $(\mathbb{S}, \cdot)$ is a monoid with identity element 1;
(c) multiplication is distribute over addition on both side;
(d) $s0 = 0 = 0s$ for all $s \in \mathbb{S}$.

Usually $\mathbb{S}$ denotes both the semiring and the set. A semiring $\mathbb{S}$ is called *commutative* if the monoid $(\mathbb{S}, \cdot)$ is commutative; $\mathbb{S}$ is called *antinegative* if $a + b = 0$ implies $a = b = 0$ for any $a, b \in \mathbb{S}$.

Here are some examples of semirings which occur in combinatorics. They are all commutative and antinegative. For a fixed positive integer k, let $\mathbb{B}_k$ be the *Boolean algebra* of subsets of a k-element set $\mathbb{S}_k$ and $\sigma_1, \sigma_2, \ldots, \sigma_k$ denote the singleton subsets of $\mathbb{S}_k$. Union is denoted by $+$ and intersection by juxtaposition; 0 denotes the null set and 1 the set $\mathbb{S}_k$. Under these two operations, $\mathbb{B}_k$ is a semiring; all of its elements, except 0 and 1, are zero-divisors. In particular, if $k = 1$, $\mathbb{B}_1$ is called the *binary Boolean algebra.*

Let $\mathbb{K}$ be any set of two or more elements. If $\mathbb{K}$ is totally ordered by $<$ (i.e., $x < y$ or $y < x$ for all distinct elements x, y in $\mathbb{K}$), then define $x + y$ as $\max(x, y)$ and xy as $\min(x, y)$ for all $x, y \in \mathbb{K}$. If $\mathbb{K}$ has both a universal lower bound and a universal upper bound, then $\mathbb{K}$ becomes a semiring, and called a *chain semiring.* In particular, if $\mathbb{F}$ is the real interval $[0, 1]$, then $(\mathbb{F}, \max, \min)$ is a semiring, the *fuzzy semiring.*

If $\mathbb{P}$ is any subring of the reals $\mathbb{R}$ under usual addition and multiplication, then $\mathbb{P}_+$, the nonnegative part of $\mathbb{P}$, is a semiring. In particular, $\mathbb{Z}_+$ is the semiring of all nonnegative integers.

Algebraic terms such as *unit, zero divisor* and *invertibility* are defined for semirings as for rings.

Hereafter, $\mathbb{S}$ denote an arbitrary semiring which is commutative and antinegative.

Let $\mathbb{M}_n(\mathbb{S})$ denote the set of all $n \times n$ matrices over a semiring $\mathbb{S}$. The $n \times n$ identity matrix, I_n, and the $n \times n$ zero matrix, O_n, are defined as if $\mathbb{S}$ were a field. We denote the $n \times n$ matrix all of whose entries are 1 by J_n.

We define $^* : \mathbb{M}_n(\mathbb{S}) \to \mathbb{M}_n(\mathbb{B}_1)$ by $X^* = [x^*_{ij}]$ for all $X = [x_{ij}] \in \mathbb{M}_n(\mathbb{S})$, where $x^*_{ij} = 1$ if and only if $x_{ij} \neq 0$. Then * is a semiring homomorphism when $\mathbb{S}$ has no zero divisor.

Lemma 2.1. *Let $\mathbb{S}$ be a semiring without zero divisors. Then a matrix $X = [x_{ij}] \in \mathbb{M}_n(\mathbb{S})$ is invertible if and only if $X^* \in \mathbb{M}_n(\mathbb{B}_1)$ is a permutation matrix and all nonzero entries of X are units.*

Proof. Suppose that $X = [x_{ij}] \in \mathbb{M}_n(\mathbb{S})$ is invertible. Then X^* is an invertible binary Boolean matrix, and hence it is a permutation matrix. It follows from the invertibility of X that all nonzero entries of X are units. The converse is immediate. □

Let $\mathbb{E}_n = \{E_{ij} \mid i, j = 1, \ldots, n\}$, where E_{ij} is the $n \times n$ matrix whose $(i, j)^{\text{th}}$ entry is 1 and whose other entries are 0. We call each member of $\mathbb{E}_n$ a *cell.* When $i \neq j$, we say E_{ij} is an *off-diagonal* cell; E_{ii} is a *diagonal* cell. A *line* is a row or a column of a matrix. A set of cells is *collinear* if they are all in the same line.

