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COHOMOLOGY GROUPS OF RADICAL EXTENSIONS

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ABSTRACT. If k is a subfield of $\mathbb{Q}(\varepsilon_m)$ then the cohomology group $H^2(k(\varepsilon_n)/k)$ is isomorphic to $H^2(k(\varepsilon_{n'})/k)$ with $\gcd(m,n')=1$. This enables us to reduce a cyclotomic k-algebra over $k(\varepsilon_n)$ to the one over $k(\varepsilon_{n'})$. A radical extension in projective Schur algebra theory is regarded as an analog of cyclotomic extension in Schur algebra theory. We will study a reduction of cohomology group of radical extension and show that a Galois cohomology group of a radical extension is isomorphic to that of a certain subextension of radical extension. We then draw a cohomological characterization of radical group.

1. Introduction

Let k be a field, k^* be the multiplicative subgroup of k and $\mu(k)$ be the group of roots of unity in k. For a Galois extension L of k with Galois group $\mathcal{G}=\mathcal{G}(L/k)$ and for a 2-cocycle $\alpha \in Z^2(\mathcal{G},L^*)=Z^2(L/k,L^*)$, a crossed product algebra $(L/k,\alpha)=\sum_{\sigma\in\mathcal{G}} Lu_\sigma$ with $u_\sigma u_\tau=\alpha(\sigma,\tau)u_{\sigma\tau}$ and $u_\sigma x=\sigma(x)u_\sigma$ ($x\in L$, $\sigma,\tau\in\mathcal{G}$) is called a *cyclotomic algebra* if L is a cyclotomic extension of k and α has values in $\mu(L)$ (i.e., $\alpha\in Z^2(L/k,\mu(L))$). Let $H^2_\iota(L/k)$ be the image of a canonical homomorphism ι of $H^2(L/k,\mu(L))$ into $H^2(L/k,L^*)$ induced by the inclusion $\mu(L)\hookrightarrow L^*$. Since $\mu(L)$ is a subgroup of the torsion group of L^*, ι is injective ([7, p.91]), so we may identify $H^2(L/k,\mu(L))=H^2_\iota(L/k)\le H^2(L/k,L^*)$.

Suppose k is a subfield of the cyclotomic extension $\mathbb{Q}(\varepsilon_m)$ ($\mathbb{Q}$: the rational number field, ε_m: a primitive m-th root of unity). Let $L=k(\varepsilon_n)$. Due to [6], m and n are assumed to be either odd or divisible by 4. Then the Galois cohomology group $H^2_\iota(L/k)$ is isomorphic to $H^2_\iota(K/k)$ where $K=k(\varepsilon_{n'})$ is a subextension of L such that n' is a certain divisor of n which is prime to m [13, (7.12)]. Employing this result, Janusz's reduction theorem on cyclotomic algebras in [6] ([13, (7.9)]) follows that, a cyclotomic algebra $(L/k,\alpha)$ with $\alpha\in Z^2(L/k,\mu(L))$ can be reduced to the case $\gcd(m,n)=1$, i.e., to $(K/k,\beta)$ where β is a 2-cocycle in $Z^2(K/k,\mu(K))$ defined over the smaller group $\mathcal{G}(K/k)$.

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It is well known that every *Schur k-algebra* (a central simple k-algebra which is a homomorphic image of a group algebra kG for a finite group G) is similar to a cyclotomic k-algebra [13, (3.10)]. The idea of Schur algebra has been generalized to a projective Schur algebra in [9] by replacing group algebra by twisted group algebra; a *projective Schur k-algebra* is a central simple algebra that is a homomorphic image of a twisted group algebra $k^\alpha G$ for a finite group G and $\alpha \in Z^2(G, k^*)$.

The analogue of cyclotomic algebra in the theory of projective Schur algebra is the radical algebra ([1]). A *radical k-algebra* is a crossed product algebra $(L/k, \alpha)$ where $L = k(\Omega)$ is a finite Galois radical extension of k, Ω is a subgroup of L^* which is finite modulo k^* (i.e., Ω/k^* finite), and $\alpha \in Z^2(L/k, L^*)$ is represented by a 2-cocycle with values in Ω.

In this paper we study radical extensions and radical algebras, and obtain a corresponding result to Janusz's reduction theorem on radical extensions. We derive a reduction of Galois cohomology groups over radical extension fields, indeed prove that for a radical extension L of k, there exists a Galois radical extension K of k in L such that the cohomology group of $\mathcal{G}(L/k)$ is isomorphic to that of $\mathcal{G}(K/k)$ (in Theorem 10). We then verify a cohomological characterization of radical groups that a homomorphism of radical groups $R(K/k) \to R(L/k)$ commutes with certain homomorphisms of cohomology groups (in Theorem 14).

All notations are standard. $H^2(L/k, M)$ is the 2-dimensional cohomology group $H^2(G, M)$ where $G = \mathcal{G}(L/k)$ and M is a G-module, while $Z^2(L/k, M)$ is the 2-cocycle group. If $M = L^*$, we write $H^2(L/k, L^*) = H^2(L/k)$. Let ε_d $(d > 0)$ denote a primitive d-th root of unity, $a|b$ denote the division of b by a, while $a^t||b$ denote the highest power t of a to be $a^t|b$.

2. Preliminaries

Lemma 1. ([13, 7.10]) *Let H be a cyclic normal subgroup of G and M be a finite G-module. Let $N_H = \prod_{h\in H} h$. If $N_H(M) = M^H$ then $\inf\colon H^2(G/H, M^H) \to H^2(G, M)$ is an isomorphism, where inf is the inflation map from G/H to G and M^H is the subset of M consisting of elements fixed by H.*

For finite Galois extensions K and L of k with $K < L$, the norm $N_{L/K}$: $L \to K$, $x \mapsto (\prod_{\sigma\in\mathcal{G}(L/K)} \sigma)(x)$ is a homomorphism for $x \in L$. If $H = \mathcal{G}(L/K)$ is normal in $\mathcal{G} = \mathcal{G}(L/k)$, then $N_{L/K}$ corresponds to N_H in Lemma 1. In particular it is clear that $N_{\mathbb{Q}(\varepsilon_{p^{i+1}})/\mathbb{Q}(\varepsilon_{p^i})}\langle\varepsilon_{p^{i+1}}\rangle = \langle\varepsilon_{p^i}\rangle = \langle\varepsilon_{p^{i+1}}\rangle^H$ for a prime p and $i > 0$, thus the following theorem is due to Lemma 1.

Theorem 2. ([13, (7.12)]) *Let $k \le \mathbb{Q}(\varepsilon_m)$ and $L = \mathbb{Q}(\varepsilon_m, \varepsilon_n)$. Let $n' = 4^\delta p_1 \cdots p_s$ where p_i are distinct odd prime divisors of n not dividing m, and $\delta = 1$ if $4|n$, $4 \not| m$; $\delta = 0$ otherwise. Let $K = \mathbb{Q}(\varepsilon_m, \varepsilon_{n'})$. Then $H^2_\iota(K/k) \cong H^2_\iota(L/k)$.*

As a consequence of Theorem 2, Janusz proved the next theorem on algebras.

Theorem 3. ([6], [13, (7.9)]) *If $k \leq \mathbb{Q}(\varepsilon_m)$ then any cyclotomic algebra over k is similar to the cyclotomic algebra $(\mathbb{Q}(\varepsilon_m, \varepsilon_t)/k, \alpha)$ with $t = 4^{\delta} p_1 \cdots p_s$; $\delta = 0$ if $4|m$ and $\delta = 1$ otherwise, where all p_i are distinct odd primes not dividing m.*

Theorem 2 and 3 can be generalized to any cyclotomic extension field L containing finitely many roots of unity in [4].

Theorem 4. *Let $k \leq \mathbb{Q}(\varepsilon_m)$ and $L = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_1}, \ldots, \varepsilon_{n_w})$. Let*

$$n_1' = 4^{\delta_1} p_1 \cdots p_s \quad \text{with distinct odd primes} \quad p_j | n_1, \; p_j \nmid m \; (1 \leq j \leq s)$$

and $\delta_1 = 1$ if $4|n_1$, $4 \nmid m$; $\delta_1 = 0$ otherwise. And for $1 < i \leq w$, let

$$n_i' = 4^{\delta_i} p_{i1} \cdots p_{iu_i}$$

with $p_{ij}|n_i, p_{ij} \nmid m (1 \leq j \leq u_i)$, and $p_{ij} \nmid n_v (1 \leq v < i)$ where p_{ij} are distinct odd primes, and $\delta_i = 1$ if $4|n_i$, $4 \nmid m$, $4 \nmid n_v$ $(1 \leq v < i)$; $\delta_i = 0$ otherwise. Then $H_\iota^2(L/k) \cong H_\iota^2(K/k)$ where $K = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_1'}, \ldots, \varepsilon_{n_w'})$. Furthermore, a crossed product algebra $(L/k, \alpha)$ with $\alpha \in Z^2(L/k, \mu(L))$ is similar to $(K/k, \beta)$ where β is a 2-cocycle having values in $\mu(K)$.

Proof. We will prove this when $L = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_1}, \varepsilon_{n_2})$. Write $n_1' = 4^{\delta_1} p_1 \cdots p_s$ and $n_2' = 4^{\delta_2} q_1 \cdots q_u$ where p_i [resp. q_j] are distinct odd prime divisors of n_1 [resp. n_2] with $p_i \nmid m$, $q_j \nmid m$ and $q_j \nmid n_1$ for $1 \leq i \leq s$, $1 \leq j \leq u$. And $\delta_1 = 1$ if $4|n_1, 4 \nmid m$; $\delta_2 = 1$ if $4|n_2$, $4 \nmid m$, $4 \nmid n_1$; and $\delta_1 = \delta_2 = 0$ otherwise.

Let $K = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_1'}, \varepsilon_{n_2'})$ and $E = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_1'}, \varepsilon_{n_2})$. Then

$$k < K = \mathbb{Q}(\varepsilon_{mn_1'}, \varepsilon_{n_2'}) \leq E = \mathbb{Q}(\varepsilon_{mn_1'}, \varepsilon_{n_2})$$

for, $\gcd(m, n_i') = 1 (i = 1, 2)$. Since each q_j in the factorization of n_2' satisfies

$$q_j \nmid mn_1'; \quad \delta_2 = 1 \;\text{ if } 4|n_2 \text{ and } 4 \nmid mn_1'; \;\text{ and }\; \delta_2 = 0 \;\text{ otherwise,}$$

we are able to use Theorem 2 on K and E to get $H_\iota^2(K/k) \cong H_\iota^2(E/k)$.

Moreover with $l = \operatorname{lcm}(m, n_2)$, we have

$$E = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_2})(\varepsilon_{n_1'}) = \mathbb{Q}(\varepsilon_l, \varepsilon_{n_1'}) \leq L = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_2})(\varepsilon_{n_1}) = \mathbb{Q}(\varepsilon_l, \varepsilon_{n_1})$$

and each p_i in the factorization of n_1' satisfies

$$p_i \nmid l; \quad \delta_1 = 1 \;\text{ if } 4|n_1 \text{ and } 4 \nmid l; \;\text{ and }\; \delta_1 = 0 \;\text{ otherwise.}$$

Thus by applying Theorem 2 to L and E, we have $H_\iota^2(E/k) \cong H_\iota^2(L/k)$.

Now the isomorphism $(L/k, \alpha) \cong (K/k, \beta)$ of crossed product algebra with $\alpha \in Z^2(L/k, \mu(L))$, $\beta \in Z^2(K/k, \mu(K))$ follows immediately by Theorem 3. □

3. Norm on radical extension fields

A field L is a (finite) *radical extension* of k if there is a subgroup Ω of L^* such that $L = k(\Omega)$ and Ω/k^* is a (finite) torsion group. We may exhibit L by $k(\sqrt[n_1]{a_1}, \ldots, \sqrt[n_w]{a_w})$ with $a_i \in k^*$ and $n_i > 0$ $(1 \le i \le w)$. Moreover if $L = k(\Omega)$ is a Galois extension over k, then Ω is $\mathcal{G}(L/k)$-invariant, so $k(\Omega)$ contains enough roots of unity, i.e., $L = k(\{\varepsilon_{n_i}, \sqrt[n_i]{a_i} | 1 \le i \le w\})$ with primitive n_i-th roots of unity ε_{n_i}. Clearly Ω is not determined uniquely, while Ω/k^* is unique and finite.

The most interesting case of radical extension is $L = k(\lambda)$ with $\lambda \in L^*$ and $\lambda^n \in k$. If L/k is Galois radical then we may regard $L = k(\varepsilon_n, \lambda)$. In particular if $\varepsilon_n \in k$ then L/k is a cyclic extension of degree dividing n(see [8, Theorem 14.4]). The radical extension $L = k(\lambda)$ is said to be *irreducible* if degree $[L : k] = n$. Thus L/k is irreducible radical if and only if $L = k(\lambda)$ and $\lambda \in L^*$ is a root of an irreducible polynomial $X^n - a \in k[X]$, and, if and only if the order of λk^* in L^*/k^* is equal to the degree of λ over k.

Remark that, in case of a cyclotomic extension $L = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_1}, \ldots, \varepsilon_{n_w})$, the reduction in Theorem 4 would follow immediately by taking $n = \mathrm{lcm}(n_1, \ldots, n_w)$ and applying Theorem 2 to m and n. However when $L = k(\sqrt[n_1]{a_1}, \ldots, \sqrt[n_w]{a_w})$ is a Galois radical extension, to get a kind of reduction we need to know how to choose n_i' from n_i each other explicitly, not just taking n' from n =lcm of n_i's.

Lemma 5. *Let $L = k(\Omega)$ be a finite Galois radical extension of k. Then Ω/k^* is a finite G-module where $G = \mathcal{G}(L/k)$.*

Proof. The finite group Ω/k^* has a G-module structure by defining an action $\sigma\lambda = \sigma(\lambda')k^*$ where $\sigma \in G$ and $\lambda \in \Omega/k^*$ such that $\lambda = \lambda' k^*$ for $\lambda' \in \Omega$. □

We begin with a radical extension containing an n-th root of $1 \ne a \in k^*$.

Theorem 6. *Let $k = \mathbb{Q}(\varepsilon_m)$ and $K = k(\sqrt[n]{a})$ be a Galois radical extension of k, where m and n are positive odd integers. For a prime p dividing n, let $E = k(\sqrt[pn]{a})$ and $F = k(\varepsilon_{pn}, \sqrt[n]{a})$ be Galois radical extensions of k. Then*

(i) *$E = F$, or E/F is cyclic of degree p and $N_{E/F}\langle \varepsilon_{pn}, \sqrt[pn]{a} \rangle = \langle \varepsilon_n, \sqrt[n]{a} \rangle$.*

(ii) *Furthermore we assume $p^z || m$ and $p^b || n$.*

a) *If $0 \le z \le b$ then F/K is cyclic of degree p, and $N_{F/K}\langle \varepsilon_{pn}, \sqrt[n]{a} \rangle = \langle \varepsilon_n, \sqrt[n]{a^p} \rangle < \langle \varepsilon_n, \sqrt[n]{a} \rangle$.*

b) *If $b < z$ then $E = K$, or $F = K$ so $N_{E/K}\langle \varepsilon_{pn}, \sqrt[pn]{a} \rangle = \langle \varepsilon_n, \sqrt[n]{a} \rangle$.*

Proof. The Galois radical extensions K and E form

$$k < K = k(\varepsilon_n, \sqrt[n]{a}) < F = k(\varepsilon_{pn}, \sqrt[n]{a}) < E = k(\varepsilon_{pn}, \sqrt[pn]{a}).$$

Since $E = k(\sqrt[pn]{a}) = F(\sqrt[p]{\lambda})$ where $\lambda = \sqrt[n]{a} \in F$ and $\sqrt[p]{\lambda}$ is a root of $X^p - \sqrt[n]{a} \in F[X]$, and since $\varepsilon_{pn} \in F$, E/F is a cyclic extension of degree dividing p [8, Theorem 14.4]. Hence $E = F$ or $[E : F] = p$.

If $[E : F] = p$, any $\sigma \in \mathcal{G}(E/F)$ maps $\sqrt[pn]{a}$ to a zero of $X^{pn} - a$, say $\sigma(\sqrt[pn]{a}) = \varepsilon_{pn}^{l} \sqrt[pn]{a}$ for $0 \leq l < pn$. But $\sqrt[pn]{a}^p$ is a zero of $X^n - a$, so $\sqrt[pn]{a}^p \in F^*$, $\sqrt[pn]{a}^p = \sigma(\sqrt[pn]{a}^p) = \varepsilon_{pn}^{pl} \sqrt[pn]{a}^p$ and $\varepsilon_{pn}^{pl} = 1$. Thus $l = tn$ for some $t = 0, \ldots, p-1$. Moreover since each $\sigma \in \mathcal{G}(E/F)$ leaves ε_{pn} fixed, we have

$$\sigma(\sqrt[pn]{a})\varepsilon_{pn}^{tn} \sqrt[pn]{a}, \sigma^2(\sqrt[pn]{a}) = \varepsilon_{pn}^{2tn} \sqrt[pn]{a}, \ldots, \sigma^{p-1}(\sqrt[pn]{a}) = \varepsilon_{pn}^{(p-1)tn} \sqrt[pn]{a},$$

hence it follows that

$$N_{E/F}(\sqrt[pn]{a}) = \prod_{\sigma \in \mathcal{G}(E/F)} \sigma(\sqrt[pn]{a}) = \varepsilon_{pn}^{tn(1+\cdots+(p-1))} \sqrt[pn]{a}^p = \sqrt[pn]{a}^p.$$

Thus $N_{E/F}\langle \sqrt[pn]{a}\rangle = \langle \sqrt[pn]{a}^p\rangle = \langle \sqrt[n]{a}\rangle$ and $N_{E/F}\langle \varepsilon_{pn}\rangle = \langle \varepsilon_{pn}^p\rangle = \langle \varepsilon_n\rangle$.