The following proposition is an immediate consequence of the rules of matrix multiplication.

Proposition 2.2. *For two cells E_{ij} and E_{rs} in $\mathbb{E}_n$, we have $E_{ij}E_{rs} = E_{is}$ or O_n according as $j = r$ or $j \neq r$.*

Let $\mathbb{D}_n(\mathbb{S}) = \{A \in \mathbb{M}_n(\mathbb{S}) \mid A^2 = A\}$. We call each member of $\mathbb{D}_n(\mathbb{S})$ an *idempotent matrix.* Then we can easily show that all diagonal cells are idempotent but all off-diagonal cells are not idempotent. Furthermore, any sum of distinct diagonal cells is idempotent.

Lemma 2.3. *Suppose that E is a diagonal cell and E_1, E_2 are distinct off-diagonal cells. Then their sum is idempotent if and only if they are collinear.*

Proof. It is an easy exercise. □

A mapping $T : \mathbb{M}_n(\mathbb{S}) \to \mathbb{M}_n(\mathbb{S})$ is said to be a *linear operator* on $\mathbb{M}_n(\mathbb{S})$ if $T(s_1X_1 + s_2X_2) = s_1T(X_1) + s_2T(X_2)$ for all X_1, X_2 in $\mathbb{M}_n(\mathbb{S})$ and for all s_1, s_2 in $\mathbb{S}$. A linear operator T on $\mathbb{M}_n(\mathbb{S})$ is said to be a *preserver* of $\mathbb{D}_n(\mathbb{S})$ (or *T preserves idempotent matrices*) if $T(X) \in \mathbb{D}_n(\mathbb{S})$ whenever $X \in \mathbb{D}_n(\mathbb{S})$.

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices in $\mathbb{M}_n(\mathbb{S})$, we shall use the notation $A \geq B$(or $B \leq A$) if $b_{ij} \neq 0$ implies $a_{ij} \neq 0$ for all i and j. This provides a reflexive and transitive relation on $\mathbb{M}_n(\mathbb{S})$. If A and B are matrices in $\mathbb{M}_n(\mathbb{S})$ with $A \geq B$, it follows from the linearity of T that $T(A) \geq T(B)$ for any linear operator T on $\mathbb{M}_n(\mathbb{S})$.

Lemma 2.4. *Let $\mathbb{S}$ be a semiring, and let T be a linear operator on $\mathbb{M}_n(\mathbb{S})$. Then T is invertible if and only if there exist a permutation α on the set $\{(i, j) \mid i, j = 1, \ldots, n\}$ and unit elements $b_{ij} \in \mathbb{S}$, $i, j = 1, \ldots, n$ such that $T(E_{ij}) = b_{ij}E_{\alpha(i,j)}$.*

Proof. Suppose that T is invertible on $\mathbb{M}_n(\mathbb{S})$. Let E_{ij} be any cell in $\mathbb{E}_n$. By invertibility of T, there exists at least one cell $E_{rs} \in \mathbb{E}_n$ such that $T(E_{ij}) \geq E_{rs}$. Thus we have $E_{ij} \geq T^{-1}(E_{rs})$ because T^{-1} is also linear. This implies that

$T^{-1}(E_{rs}) = xE_{ij}$ for some nonzero scalar $x \in \mathbb{S}$, equivalently $E_{rs} = xT(E_{ij})$, equivalently $T(E_{ij}) = b_{ij}E_{rs}$ for some nonzero scalar $b_{ij} \in \mathbb{S}$. Since E_{ij} is an arbitrary cell, T permutes $\mathbb{E}_n$ with nonzero scalar multiplication. That is, there exists a permutation α on the set $\{(i,j) \mid i,j = 1,\dots,n\}$ such that $T(E_{ij}) = b_{ij}E_{\alpha(i,j)}$ for some nonzero scalar b_{ij}, $i,j = 1,\dots,n$.

We now need to show that the b_{ij} are all units. Since T is surjective and $T(E_{rs}) \not\leq E_{\alpha(i,j)}$ for $(r,s) \neq (i,j)$, there is some nonzero scalar x in $\mathbb{S}$ such that $T(xE_{ij}) = E_{\sigma(i,j)}$. Then we have that $T(xE_{ij}) = xT(E_{ij}) = xb_{ij}E_{\sigma(i,j)} = E_{\sigma(i,j)}$ because T is linear. That is, $xb_{ij} = 1$, and hence b_{ij} is a unit.

The converse is immediate. □

The *Schur* (or *Hadamard*) *product*, $A \circ B$ in $\mathbb{M}_n(\mathbb{S})$ is defined by $A \circ B = [a_{ij}\, b_{ij}]$.

Lemma 2.5. *For a semiring $\mathbb{S}$ without zero divisors, let T be an invertible linear operator on $\mathbb{M}_n(\mathbb{S})$ that preserves $\mathbb{D}_n(\mathbb{S})$. Then*

(a) $T(I) = I$;

(b) *there exist a permutation matrix P and a matrix $B = [b_{ij}]$ such that $T(X) = P(X \circ B)P^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$ or $T(X) = P(X^t \circ B)P^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$, where all b_{ij} are unit elements with $b_{ii} = 1$ for $i = 1,\dots,n$*

Proof. Let T be an invertible linear operator on $\mathbb{M}_n(\mathbb{S})$ that preserves $\mathbb{D}_n(\mathbb{S})$. By Lemma 2.4, there exist a permutation α on the set $\{(i,j) \mid i,j = 1,\dots,n\}$ and unit elements $c_{ij} \in \mathbb{S}$, $i,j = 1,\dots,n$ such that $T(E_{ij}) = c_{ij}E_{\alpha(i,j)}$. Now, we will show that $T(I) \geq I$. If not, there exist a diagonal cell E_{ii} and an off-diagonal cell E_{rs} such that $T(E_{ii}) \geq E_{rs}$, equivalently $T(E_{ii}) = c_{ii}E_{rs}$. Then we have that E_{ii} is idempotent while $T(E_{ii})$ is not, a contradiction. Thus we obtain $T(I) = \sum_{i=1}^{n} c_{ii}E_{ii}$. Since T preserves idempotent matrices, it follows that all c_{ii} are idempotent elements. That is, c_{ii} are unit and idempotent, and hence we have $c_{ii} = 1$ for all $i = 1,\dots,n$, equivalently $T(I) = I$. Therefore there is a permutation β of $\{1,\dots,n\}$ such that $T(E_{ii}) = E_{\beta(i)\beta(i)}$ for each $i = 1,\dots,n$. Define an operator L on $\mathbb{M}_n(\mathbb{S})$ by

$$L(X) = P^tT(X)P$$

for all $X \in \mathbb{M}_n(\mathbb{S})$, where P is the permutation matrix corresponding to β so that $L(E_{ii}) = E_{ii}$ for each $i = 1,\dots,n$. Then we can easily show that L is an invertible linear operator on $\mathbb{M}_n(\mathbb{S})$ that preserves idempotent matrices. By Lemma 2.4, L permutes $\mathbb{E}_n$ with some unit scalar multiplication. Therefore for any cell E_{rs} in $\mathbb{E}_n$, there exist exactly one cell E_{pq} and a unit element b_{rs} such that $L(E_{rs}) = b_{rs}E_{pq}$. It follows from $L(I) = I$ that $b_{ii} = 1$ for each $i = 1,\dots,n$.

Suppose that $r \neq s$. Then we have $p \neq q$ because L is injective. Consider an idempotent matrix $A = E_{rs} + E_{rr}$ so that $L(A) = b_{rs}E_{pq} + E_{rr}$ is also idempotent. This implies that $r = p$ or $r = q$. Similarly we have $s = p$ or

$s = q$. Therefore we obtain that for each cell E_{rs} in $\mathbb{E}_n$, there exists a unit element b_{rs} such that

$$L(E_{rs}) = b_{rs}E_{rs} \quad \text{or} \quad L(E_{rs}) = b_{rs}E_{sr}.$$

Suppose that $L(E_{rs}) = b_{rs}E_{rs}$ with $r \neq s$ and $L(E_{rt}) = b_{rt}E_{tr}$ for some $t \neq r, s$. Let $B = E_{rr} + E_{rs} + E_{rt}$. By Lemma 2.3, we have that B is idempotent, while $L(B) = E_{rr} + b_{rs}E_{rs} + b_{rt}E_{tr}$ is not, a contradiction. It follows that if $L(E_{ij}) = b_{ij}E_{ij}$ for some cell $E_{ij} \in \mathbb{E}_n$ with $i \neq j$, then we have $L(E_{rs}) = b_{rs}E_{rs}$ for all cell $E_{rs} \in \mathbb{E}_n$. Similarly, if $L(E_{ij}) = b_{ij}E_{ji}$ for some cell $E_{ij} \in \mathbb{E}_n$ with $i \neq j$, then we have $L(E_{rs}) = b_{rs}E_{sr}$ for all cell $E_{rs} \in \mathbb{E}_n$.