Suppose that $m = p^z m'$ and $n = p^b n'$ with $p \nmid m'n'$. If $0 \leq z \leq b$ then

$$\operatorname{lcm}(m, pn) = \frac{p^z m' p^{b+1} n'}{p^z \gcd(m', p^{b+1-z} n')} = \frac{m' p^{b+1} n'}{\gcd(m', n')} = p^{b+1} \operatorname{lcm}(m', n')$$

and similarly $\operatorname{lcm}(m, n) = p^b \operatorname{lcm}(m', n')$. Set $\operatorname{lcm}(m', n') = l$. Then $\mathbb{Q}(\varepsilon_m, \varepsilon_{pn}) = \mathbb{Q}(\varepsilon_{p^{b+1}l})$ and $\mathbb{Q}(\varepsilon_m, \varepsilon_n) = \mathbb{Q}(\varepsilon_{p^b l})$ because $p \nmid l$, hence

$$[\mathbb{Q}(\varepsilon_m, \varepsilon_{pn}) : \mathbb{Q}(\varepsilon_m, \varepsilon_n)] = \frac{\phi(p^{b+1} l)}{\phi(p^b l)} = \frac{\phi(p^{b+1})}{\phi(p^b)} = p$$

(where ϕ is the Euler phi function), which shows that $[F : K] = p$.

Now each $\tau \in \mathcal{G}(F/K)$ maps $\varepsilon_{pn} \in F$ to ε_{pn}^l for some $0 \leq l < pn$. Because $\varepsilon_{pn}^p \in K^*$, we have $\varepsilon_{pn}^p = \tau(\varepsilon_{pn}^p) = \varepsilon_{pn}^{pl}$, hence $p(l-1) \equiv 0 \pmod{pn}$ and $l = tn + 1$ with $0 \leq t < p$. Indeed, the cyclic group $\mathcal{G}(F/K)$ of order p consists of automorphisms τ_t such that $\tau_t(\varepsilon_{pn}) = \varepsilon_{pn}^{tn+1}$ for $0 \leq t < p$. Thus we have

$$N_{F/K}(\varepsilon_{pn}) = \varepsilon_{pn}^{\sum_{t=0}^{p-1}(tn+1)} = \varepsilon_{pn}^{p+pn\frac{(p-1)}{2}} = \varepsilon_{pn}^p,$$

so $N_{F/K}\langle \varepsilon_{pn}\rangle = \langle \varepsilon_{pn}^p\rangle \leq \langle \varepsilon_n\rangle$. Comparing the orders, we get $N_{F/K}\langle \varepsilon_{pn}\rangle = \langle \varepsilon_n\rangle$. And $N_{F/K}\langle \sqrt[n]{a}\rangle = \langle \prod_{\tau \in \mathcal{G}(F/K)} \tau(\sqrt[n]{a})\rangle = \langle \sqrt[n]{a}^p\rangle < \langle \varepsilon_n, \sqrt[n]{a}\rangle$, this is (ii-a).

In case of $m = p^z m'$, $n = p^b n'$ with $0 < b < z$, $\gcd(m, pn) = p\gcd(m, n)$, $\operatorname{lcm}(m, pn) = \operatorname{lcm}(m, n)$, so $\mathbb{Q}(\varepsilon_m, \varepsilon_{pn}) = \mathbb{Q}(\varepsilon_m, \varepsilon_n)$ and $[F : K] = 1$. Hence $E = K$, or $[E : K] = p$ and $N_{E/K}\langle \varepsilon_{pn}, \sqrt[pn]{a}\rangle = N_{E/F}\langle \varepsilon_{pn}, \sqrt[pn]{a}\rangle = \langle \varepsilon_n, \sqrt[n]{a}\rangle$. □

We observe that the assumption in (ii-b), i.e., $m = p^z m'$, $n = p^b n'$ with $b < z$ implies that $k = \mathbb{Q}(\varepsilon_m)$ already contains enough p-th roots of unity.

Corollary 7. *Assume the same context as in* (ii-b) *Theorem 6 for radical extensions* $K = k(\sqrt[n]{a}) = k(\Omega_K)$ *and* $E = k(\sqrt[pn]{a}) = k(\Omega_E)$ *with finite* Ω_K/k^* *and* Ω_E/k^*. *If* $H = \mathcal{G}(E/K) \neq 1$ *then* $N_H(\Omega_E/k^*) = \Omega_K/k^*$.

Proof. $E = k(\varepsilon_{pn}, \sqrt[pn]{a})$ is a cyclic extension of $K = k(\varepsilon_n, \sqrt[n]{a})$ of degree p with cyclic Galois group H. The mapping $N_H = \prod_{\sigma \in H} \sigma$ in Lemma 1 determines $N_H\langle \varepsilon_{pn}, \sqrt[pn]{a}\rangle = \langle \varepsilon_n, \sqrt[n]{a}\rangle$ by Theorem 6. By the action in Lemma 5, we have

$$N_H(\Omega_E/k^*) = N_H\langle \varepsilon_{pn}k^*,\ \sqrt[pn]{a}k^*\rangle = \langle \varepsilon_n k^*,\ \sqrt[n]{a}k^*\rangle = \Omega_K/k^*.$$

□

We shall denote $N_H(\Omega_E/k^*)$ by the same notation $N_{E/K}(\Omega_E/k^*)$ which is the norm map. As a generalization of Theorem 6, we have the following theorem.

Theorem 8. *Let $k = \mathbb{Q}(\varepsilon_m)$ and $L = k(\sqrt[n]{a}) = k(\Omega)$ be a Galois radical extension of k. Assume a prime p with $p^z||m$ and $p^b||n$. Let $F = k(\sqrt[n/p]{a}) = k(\Omega_F)$ be a Galois radical extension with $L \neq F$. If $0 < b \leq z$ then $N_{L/F}(\Omega/k^*) = \Omega_F/k^*$.*

Proof. Let $E = k(\varepsilon_n,\ \sqrt[n/p]{a})$ be a Galois radical extension. Then

$$k < F = k(\varepsilon_{n/p},\ \sqrt[n/p]{a}) < E = k(\varepsilon_n,\ \sqrt[n/p]{a}) < L = k(\varepsilon_n,\ \sqrt[n]{a}),$$

and $L = E(\sqrt[p]{\lambda})$ where $\lambda = \sqrt[n/p]{a} \in E$ and $\sqrt[p]{\lambda}$ is a root of a polynomial $X^p - \sqrt[n/p]{a}$ in $E[X]$. Since $\varepsilon_p \in E$, L/E is a cyclic radical extension of degree dividing p. So $L = E$ or $[L : E] = p$. Because $b \leq z$, k contains enough p-th roots of unity so $F = E$. But since $L \neq F$, we have $[L : F] = p$ and

$$N_{L/F}\langle \varepsilon_n,\ \sqrt[n]{a}\rangle = \langle \varepsilon_n^p,\ \sqrt[n]{a}^p\rangle = \langle \varepsilon_{n/p},\ \sqrt[n/p]{a}\rangle.$$

Similar to Corollary 7, we consequently have $N_{L/F}(\Omega/k^*) = \Omega_F/k^*$. □

4. Cohomology group on radical extension fields

We shall discuss cohomology groups over Galois radical extension fields, and have a reduction of cohomology group that is an analog of Theorem 2. Let H be a normal subgroup of G and M be a G-module. The inflation-restriction sequence on cohomology group

$$1 \to H^r(G/H, M^H) \overset{\text{inf}}{\to} H^r(G, M) \overset{\text{res}}{\to} H^r(H, M) \tag{1}$$

is exact if $r = 1$. When $r > 1$, the sequence is exact if $H^i(H, M) = 1$ for all $1 \leq i \leq r-1$ (refer to [12, (3.4.2), (3.2.3)]).

Theorem 9. *Let $n = p^b n_0$ and $m = p^z m_0$ with an odd prime $p \nmid n_0 m_0$. Let $k = \mathbb{Q}(\varepsilon_m)$, and $L = k(\sqrt[n]{a}) = k(\Omega)$ and $L_0 = k(\sqrt[n_0]{a}) = k(\Omega_0)$ be Galois radical extensions of k with $L \neq L_0$. If $0 < b \leq z$ then the inflation map is an isomorphism on cohomology groups*

$$H^2(L_0/k, \Omega_0/k^*) \overset{inf}{\cong} H^2(L/k, \Omega_0/k^*).$$

Proof. Obviously $L = k(\varepsilon_n,\ \sqrt[n]{a})$ and $L_0 = k(\varepsilon_{n_0},\ \sqrt[n_0]{a})$. Since $b \leq z$, ε_{p^b} is contained in $k < L_0$ so $\varepsilon_n \in L_0$. Thus $L = L_0(\sqrt[p^b]{\lambda})$ where $\lambda = \sqrt[n_0]{a}$ and $\sqrt[p^b]{\lambda}$ is a root of $X^{p^b} - \sqrt[n_0]{a}$ in $L_0[X]$. So L/L_0 is cyclic of degree p^c for some $c \leq b$.

Due to Lemma 5, Ω/k^* is a $\mathcal{G}(L/k)$-module with module action that, for any $\tau \in \mathcal{G}(L/k)$ and $\sqrt[n]{a}k^* \in \Omega/k^*$, $\tau(\sqrt[n]{a}k^*) = \tau(\sqrt[n]{a})k^* = \varepsilon_n^l\,\sqrt[n]{a}k^* \in \Omega/k^*$

for some $l > 0$. Similarly Ω_0/k^* is a $\mathcal{G}(L_0/k)$-module. Moreover it is easy to see that Ω_0/k^* is also a $\mathcal{G}(L/k)$-module by regarding $\sqrt[n_0]{a}$ as $(\sqrt[n]{a})^{p^b}$.