We have established that $L(X) = X \circ B$ for all $X \in \mathbb{M}_n(\mathbb{S})$ or $L(X) = X^t \circ B$ for all $X \in \mathbb{M}_n(\mathbb{S})$, where $B = [b_{ij}]$ is a matrix whose entries are unit elements with $b_{ii} = 1$ for each $i = 1, \ldots, n$. Since $L(X) = P^tT(X)P$, we have that $T(X) = P(X \circ B)P^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$ or $T(X) = P(X^t \circ B)P^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$. □

3. The binary Boolean case and others

In this section we obtain characterizations of the linear operators that preserve idempotent matrices over the binary Boolean algebra $\mathbb{B}_1$, a chain semiring $\mathbb{K}$, the nonnegative integers $\mathbb{Z}_+$ and the nonnegative reals $\mathbb{R}_+$.

Theorem 3.1. *Let T be a linear operator on $\mathbb{M}_n(\mathbb{B}_1)$. Then T is an invertible linear operator that preserves $\mathbb{D}_n(\mathbb{B}_1)$ if and only if there exists a permutation matrix P such that either*

(a) $T(X) = PXP^t$ *for all* $X \in \mathbb{M}_n(\mathbb{B}_1)$, *or*
(b) $T(X) = PX^tP^t$ *for all* $X \in \mathbb{M}_n(\mathbb{B}_1)$.

Proof. Let T be an invertible linear operator on $\mathbb{M}_n(\mathbb{B}_1)$ that preserves $\mathbb{D}_n(\mathbb{B}_1)$. By Lemma 2.5, there exist a permutation matrix P and a matrix $B = [b_{ij}]$ such that $T(X) = P(X \circ B)P^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$ or $T(X) = P(X^t \circ B)P^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$, where all b_{ij} are unit elements with $b_{ii} = 1$ for $i = 1, \ldots, n$. Since the element "1" is the only unit element in $\mathbb{B}_1$, we have $b_{ij} = 1$ for all $i, j = 1, \ldots, n$. It follows that $B = J_n$, and thus the necessity is satisfied. The sufficiency is obvious. □

Corollary 3.2. *Let T be a linear operator on $\mathbb{M}_n(\mathbb{S})$, where $\mathbb{S} = \mathbb{K}$ or $\mathbb{Z}_+$. Then T is an invertible linear operator that preserves $\mathbb{D}_n(\mathbb{S})$ if and only if there exists a permutation matrix P such that either*

(a) $T(X) = PXP^t$ *for all* $X \in \mathbb{M}_n(\mathbb{S})$, *or*
(b) $T(X) = PX^tP^t$ *for all* $X \in \mathbb{M}_n(\mathbb{S})$.

Proof. The proof is similar to that of Theorem 3.1. □

Theorem 3.3. *Let T be a linear operator on $\mathbb{M}_n(\mathbb{R}_+)$. Then T is an invertible linear operator that preserves $\mathbb{D}_n(\mathbb{R}_+)$ if and only if there exists an invertible matrix U such that either*

(a) $T(X) = UXU^{-1}$ *for all* $X \in \mathbb{M}_n(\mathbb{R}_+)$, *or*
(b) $T(X) = UX^tU^{-1}$ *for all* $X \in \mathbb{M}_n(\mathbb{R}_+)$.

Proof. Suppose that T is an invertible linear operator on $\mathbb{M}_n(\mathbb{R}_+)$ that preserves $\mathbb{D}_n(\mathbb{R}_+)$. By Lemma 2.5, there exist a permutation matrix P and a matrix $B = [b_{ij}]$ such that $T(X) = P(X \circ B)P^t$ for all $X \in \mathbb{M}_n(\mathbb{R}_+)$ or $T(X) = P(X^t \circ B)P^t$ for all $X \in \mathbb{M}_n(\mathbb{R}_+)$, where all b_{ij} are unit elements with $b_{ii} = 1$ for $i = 1, \ldots, n$. Let β be a permutation of $\{1, \ldots, n\}$ corresponding to P. Assume that $T(X) = P(X \circ B)P^t$ for all $X \in \mathbb{M}_n(\mathbb{R}_+)$. Then we have $T(E_{ij}) = b_{ij}E_{\beta(i)\beta(j)}$ for any cell E_{ij} in $\mathbb{E}_n$.