Write $H = \mathcal{G}(L/L_0)$. From the sequence of groups

$$1 \to \mathcal{G}(L/L_0) \to \mathcal{G}(L/k) \to \mathcal{G}(L_0/k) \to 1,$$

we consider the inflation-restriction sequence

$$H^2(L_0/k, (\Omega_0/k^*)^H) \overset{\text{inf}}{\to} H^2(L/k, \Omega_0/k^*) \overset{\text{res}}{\to} H^2(L/L_0, \Omega_0/k^*).$$

Since H is cyclic, it follows from [12, (1.5.6)] and [12, (3.2.1)] that

$$H^2(L/L_0, \Omega_0/k^*) = H^0(H, \Omega_0/k^*) = \frac{(\Omega_0/k^*)^H}{N_{L/L_0}(\Omega_0/k^*)}.$$

Every $\sigma \in H$ leaves all elements in Ω_0/k^* fixed, so $(\Omega_0/k^*)^H = \Omega_0/k^*$. Moreover due to Theorem 8, we compute the norm N_{L/L_0} directly to get

$$N_{L/L_0}(\Omega_0/k^*) = \langle \varepsilon_{n_0}^{p^c} k^*, \ \sqrt[n_0]{a}^{p^c} k^* \rangle = \langle \varepsilon_{n_0} k^*, \ \sqrt[n_0]{a} k^* \rangle = \Omega_0/k^*$$

for $\gcd(p^c, n_0) = 1$. Hence $H^2(L/L_0, \Omega_0/k^*) = 1$.

Again since H is finite cyclic and Ω_0/k^* is a finite H-module, we use the Herbrand's quotient of Ω_0/k^* (refer to [3, (23.2)]) that

$$1 = h_{2/1}(\Omega_0/k^*) = \frac{|H^2(L/L_0, \Omega_0/k^*)|}{|H^1(L/L_0, \Omega_0/k^*)|}$$

hence it follows that $H^1(L/L_0, \Omega_0/k^*) = H^2(L/L_0, \Omega_0/k^*) = 1$. Therefore we can conclude from (1) that the sequence

$$1 \to H^2(L_0/k, (\Omega_0/k^*)^H) \overset{\text{inf}}{\to} H^2(L/k, \Omega_0/k^*) \overset{\text{res}}{\to} H^2(L/L_0, \Omega_0/k^*) = 1$$

is exact, so have an isomorphism $H^2(L_0/k, \Omega_0/k^*) \cong H^2(L/k, \Omega_0/k^*)$. $\square$

For a given finite radical extension L over k, we observed in Theorem 9 that the cohomology group over $\mathcal{G}(L/k)$ can be decreased down to that over $\mathcal{G}(L_0/k)$, where L_0 is the Galois radical extension of k smaller than L by 'one' prime factor power p^b. We can strengthen this observation with the following theorem which will go to reduction of Galois cohomology groups.

Theorem 10. *Let $k = \mathbb{Q}(\varepsilon_m)$ and $L = k(\sqrt[n]{a})$ be the same fields as in Theorem 9. Assume $n = p_1^{b_1} \cdots p_u^{b_u}$ with distinct primes p_i and $b_i > 0$. For each p_i $(1 \le i \le u)$, write $m = p_i^{z_i} m_i$ (with $p_i \nmid m_i$ and $z_i \ge 0$). Suppose $b_i \le z_i$ for some $1 \le i \le u$, and after appropriate renumbering, we assume that $b_i \le z_i$ for all $s+1 \le i \le u$. Let $n_0 = p_1^{b_1} \cdots p_s^{b_s}$, and let the Galois extensions be*

$$L_0 = k(\sqrt[n_0]{a}) = k(\Omega_0),$$

$$L_j = k(\sqrt[n_0 p_{s+1}^{b_{s+1}} \cdots p_{s+j}^{b_{s+j}}]{a}) = k(\Omega_j) \quad \text{for } 1 \le j \le u - s.$$

Then $H^2(L_j/k, \Omega_{j-1}/k^*) \cong H^2(L_{j-1}/k, \Omega_{j-1}/k^*)$ for all $1 \le j \le u - s$, so $H^2(L/k, \Omega_0/k^*)$ and $H^2(L_0/k, \Omega_0/k^*)$ are isomorphic.

Proof. Since L_0, L_j $(1 \le j \le u-s)$ are all Galois radical extensions of k, and since $n = p_1^{b_1} \cdots p_s^{b_s} \cdot p_{s+1}^{b_{s+1}} \cdots p_u^{b_u} = n_0 \cdot p_{s+1}^{b_{s+1}} \cdots p_u^{b_u}$, we have $L_{u-s} = L$ and

$$\begin{aligned} k &< L_0 = k(\varepsilon_{n_0}, \sqrt[n_0]{a}) < L_1 = k(\varepsilon_{n_0 p_{s+1}^{b_{s+1}}}, \sqrt[n_0 p_{s+1}^{b_{s+1}}]{a}) < \cdots \\ &< L_j = k(\varepsilon_{n_0 p_{s+1}^{b_{s+1}} \cdots p_{s+j}^{b_{s+j}}}, \sqrt[n_0 p_{s+1}^{b_{s+1}} \cdots p_{s+j}^{b_{s+j}}]{a}) < \cdots < L_{u-s} = L. \end{aligned}$$

Since $m = p_i^{z_i} m_i$ and $0 < b_i \le z_i$ for $s+1 \le i \le u$, each $\varepsilon_{p_i^{b_i}} \in k$. Together with $\varepsilon_{n_0} \in L_0$, $\varepsilon_{n_0 p_{s+1}^{b_{s+1}} \cdots p_{s+j}^{b_{s+j}}}$ belong to L_0 for all $1 \le j \le u-s$. Hence

$$L_j = L_{j-1}(\lambda_j), \text{ where } \lambda_j \in L_j \text{ and } \lambda_j^{p_{s+j}^{b_{s+j}}} \in L_{j-1} \ (1 \le j \le u-s),$$

and L_j/L_{j-1} is cyclic of degree $p_{s+j}^{c_j}$ for some $c_j \le b_{s+j}$. Thus we have the isomorphism on cohomology groups

$$H^2(L_1/k,\ \Omega_0/k^*) \cong H^2(L_0/k,\ \Omega_0/k^*)$$

due to Theorem 9, and proceeding inductively we get

(2) $\quad H^2(L_j/k,\ \Omega_{j-1}/k^*) \cong H^2(L_{j-1}/k,\ \Omega_{j-1}/k^*)$ for each $1 \le j \le u-s$.

Let $H = \mathcal{G}(L_2/L_0)$, and consider the inflation-restriction sequence

(3) $$H^2(L_0/k, (\Omega_0/k^*)^H) \overset{\inf}{\to} H^2(L_2/k, \Omega_0/k^*) \overset{\text{res}}{\to} H^2(L_2/L_0, \Omega_0/k^*).$$

Since $L_2 = L_0(\xi)$ with $\xi^{p_{s+1}^{b_{s+1}} p_{s+2}^{b_{s+2}}} \in L_0$, L_2/L_0 is a cyclic radical extension of degree dividing $p_{s+1}^{b_{s+1}} p_{s+2}^{b_{s+2}}$, and in fact $[L_2 : L_0] = p_{s+1}^{c_1} p_{s+2}^{c_2}$. Thus

$$H = \mathcal{G}(L_2/L_0) \cong Z_{p_{s+1}^{c_1} p_{s+2}^{c_2}} \cong Z_{p_{s+1}^{c_1}} \times Z_{p_{s+2}^{c_2}} \cong \mathcal{G}(L_2/L_1) \times \mathcal{G}(L_1/L_0),$$

and by [7, (2.3.14)], we have

$$H^2(L_2/L_0, \Omega_0/k^*) \cong H^2(L_2/L_1, \Omega_0/k^*) \times H^2(L_1/L_0, \Omega_0/k^*).$$

But since $p_i \nmid n_0$ for $s+1 \le i \le u$, we obtain

$$N_{L_1/L_0}(\Omega_0/k^*) = \langle \varepsilon_{n_0} k^*,\ \sqrt[n_0]{a} k^* \rangle^{p_{s+1}^{c_1}} = \Omega_0/k^* = (\Omega_0/k^*)^{L_1/L_0}$$

and similarly, $N_{L_2/L_1}(\Omega_0/k^*) = (\Omega_0/k^*)^{p_{s+2}^{c_2}} = \Omega_0/k^* = (\Omega_0/k^*)^{L_2/L_1}$.

By the proof of Theorem 9, both $H^2(L_1/L_0, \Omega_0/k^*)$ and $H^2(L_2/L_1,\ \Omega_0/k^*)$ are trivial groups, so that $H^2(L_2/L_0, \Omega_0/k^*) = 1$. Then again the Herbrand quotient of Ω_0/k^* is equal to 1, i.e.,

$$1 = |H^2(L_2/L_0, \Omega_0/k^*)| / |H^1(L_2/L_0, \Omega_0/k^*)|.$$

So $H^1(L_2/L_0, \Omega_0/k^*)$ is trivial, and the inflation-restriction sequence (3) is exact. We thus obtain the isomorphism on cohomology groups

$$H^2(L_0/k, \Omega_0/k^*) \overset{\inf}{\cong} H^2(L_2/k, \Omega_0/k^*).$$

And together with (2), it follows that

$$H^2(L_2/k, \Omega_0/k^*) \cong H^2(L_1/k, \Omega_0/k^*) \cong H^2(L_0/k, \Omega_0/k^*).$$

Applying this process to the cyclic group $H = \mathcal{G}(L_3/L_0)$ of order $p_{s+1}^{c_1}p_{s+2}^{c_2}$ $p_{s+3}^{c_3}$ and to the sequence

$$H^2(L_0/k, (\Omega_0/k^*)^H) \overset{\inf}{\to} H^2(L_3/k, \Omega_0/k^*) \overset{\mathrm{res}}{\to} H^2(L_3/L_0, \Omega_0/k^*),$$

we also get $H^2(L_3/L_0, \Omega_0/k^*) = 1$, for $\mathcal{G}(L_3/L_0) \cong \mathcal{G}(L_3/L_2) \times \mathcal{G}(L_2/L_0)$ and $H^2(L_3/L_2, \Omega_0/k^*) = 1 = H^2(L_2/L_0, \Omega_0/k^*)$.

The exactness of the inflation-restriction sequence guarantees the isomorphism $H^2(L_3/k, \Omega_0/k^*) \cong H^2(L_0/k, \Omega_0/k^*)$. Continuing, we eventually get

$$H^2(L/k, \Omega_0/k^*) = H^2(L_{u-s}/k, \Omega_0/k^*) \cong H^2(L_0/k, \Omega_0/k^*).$$

□

Remark 1. In Theorem 10, we assume $a = 1$, i.e., $\sqrt[n]{a} = \varepsilon_n$ for $n = p_1^{b_1} \cdots p_u^{b_u}$ (odd primes). Suppose $m = p_i^{z_i} m_i$ with $z_i < b_i$ for $1 \le i \le s$, and $b_i \le z_i$ for $s + 1 \le i \le u$. Let $n_0 = p_1^{b_1} \cdots p_s^{b_s}$, $L = \mathbb{Q}(\varepsilon_m, \varepsilon_n)$ and $L_0 = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_0})$. Then $L = L_0$ and $H^2(L/k, \Omega_L/k^*) \cong H^2(L_0/k, \Omega_{L_0}/k^*)$ (this is Theorem 10). Moreover, by rearrangement if necessary, we assume $z_1 = \cdots = z_t = 0$ for $t \le s$, i.e., $p_i \nmid m$ for $1 \le i \le t$. By letting $n_0' = p_1 \cdots p_t$, we can further reduce the cohomology group by Janusz theorem that

$$H^2(L/k, \mu(L)) = H^2(L_0/k, \mu(L_0)) \cong H^2(k(\varepsilon_{n_0'})/k, \mu(k(\varepsilon_{n_0'}))).$$

Remark 2. It is natural to ask whether the Remark 1 is true for $a \ne 1$ with the same n and n_0', i.e., is $H^2(k(\sqrt[n]{a})/k, \langle \sqrt[n]{a} \rangle/k^*) \cong H^2(k(\sqrt[n_0']{a})/k, \langle \sqrt[n_0']{a} \rangle/k^*)$? The following proposition provides a partial answer.

Proposition 11. *Let $L = k(\sqrt[n]{a})$ be a Galois radical extension of $k = \mathbb{Q}(\varepsilon_m)$. Let $n = p_1^{b_1} \cdots p_u^{b_u}$ ($b_i > 0$, odd primes p_i) and $m = p_i^{z_i} m_i$ ($z_i \ge 0$, $p_i \nmid m_i$) for all i. By rearrangement, assume $z_i \le b_i$ for $1 \le i \le s (\le u)$, $z_i > b_i$ for $s + 1 < i \le u$, and moreover $z_j = 0$ for $1 \le j \le t (\le s)$. Set $n_0 = p_1^{b_1} \cdots p_s^{b_s}$ and $n_0' = p_1 \cdots p_t$. Let $K = k(\sqrt[n_0']{a}) = k(\Omega)$, and $F_v = k(\sqrt[n_0/p_v^{b_v - z_v}]{a})$ and $B_v = k(\varepsilon_{n_0/p_v^{b_v - z_v}}, \sqrt[n_0']{a})$ be Galois radical extensions of k for $t + 1 \le v \le s$. Then $H^2(L/k, \Omega/k^*) \cong H^2(F_v/k, \Omega/k^*)$, $H^2(K/k, \Omega/k^*) \cong H^2(B_v/k, \Omega/k^*)$, but $H^2(F_v/k, \Omega/k^*) \not\cong H^2(B_v/k, \Omega/k^*)$.*

Proof. Let $L_0 = k(\sqrt[n_0]{a}) = k(\Omega_{L_0})$ be a Galois radical extension of k. With the integers $n = p_1^{b_1} \cdots p_u^{b_u}$, $n_0 = p_1^{b_1} \cdots p_s^{b_s}$ and $n_0' = p_1 \cdots p_t$ for $t \le s \le u$, it is clear that $k < K < L_0 < L$ and $H^2(L/k, \Omega_{L_0}/k^*) \cong H^2(L_0/k, \Omega_{L_0}/k^*)$ due to Theorem 10. Hence it is follows that

$$H^2(L/k, \Omega/k^*) \cong H^2(L_0/k, \Omega/k^*).$$

Let p_v be one of $p_{t+1}, \ldots, p_s$. Since $m = p_v^{z_v} m_v$ with $0 < z_v \le b_v$, the Galois radical extensions of k are

$$K = k(\varepsilon_{n_0'}, \sqrt[n_0']{a}) < F_v = k(\varepsilon_{n_0/p_v^{b_v - z_v}}, \sqrt[n_0/p_v^{b_v - z_v}]{a}) < L_0 = k(\varepsilon_{n_0}, \sqrt[n_0']{a}).$$

Now let $\mathcal{K}$, $\mathcal{F}_v$ and $\mathcal{L}_0$ denote the cyclotomic extensions

$$\mathcal{K} = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_0'}) < \mathcal{F}_v = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_0/p_v^{b_v-z_v}}) < \mathcal{L}_0 = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_0}).$$

Owing to $p_v^{z_v} \| \frac{n_0}{p_v^{b_v-z_v}}$ and $\varepsilon_{p_v^{z_v}} \in \mathcal{K}$, we have $p_v \nmid [\mathcal{F}_v : \mathcal{K}]$ and $H^2(\mathcal{F}_v/\mathcal{K}, \mu(\mathcal{F}_v)) = H^2(\mathcal{L}_0/\mathcal{F}_v, \mu(\mathcal{L}_0)) = 1$. Thus due to Janusz theorem we have

$$H^2(\mathcal{K}/k, \mu(\mathcal{K})) \cong H^2(\mathcal{F}_v/k, \mu(\mathcal{F}_v)) \cong H^2(\mathcal{L}_0/k, \mu(\mathcal{L}_0)).$$

On the other hand, from the tower of fields

$$k < K = \mathcal{K}(\sqrt[n_0']{a}) < F_v = \mathcal{F}_v(\sqrt[n_0/p_v^{b_v-z_v}]{a}) < L_0 = \mathcal{L}_0(\sqrt[n_0]{a}),$$

consider a field $A_v = k(\varepsilon_{n_0}, \sqrt[n_0/p_v^{b_v-z_v}]{a})$. We now observe the followings.

(i) $K < B_v < F_v < A_v < L_0$

(ii) $L_0 = A_v(\sqrt[p_v^{z_v}]{\lambda})$ where $\lambda = \sqrt[n_0/p_b^{b_v-z_v}]{a}$ and $\sqrt[p_v^{z_v}]{\lambda}$ is a root of $X^{p_v^{z_v}} - \lambda \in A_v[X]$. Since $\varepsilon_{p_v^{z_v}} \in A_v$, L_0/A_v is a cyclic extension of order $p_v^{w_v}$ with $w_v \leq z_v$.

(iii) Similar to (ii), it can be seen that $F_v = B_v(\sqrt[n_0/n_0' p_v^{b_v-z_v}]{\theta})$ where $\theta = \sqrt[n_0']{a} \in B_v$ and $\sqrt[n_0/n_0' p_v^{b_v-z_v}]{\theta}$ is a root of $X^{n_0/n_0' p_v^{b_v-z_v}} - \theta \in B_v[X]$. Since $\varepsilon_{n_0/n_0' p_v^{b_v-z_v}}$ belongs to $B_v = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_0/p_v^{b_v-z_v}}, \sqrt[n_0']{a})$, F_v/B_v is cyclic of degree dividing $p_1^{b_1-1} \cdots p_t^{b_t-1} \cdot p_{t+1}^{b_{t+1}} \cdots p_v^{z_v} \cdots p_s^{b_s}$.