Consider a matrix $X = E_{11} + E_{1j} + E_{i1} + E_{ij}$, where $i, j \neq 1$. Then by Proposition 2.2, X is idempotent. Since T preserves idempotent matrices, we have $T(X) = T(X)^2$ so that

$$\begin{aligned} & E_{\beta(1)\beta(1)} + b_{1j}E_{\beta(1)\beta(j)} + b_{i1}E_{\beta(i)\beta(1)} + b_{ij}E_{\beta(i)\beta(j)} \\ = \quad & E_{\beta(1)\beta(1)} + b_{1j}E_{\beta(1)\beta(j)} + b_{i1}E_{\beta(i)\beta(1)} + b_{i1}b_{1j}E_{\beta(i)\beta(j)}, \end{aligned}$$

equivalently,

$$b_{i1}\, b_{1j} = b_{ij}. \tag{3.1}$$

Now we will show that $b_{ij}\, b_{ji} = 1$ for all $i, j = 1, \ldots, n$. Consider a matrix

$$Y = \frac{1}{2}E_{ii} + \frac{1}{2}E_{ij} + \frac{1}{2}E_{ji} + \frac{1}{2}E_{jj}.$$

Then by Proposition 2.2, we can easily show that Y is idempotent so that

$$T(Y) = \frac{1}{2}E_{\beta(i)\beta(i)} + \frac{1}{2}b_{ij}E_{\beta(i)\beta(j)} + \frac{1}{2}b_{ji}E_{\beta(j)\beta(i)} + \frac{1}{2}E_{\beta(j)\beta(j)}.$$

It follows from Proposition 2.2 and $T(Y)^2 = T(Y)$ that $\frac{1+b_{ij}b_{ji}}{4} = \frac{1}{2}$, equivalently $b_{ij}\, b_{ji} = 1$ for all $i, j = 1, \ldots, n$.

Let $D = [d_{ij}] = \mathrm{diag}(b_{11}, b_{21}, \ldots, b_{n1})$. By Lemma 2.1, D is invertible and $D^{-1} = [e_{ij}] = \mathrm{diag}(b_{11}, b_{12}, \ldots, b_{1n})$. Now for any $X = [x_{ij}] \in \mathbb{M}_n(\mathbb{R}_+)$, the $(i, j)^{\text{th}}$ entry of DXD^{-1} is

$$x_{ij}\, b_{i1}\, b_{1j} = x_{ij}\, b_{ij}$$

by (3.1), which is the $(i, j)^{\text{th}}$ entry of $X \circ B$. Therefore $X \circ B = DXD^{-1}$. It follows from $T(X) = P(X \circ B)P^t$ that $T(X) = P(DXD^{-1})P^t$. If we let $U = PD$, then U is invertible and $T(X) = UXU^{-1}$ for all $X \in \mathbb{M}_n(\mathbb{R}_+)$. Similarly, if $T(X) = P(X^t \circ B)P^t$, we obtain that $T(X) = UX^tU^{-1}$ for all $X \in \mathbb{M}_n(\mathbb{R}_+)$.

The converse is obvious. □

Thus we have characterized the linear operators that preserve idempotent matrices over the binary Boolean algebra $\mathbb{B}_1$, a chain semiring $\mathbb{K}$, the nonnegative integers $\mathbb{Z}_+$, and the nonnegative reals $\mathbb{R}_+$.

4. The general Boolean case

In this section, we study idempotent matrices over a general Boolean algebra $\mathbb{B}_k$ with $k \geq 1$. Furthermore, using Theorem 3.1, we obtain characterizations of linear operators that preserve idempotent matrices over a general Boolean algebra.

For any matrix $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{B}_k)$, the l^{th} *constituent*, A_l, of A is the $n \times n$ binary Boolean matrix whose $(i,j)^{\text{th}}$ entry is 1 if and only if $a_{ij} \supseteq \sigma_l$. Via the constituents, A can be written uniquely as

$$A = \sum_{l=1}^{k} \sigma_l A_l,$$

which is called the *canonical form* of A (see [8]).