(iv) $A_v = F_v(\varepsilon_{n_0}) = F_v(\varepsilon_{p_v^{b_v-z_v}})$, so $\mathcal{G}(A_v/F_v) \cong \mathcal{G}(\mathcal{L}_0/\mathcal{F}_v)$.

(v) $B_v = K(\varepsilon_{n_0/p_v^{b_v-z_v}})$, so $\mathcal{G}(B_v/K) \cong \mathcal{G}(\mathcal{F}_v/\mathcal{K})$.

$$\begin{array}{lcl}
\mathcal{L}_0 = k(\varepsilon_{n_0}) & \text{———} & L_0 = k(\varepsilon_{n_0}, \sqrt[n_0']{a}) = \mathcal{L}_0(\sqrt[n_0]{a}) \\
\mid & & \mid \\
 & & A_v = k(\varepsilon_{n_0}, \sqrt[n_0/p_v^{b_v-z_v}]{a}) \\
 & & \mid \\
\mathcal{F}_v = k(\varepsilon_{n_0/p_v^{b_v-z_v}}) & \text{———} & F_v = k(\sqrt[n_0/p_v^{b_v-z_v}]{a}) = \mathcal{F}_v(\sqrt[n_0/p_b^{b_v-z_v}]{a}) \\
\mid & & \mid \\
 & & B_v = k(\varepsilon_{n_0/p_v^{b_v-z_v}}, \sqrt[n_0']{a}) \\
 & & \mid \\
\mathcal{K} = k(\varepsilon_{n_0'}) & \text{———} & K = k(\sqrt[n_0']{a}) = k(\Omega) = \mathcal{K}(\sqrt[n_0']{a})
\end{array}$$

Thus from (ii), L_0/A_v is cyclic of order $p_v^{w_v}$ with $w_v \leq z_v$, so

$$N_{L_0/A_v}(\Omega/k^*) = N_{L_0/A_v}\langle \varepsilon_m, \varepsilon_{n_0'}, \sqrt[n_0']{a}\rangle k^* = \langle \varepsilon_m k^*, \varepsilon_{n_0'}^{p_v^{w_v}} k^*, \sqrt[n_0']{a}^{p_v^{w_v}} k^* \rangle,$$

and this is equal to Ω/k^* because $\gcd(n_0', p_v) = 1$ for $t+1 \leq v \leq s$. Hence

$$\begin{aligned}
H^2(L_0/A_v, \Omega/k^*) &= \frac{(\Omega/k^*)^{\mathcal{G}(L_0/A_v)}}{N_{L_0/A_v}(\Omega/k^*)} = 1 \\
&= H^0(L_0/A_v, \Omega/k^*) = H^1(L_0/A_v, \Omega/k^*).
\end{aligned}$$

So the exact sequence

$$H^2(A_v/k, \Omega/k^*) \to H^2(L_0/k, \Omega/k^*) \to H^2(L_0/A_v, \Omega/k^*)$$

yields the isomorphism $H^2(A_v/k, \Omega/k^*) \cong H^2(L_0/k, \Omega/k^*)$.

From (iv), $A_v = F_v(\varepsilon_{p_v^{b_v - z_v}})$ and $\mathcal{G}(A_v/F_v) \cong \mathcal{G}(\mathcal{L}_0/\mathcal{F}_v)$ cyclic, so the invariant set Ω/k^* by $\mathcal{G}(A_v/F_v)$ corresponds to $\mu(\mathcal{K})/k^*$ by $\mathcal{G}(\mathcal{L}_0/\mathcal{F}_v)$. Thus

$$H^2(A_v/F_v, \Omega/k^*) \cong H^2(\mathcal{L}_0/\mathcal{F}_v, \mu(\mathcal{K})/k^*).$$

Since $H^2(\mathcal{L}_0/\mathcal{F}_v, \mu(\mathcal{K})) \hookrightarrow H^2(\mathcal{L}_0/\mathcal{F}_v, \mu(\mathcal{L}_0)) = 1$ by Janusz theorem, we have $H^2(\mathcal{L}_0/\mathcal{F}_v, \mu(\mathcal{K})) = 1$. Moreover since $H^2(\mathcal{L}_0/\mathcal{F}_v, \mu(\mathcal{K})) \to H^2(\mathcal{L}_0/\mathcal{F}_v, \mu(\mathcal{K})/k^*)$, we have

$$H^2(\mathcal{L}_0/\mathcal{F}_v, \mu(\mathcal{K})/k^*) = 1 = H^2(A_v/F_v, \Omega/k^*).$$

So we obtain the isomorphism $H^2(F_v/k, \Omega/k^*) \cong H^2(A/k, \Omega/k^*)$ from the exact sequence $H^2(F_v/k, \Omega/k^*) \to H^2(A_v/k, \Omega/k^*) \to H^2(A_v/F_v, \Omega/k^*) = 1$.

We therefore have the isomorphisms

$$\begin{aligned} H^2(L/k, \Omega/k^*) &\cong H^2(L_0/k, \Omega/k^*) \cong H^2(A_v/k, \Omega/k^*) \\ &\cong H^2(F_v/k, \Omega/k^*). \end{aligned}$$

Now from (v), $B_v = K(\varepsilon_{n_0/p_v^{b_v - z_v}})$ and $\mathcal{G}(B_v/K) \cong \mathcal{G}(\mathcal{F}_v/\mathcal{K})$ cyclic. As above,

$$H^2(B_v/K, \Omega/k^*) \cong H^2(\mathcal{F}_v/\mathcal{K}, \mu(\mathcal{K})/k^*) = 1.$$

Thus $H^2(K/k, \Omega/k^*) \to H^2(B_v/k, \Omega/k^*) \to H^2(B_v/K, \Omega/k^*) = 1$ is exact, so the isomorphism $H^2(K/k, \Omega/k^*) \cong H^2(B_v/k, \Omega/k^*)$ follows.

However we observe that $H^2(B_v/k, \Omega/k^*)$ is not isomorphic to $H^2(F_v/k, \Omega/k^*)$. In fact, F_v/B_v is cyclic of degree d dividing $p_1^{b_1-1} \cdots p_t^{b_t-1} \cdot p_{t+1}^{b_{t+1}} \cdots p_v^{z_v} \cdots p_s^{b_s}$ by (iii). Thus the $N_{F_v/B_v}(\Omega/k^*) = N_{F_v/B_v}\langle \varepsilon_m k^*, \varepsilon_{n_0'} k^*, \sqrt[n_0']{a} k^* \rangle = \langle \varepsilon_m k^*, \varepsilon_{n_0'}^d k^*, \sqrt[n_0']{a}^d k^* \rangle \neq \Omega/k^*$, because $\gcd(d, n_0')$ need not be 1. □

The exact correspondence of Theorem 2 with respect to radical extension is to show $H^2(K/k, \Omega_K/k^*) \cong H^2(L/k, \Omega_L/k^*)$ where $K = k(\Omega_K) < L = k(\Omega_L)$. Instead of this, we proved in Theorem 10 that $H^2(K/k, \Omega_K/k^*) \cong H^2(L/k, \Omega_K/k^*)$ which is a subgroup of $H^2(L/k, \Omega_L/k^*)$. We have discussed a radical extension field with one n-th root of an element in k. The next theorem is about a radical extension having more than one n-th root.

Theorem 12. *Let $k = \mathbb{Q}(\varepsilon_m)$. Write $m = p^z m'$ and $n_i = p^{b_i} n_i'$ $(i = 1, 2)$ with an odd prime $p \nmid m' n_1' n_2'$, and $z, b_i \geq 0$. Let $L = k(\sqrt[n_1]{a_1}, \sqrt[n_2]{a_2}) = k(\Omega_L)$, $F = k(\sqrt[n_1']{a_1}, \sqrt[n_2]{a_2}) = k(\Omega_F)$, and $K = k(\sqrt[n_1']{a_1}, \sqrt[n_2']{a_2}) = k(\Omega)$ be Galois radical extensions of k. Assume $b_i \leq z$ for $i = 1, 2$. Then*

(i) $N_{F/K}(\Omega_F/k^*) = \Omega/k^* = N_{L/K}(\Omega_L/k^*)$.

(ii) *Moreover,* $H^2(L/k, \Omega/k^*) \cong H^2(F/k, \Omega/k^*) \cong H^2(K/k, \Omega/k^*)$.

Proof. We may write the Galois radical extensions of k by

$$\begin{aligned} K = k(\varepsilon_{n_i'}, \sqrt[n_1']{a_1}, \sqrt[n_2']{a_2}) &< F = k(\varepsilon_{n_1'}, \varepsilon_{n_2}, \sqrt[n_1']{a_1}, \sqrt[n_2]{a_2}) \\ &< L = k(\varepsilon_{n_i}, \sqrt[n_1]{a_1}, \sqrt[n_2]{a_2}) \end{aligned}$$

$(i = 1, 2)$. Since $p^{b_i} \le p^z$ and $\varepsilon_{p^z} \in k$, we have $\varepsilon_{p^{b_i}} \in k < K$. Together with $\varepsilon_{n_i'} \in K$, it follows that ε_{n_i} belongs to K. Hence

$$F = k(\sqrt[n_1]{a_1}, \sqrt[n_2]{a_2}) = K(\sqrt[p^{b_2}]{\lambda_2}), \quad \text{where } \lambda_2 = \sqrt[n_2']{a_2} \in K$$

and $\sqrt[p^{b_2}]{\lambda_2}$ is a root of $X^{p^{b_2}} - \lambda_2 \in K[X]$. Thus F/K is a cyclic extension of degree p^{c_2} with $c_2 \le b_2$. And the minimal polynomial over K of $\sqrt[p^{b_2}]{\lambda_2} \in F$ is $X^{p^{c_2}} - \sqrt[p^{b_2}]{\lambda_2}^{p^{c_2}} \in K[X]$. Thus $\sqrt[n_2]{a_2}^{p^{c_2}} \in K$ and $\sqrt[n_2]{a_2}^{p^{c_2}} \in \langle \sqrt[n_2']{a_2} \rangle$. Moreover the cyclic group $\mathcal{G}(F/K)$ is generated by σ such that

$$\sigma(\sqrt[p^{b_2}]{\lambda_2}) = \varepsilon_{p^{c_2}} \sqrt[p^{b_2}]{\lambda_2}, \quad \text{i.e.,} \quad \sigma(\sqrt[n_2]{a_2}) = \varepsilon_{n_2}^{p^{b_2-c_2} n_2'} \sqrt[n_2]{a_2}.$$

Now for the Galois extension L over F,

$$L = k(\sqrt[n_1]{a_1}, \sqrt[n_2]{a_2}) = F(\sqrt[p^{b_1}]{\lambda_1}), \quad \text{where } \lambda_1 = \sqrt[n_1']{a_1} \in F$$

and $\sqrt[p^{b_1}]{\lambda_1}$ is a root of $X^{p^{b_1}} - \lambda_1 \in F[X]$. Since $\varepsilon_{p^{b_1}} \in F$, L/F is cyclic of degree p^{c_1} for $c_1 \le b_1$. Then $\sqrt[n_1]{a_1}^{c_1} \in F$ and

$$\mathcal{G}(L/F) = \langle \tau \rangle \quad \text{such that} \quad \tau(\sqrt[n_1]{a_1}) = \varepsilon_{n_1}^{p^{b_1-c_1} n_1'} \sqrt[n_1]{a_1}.$$

We shall compute the norm $N_{F/K}$ on $\Omega_F/k^* = \langle \varepsilon_{n_1'}, \varepsilon_{n_2}, \sqrt[n_1']{a}, \sqrt[n_2]{a} \rangle k^*$ that

$$N_{F/K}\langle \varepsilon_{n_1'} k^* \rangle = \langle \prod_{i=0}^{p^{c_2}-1} \sigma^i(\varepsilon_{n_1'} k^*) \rangle = \langle \varepsilon_{n_1'}^{p^{c_2}} k^* \rangle \le \langle \varepsilon_{n_1'} k^* \rangle,$$

and the equality holds because $1 = \gcd(n_1', p)$. Similarly

$$\begin{aligned} N_{F/K}\langle \varepsilon_{n_2} k^* \rangle &= N_{F/K}\langle \varepsilon_{n_2'} k^* \rangle = \langle \varepsilon_{n_2}^{p^{c_2}} k^* \rangle = \langle \varepsilon_{n_2'} k^* \rangle, \\ N_{F/K}\langle \sqrt[n_1']{a_1} k^* \rangle &= \langle \prod_{i=0}^{p^{c_2}-1} \sigma^i(\sqrt[n_1']{a_1} k^*) \rangle = \langle \sqrt[n_1']{a_1}^{p^{c_2}} k^* \rangle \le \langle \sqrt[n_1']{a_1} k^* \rangle, \end{aligned}$$

and the equality $N_{F/K}\langle \sqrt[n_1']{a_1} k^* \rangle = \langle \sqrt[n_1']{a_1} k^* \rangle$ holds, for $1 = \gcd(n_1', p)$. Moreover

$$\begin{aligned} N_{F/K}\langle \sqrt[n_2]{a_2}\, k^* \rangle &= \langle \prod_{i=0}^{p^{c_2}-1} \sigma^i(\sqrt[n_2]{a_2} k^*) \rangle \\ &= \langle \varepsilon_{n_2}^{p^{b_2-c_2} n_2'(1+\cdots+(p^{c_2}-1))} \sqrt[n_2]{a_2}^{\,p^{c_2}} k^* \rangle \\ &\le \langle \sqrt[n_2]{a_2}^{p^{c_2}} \rangle \le K, \quad \text{i.e.,} \ \langle \sqrt[n_2]{a_2}^{p^{c_2}} \rangle \le \langle \sqrt[n_2']{a_2} \rangle \le K. \end{aligned}$$

Since orders of $\sqrt[n_2]{a_2}^{p^{c_2}}$ and $\sqrt[n_2']{a_2}$ over K are $n_2' p^{b_2-c_2}$ and n_2' respectively, and $n_2' p^{b_2-c_2} \ge n_2'$, the equality $N_{F/K}\langle \sqrt[n_2]{a_2}\, k^* \rangle = \langle \sqrt[n_2']{a_2} \rangle$ follows. Thus we have

$$N_{F/K}(\Omega_F/k^*) = \langle \varepsilon_{n_1'} k^*, \varepsilon_{n_2'} k^*, \sqrt[n_1']{a_1} k^*, \sqrt[n_2']{a_2} k^* \rangle = \Omega/k^*.$$

On the other hand, we shall observe that $N_{L/F}(\Omega_L/k^*) \neq \Omega_F/k^*$. In fact, since $\mathcal{G}(L/F) = \langle\tau\rangle$ with $\tau(\sqrt[n_1]{a_1}) = \varepsilon_{n_1}^{(p_1^{b_1-c_1})n_1'} \sqrt[n_1]{a_1}$ for $0 \leq i \leq p^{c_1} - 1$, and since $\varepsilon_{n_1}, \varepsilon_{n_2} \in K < F$, it is easy to see that

$$\begin{aligned} N_{L/F}\langle\varepsilon_{n_1}k^*\rangle &= \langle\prod_i \tau^i(\varepsilon_{n_1}k^*)\rangle = \langle\prod_i \tau^i(\varepsilon_{n_1'}k^*)\rangle \\ &= \langle\varepsilon_{n_1'}^{p^{c_1}} k^*\rangle = \langle\varepsilon_{n_1'}k^*\rangle, \\ N_{L/F}\langle\varepsilon_{n_2}k^*\rangle &= \langle\prod_i \tau^i(\varepsilon_{n_2}k^*)\rangle = \langle\prod_i \tau^i(\varepsilon_{n_2'}k^*)\rangle \\ &= \langle\varepsilon_{n_2'}^{p^{c_1}} k^*\rangle = \langle\varepsilon_{n_2'}k^*\rangle, \end{aligned}$$

and

$$\begin{aligned} N_{L/F}\langle \sqrt[n_1]{a_1}k^*\rangle &= \langle\prod_i \tau^i(\sqrt[n_1]{a_1}k^*)\rangle \\ &= \langle\varepsilon_{n_1}^{p_1^{b_1-c_1}n_1'(1+\cdots+(p^{c_1}-1))} \sqrt[n_1]{a_1}^{p^{c_1}} k^*\rangle \\ &= \langle \sqrt[n_1]{a_1}^{p^{c_1}} k^*\rangle \leq \langle \sqrt[n_1']{a_1}k^*\rangle, \quad \text{for } \sqrt[n_1]{a_1}^{p^{c_1}} \in F, \end{aligned}$$

(products run over $0 \leq i \leq p^{c_1} - 1$). Comparing the orders $|\langle \sqrt[n_1]{a_1}^{p^{c_1}} k^*\rangle| = n_1'p^{b_1-c_1} \geq n_1' = |\langle \sqrt[n_1']{a_1}k^*\rangle|$ over F, we have $N_{L/F}\langle \sqrt[n_1]{a_1}k^*\rangle = \langle \sqrt[n_1']{a_1}k^*\rangle$. But

$$N_{L/F}\langle \sqrt[n_2]{a_2}k^*\rangle = \langle \prod_{i=0}^{p^{c_1}-1} \tau^i(\sqrt[n_2]{a_2}k^*)\rangle = \langle \sqrt[n_2]{a_2}^{p^{c_1}} k^*\rangle < \langle \sqrt[n_2]{a_2}k^*\rangle.$$

However, we will show that $N_{L/K}(\Omega_L/k^*) = \Omega/k^*$. Owing to the chain rule of the norm map, we obtain
$N_{L/K}\langle\varepsilon_{n_1}k^*\rangle = N_{F/K}\langle\varepsilon_{n_1'}k^*\rangle = \langle\varepsilon_{n_1'}k^*\rangle$, and similarly
$N_{L/K}\langle\varepsilon_{n_2}k^*\rangle = \langle\varepsilon_{n_2'}k^*\rangle$ and $N_{L/K}\langle \sqrt[n_1]{a_1}k^*\rangle = \langle \sqrt[n_1']{a_1}k^*\rangle$. Furthermore

$$\begin{aligned} N_{L/K}\langle \sqrt[n_2]{a_2}k^*\rangle = N_{F/K}\langle \sqrt[n_2]{a_2}^{p^{c_1}} k^*\rangle &= \langle \prod_{i=0}^{p^{c_2}-1} \sigma^i(\sqrt[n_2]{a_2}^{p^{c_1}} k^*)\rangle \\ &= \langle\varepsilon_{n_2}^{p^{b_2-c_2}n_2'(1+\cdots+(p^{c_2}-1))\cdot p^{c_1}} \sqrt[n_2]{a_2}^{p^{c_1}p^{c_2}} k^*\rangle \\ &= \langle \sqrt[n_2']{a_2}^{p^{c_1}} k^*\rangle = \langle \sqrt[n_2']{a_2}k^*\rangle. \end{aligned}$$