It follows from the uniqueness of the decomposition and the fact that the singletons are mutually orthogonal idempotents that for all matrices $A, B \in \mathbb{M}_n(\mathbb{B}_k)$ and all $\alpha \in \mathbb{B}_k$,

$$(4.1) \qquad (AB)_l = A_l B_l, \qquad (A+B)_l = A_l + B_l \quad \text{and} \quad (\alpha A)_l = \alpha_l A_l$$

for all $1 \leq l \leq k$.

Theorem 4.1. *Let A be a matrix in $\mathbb{M}_n(\mathbb{B}_k)$ with $k \geq 1$. Then A is idempotent if and only if all l^{th} constituents of A are idempotent in $\mathbb{M}_n(\mathbb{B}_1)$.*

Proof. Let A be idempotent in $\mathbb{M}_n(\mathbb{B}_k)$ which has the canonical form $\sum_{l=1}^{k} \sigma_l A_l$. Then we have

$$A^2 = \sigma_1 {A_1}^2 + \cdots + \sigma_k {A_k}^2 = \sigma_1 A_1 + \cdots + \sigma_k A_k = A.$$

Suppose that an l^{th} constituent, A_l, of A is not idempotent in $\mathbb{M}_n(\mathbb{B}_1)$ for some $1 \leq l \leq k$. Then there exist indices i and j such that $(i,j)^{\text{th}}$ entries of A_l and ${A_l}^2$ are different in $\mathbb{B}_1 = \{0,1\}$. If the $(i,j)^{\text{th}}$ entry of A_l is 1, then that of ${A_l}^2$ is 0. Thus the $(i,j)^{\text{th}}$ entry of A contains σ_l, but A^2 does not. Therefore we have $A^2 \neq A$, a contradiction. Similarly, if the $(i,j)^{\text{th}}$ entry of A_l is 0, we have $A^2 \neq A$, a contradiction. Therefore all l^{th} constituents of A are idempotent in $\mathbb{M}_n(\mathbb{B}_1)$. The converse follows from the definition of the canonical form of A. □

Lemma 4.2. *For any matrix $A \in \mathbb{M}_n(\mathbb{B}_k)$ with $k \geq 1$, A is invertible if and only if all its constituents are permutation matrices. In particular, if A is invertible, then $A^{-1} = A^t$.*

Proof. If A is invertible in $\mathbb{M}_n(\mathbb{B}_k)$, there exists a matrix B in $\mathbb{M}_n(\mathbb{B}_k)$ such that $AB = I_n$. The equality (4.1) implies that $(AB)_l = A_l B_l = I_n$ for all $l = 1, \ldots, k$. It follows that all constituents of A are permutation matrices.

Conversely, assume that each l^{th} constituent, A_l, of A is a permutation matrix. Then we have $A_l A_l^t = I_n$ for all $l = 1, \dots, k$ and hence

$$\begin{aligned} AA^t &= \Big(\sum_{l=1}^{k} \sigma_l A_l\Big)\Big(\sum_{l=1}^{k} \sigma_l A_l\Big)^t \\ &= \sum_{l=1}^{k} \sigma_l A_l A_l^t = \sum_{l=1}^{k} \sigma_l I_n = I_n. \end{aligned}$$

Therefore A is invertible. □

If T is a linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ with $k \geq 1$, for each $1 \leq l \leq k$ define its l^{th} *constituent operator*, T_l, by

$$T_l(B) = (T(B))_l$$

for all $B \in \mathbb{M}_n(\mathbb{B}_k)$ (see [8]). By the linearity of T, we have

$$T(A) = \sum_{l=1}^{k} \sigma_l T_l(A_l)$$

for any matrix $A \in \mathbb{M}_n(\mathbb{B}_k)$.

Lemma 4.3. *If T is an invertible linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ with $k \geq 1$, then each l^{th} constituent operator, T_l, is also invertible linear operator on $\mathbb{M}_n(\mathbb{B}_1)$.*

Proof. It follows from Lemma 2.4 and the definition of a constituent operator. □

For any fixed invertible matrix U in $\mathbb{M}_n(\mathbb{S})$, the operator $A \to UAU^t$ is called a *similarity operator*. We can easily show that any similarity operator on $\mathbb{M}_n(\mathbb{S})$ is an invertible linear operator and preserves idempotent matrices. Also, we can restate Theorem 3.1 as follows: the semigroup of linear operators that preserve idempotent matrices over $\mathbb{B}_1$ is generated by transpositions and similarity operators. But for a general Boolean algebra $\mathbb{B}_k$ with $k \geq 2$, the following example shows that there exists another invertible linear operator preserving idempotent matrices which is neither a transposition operator nor a similarity operator.

Example 4.4. Define an operator T on $\mathbb{M}_2(\mathbb{B}_2)$ by

$$T(X) = \sigma_1 X_1 + \sigma_2 X_2^t$$

for all $X = \sum_{l=1}^{2} \sigma_l X_l \in \mathbb{M}_2(\mathbb{B}_2)$. Then we can easily show that T is a linear operator on $\mathbb{M}_2(\mathbb{B}_2)$ which is neither a transposition nor a similarity.

It follows from the uniqueness of canonical form of a matrix that T is injective. Let $Y = \sum_{l=1}^{2} \sigma_l Y_l$ be any matrix in $\mathbb{M}_2(\mathbb{B}_2)$. Then we can take the matrix $X = \sigma_1 Y_1 + \sigma_2 Y_2^t \in \mathbb{M}_2(\mathbb{B}_2)$, so that $T(X) = Y$. This implies that T is surjective. Therefore T is invertible.

Let $A = \sum_{l=1}^{2} \sigma_l A_l$ be an idempotent matrix in $\mathbb{M}_2(\mathbb{B}_2)$. By Theorem 4.1, we have that $A_l^2 = A_l$ in $\mathbb{M}_2(\mathbb{B}_1)$ for $l = 1, 2$. It follows that $T(A)_1 = A_1$ and $T(A)_2 = A_2^t$ are idempotent matrices in $\mathbb{M}_2(\mathbb{B}_1)$. By Theorem 4.1, $T(A)$ is also idempotent. Therefore T preserves idempotent matrices.

Theorem 4.5. *Let T be a linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ with $k \geq 1$. Then T is an invertible linear operator preserving idempotent matrices if and only if there exists an invertible matrix U in $\mathbb{M}_n(\mathbb{B}_k)$ such that*

$$T(X) = U\Big(\sum_{l=1}^{k} \sigma_l Y_l\Big)U^t$$

for all $X \in \mathbb{M}_n(\mathbb{B}_k)$, where $Y_l = X_l$ or $Y_l = X_l^t$ for each $l = 1, \ldots, k$.

Proof. Assume that T is an invertible linear operator on $\mathbb{M}_n(\mathbb{B}_k)$ preserving idempotent matrices. By Lemma 4.3 and Theorem 4.1, we have that all its constituent operators, T_l, are invertible linear operators on $\mathbb{M}_n(\mathbb{B}_1)$ and preserve idempotent matrices for each $l = 1, \ldots, k$.

Let $X = \sum_{l=1}^{k} \sigma_l X_l$ be any matrix in $\mathbb{M}_n(\mathbb{B}_k)$. Then we have $T(X) = \sum_{l=1}^{k} \sigma_l T_l(X_l)$. By Theorem 3.1, each l^{th} constituent operator, T_l, has the form

$$T_l(X_l) = P_l X_l P_l^t \quad \text{or} \quad T_l(X_l) = P_l X_l^t P_l^t,$$

where each P_l is a permutation matrix for all $l = 1, \ldots, k$. Thus we have

$$T(X) = \sum_{l=1}^{k} \sigma_l P_l Y_l P_l^t,$$

where $Y_l = X_l$ or $Y_l = X_l^t$ for each $l = 1, \ldots, k$, equivalently

$$T(X) = \Big(\sum_{l=1}^{k} \sigma_l P_l\Big)\Big(\sum_{l=1}^{k} \sigma_l Y_l\Big)\Big(\sum_{l=1}^{k} \sigma_l P_l\Big)^t.$$

If we let $U = \sum_{l=1}^{k} \sigma_l P_l$, then U is invertible in $\mathbb{M}_n(\mathbb{B}_k)$ by Lemma 4.2, and hence the result is satisfied.

The converse is immediate. □

Thus we have obtained characterizations of invertible linear operators that preserve idempotent matrices over general Boolean algebras.

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