Hence it follows that

$$N_{L/K}(\Omega_L/k^*) = \langle\varepsilon_{n_1'}k^*, \varepsilon_{n_2'}k^*, \sqrt[n_1']{a_1}k^*, \sqrt[n_2']{a_2}k^*\rangle = \Omega/k^*.$$

Now to prove the isomorphism $H^2(L/k, \Omega/k^*) \cong H^2(K/k, \Omega/k^*)$ in (ii), we shall refer to the proof of Theorem 9. Since $\mathcal{G}(F/K)$ is cyclic of order p^{c_2}, invoking the computation of norm map before, we have

$$\begin{aligned} N_{F/K}(\Omega/k^*) &= \langle\varepsilon_{n_1'}^{p^{c_2}} k^*, \varepsilon_{n_2'}^{p^{c_2}} k^*, \sqrt[n_1']{a_1}^{p^{c_2}} k^*, \sqrt[n_2']{a_2}^{p^{c_2}} k^*\rangle \\ &= \langle\varepsilon_{n_1'}k^*, \varepsilon_{n_2'}k^*, \sqrt[n_1']{a_1}k^*, \sqrt[n_2']{a_2}k^*\rangle \\ &= \Omega/k^* = (\Omega/k^*)^{\mathcal{G}(F/K)} \end{aligned}$$

for $\gcd(p, n_1') = \gcd(p, n_2') = 1$. Thus

$$H^2(F/K, \Omega/k^*) = H^0(F/K, \Omega/k^*) = \frac{(\Omega/k^*)^{\mathcal{G}(F/K)}}{N_{F/K}(\Omega/k^*)} = 1,$$

so that $H^2(F/K, \Omega/k^*) = H^1(F/K, \Omega/k^*) = 1$ due to the Herbrant quotient. Hence the exact sequence

$$1 \to H^2(K/k, \Omega/k^*) \overset{\text{inf}}{\to} H^2(F/k, \Omega/k^*) \overset{\text{res}}{\to} H^2(F/K, \Omega/k^*) = 1$$

gives rise to the isomorphism $H^2(K/k, \Omega/k^*) \cong H^2(F/k, \Omega/k^*)$.

Similarly, with the cyclic group $\mathcal{G}(L/F)$ of order p^{c_1}, we get

$$N_{L/K}(\Omega/k^*) = \langle \varepsilon_{n_1'}^{p^{c_1}}, \varepsilon_{n_2'}^{p^{c_1}}, \sqrt[n_1']{a_1}^{p^{c_1}}, \sqrt[n_2']{a_2}^{p^{c_1}} \rangle k^* = \Omega/k^* = (\Omega/k^*)^{\mathcal{G}(L/K)},$$

so $H^1(L/F, \Omega/k^*) = H^2(L/F, \Omega/k^*) = H^0(L/F, \Omega/k^*) = 1$. Thus the sequence

$$1 \to H^2(F/k, \Omega/k^*) \overset{\text{inf}}{\to} H^2(L/k, \Omega/k^*) \overset{\text{res}}{\to} H^2(L/F, \Omega/k^*) = 1$$

yields an isomorphism $H^2(F/k, \Omega/k^*) \cong H^2(L/k, \Omega/k^*)$. □

Remark 3. Due to Theorem 12, we now can generalize the reduction of cohomology on radical groups having finitely many n-th roots.

In Theorem 11, we furthermore assume that each $\sqrt[n_i]{a_i}$ is a root of an irreducible binomial polynomial $X^{n_i} - a_i$ in $k[X]$. Then we can observe that the degrees $[F : K]$ and $[L : F]$ are exactly equal to p^{b_2} and p^{b_1} respectively. In fact, since $X^{n_2} - a_2$ is irreducible over k, a_2 does not belong to k^r for all primes divisors r of n_2 due to [8, 16.6]. But since $n_2 = p^{b_2} n_2'$, $a_2 \notin k^p$. Thus $a_2 \notin k^{pn_2'}$ and $\sqrt[n_2']{a_2} \notin k^p$. Moreover it can be seen that $\sqrt[n_2']{a_2} \notin \langle \sqrt[n_i']{a_i} \rangle^p$ for $i = 1, 2$. Thus $\sqrt[n_2']{a_2} = \lambda_2$ does not belong to $k^p(\sqrt[n_1']{a_1}, \sqrt[n_2']{a_2})^p = K^p$, so it follows from [8, 16.6] that $X^{p^{b_2}} - \lambda_2$ is irreducible over K. Hence $[F : K] = p^{b_2}$.

Similarly since $L = F(\sqrt[p^{b_1}]{\lambda_1})$ where $\sqrt[p^{b_1}]{\lambda_1}$ is a root of $X^{p^{b_1}} - \sqrt[n_1']{a_1} \in F[X]$, and $X^{n_1} - a_1$ is irreducible over k, $a_1 \notin k^p$ so $a_1 \notin k^{pn_1'}$, i.e., $\sqrt[n_1']{a_1} \notin k^p$. Clearly $\sqrt[n_1']{a_1}$ does not belong to $\langle \sqrt[n_1']{a_1} \rangle^p$ and $\langle \sqrt[n_2']{a_2} \rangle^p$, so $\lambda_1 = \sqrt[n_1']{a_1} \notin k^p(\sqrt[n_1']{a_1}, \sqrt[n_2']{a_2})^p = F^p$, so $X^{p^{b_1}} - \lambda_1$ is irreducible over F. Since $\varepsilon_{p^{b_1}} \in F$, L/F is cyclic of degree p^{b_1}.

5. Cohomological characterization of Brauer subgroups

We give our final observation with regard to Schur and radical subgroups of Brauer group. Let A be a Schur k-algebra. The set of similarity classes $[A]$ of A forms the *Schur subgroup* $S(k)$ of the Brauer group $B(k)$. Let L be a finite Galois extension of k. Then there is a restriction homomorphism $S(k) \to S(L)$ defined by the tensor product $[A] \mapsto L \otimes_k [A]$ for $[A] \in S(k)$. The kernel $S(L/k)$ of the homomorphism, called *relative Schur group*, consists of Schur k-algebra classes split by L. Analogously, the set of similarity classes of radical k-algebras forms the *radical group* $R(k)$. And for a finite Galois extension L

of k, the kernel of the restriction $R(k) \to R(L)$ is the *relative radical group* $R(L/k)$.

A well known theorem of Brauer-Witt provides an interpretation of Schur algebra as cyclotomic algebra, so $S(k)$ can be characterized cohomologically. An analog was conjectured in [2] that every projective Schur algebra is represented by a radical algebra so that a nice cohomological description can be provided on $PS(k)$. On the other hand, it has been verified cohomological characterizations for radical group in [2, 1.5] and for relative radical group in [5, Theorem 7].

Theorem 13. [5, Theorem 7] *Let $L = k(\Omega)$ be a finite Galois radical extension of k. Then $R(L/k)$ is isomorphic to $H^2_\iota(L/k)$, where $H^2_\iota(L/k)$ is the image of a canonical homomorphism ι of $H^2(L/k, \Omega)$ to $H^2(L/k, L^*)$.*

In particular if $L = k(\varepsilon_n)$ then $S(L/k)$ is isomorphic to $H^2_\iota(L/k) = H^2(L/k, \langle\varepsilon_n\rangle)$ (Corollary 8 [5]). Moreover by employing Theorem 2, if $k \leq \mathbb{Q}(\varepsilon_m)$ and n and n' are the same as in Theorem 2, then the following diagram is commutative:

$$(4)\qquad \begin{array}{ccc} H^2_\iota(k(\varepsilon_{n'})/k) & \overset{\inf_{k(\varepsilon_{n'})\to k(\varepsilon_n)}}{\longrightarrow} & H^2_\iota(k(\varepsilon_n)/k) \\ \downarrow\cong & & \cong\downarrow \\ S(k(\varepsilon_{n'})/k) & \overset{\cong}{\longrightarrow} & S(k(\varepsilon_n)/k) \end{array}$$

where all vertical and horizontal arrows are isomorphisms. This diagram provides a stronger relationship than that of Brauer and cohomology groups: for a Galois extension $k < L < E$, the diagram is commutative: (see [10, p.252], [11, p.159])

$$\begin{array}{ccc} H^2(L/k) & \overset{\inf}{\longrightarrow} & H^2(E/k) \\ \downarrow\cong & & \cong\downarrow \\ B(L/k) & \longrightarrow & B(E/k) \end{array}$$

in which only vertical arrows are isomorphisms. Owing to Theorem 10, we obtain a diagram of radical and cohomology groups as following.

Theorem 14. *Let $L = k(\Omega)$ and $L_0 = k(\Omega_0)$ be radical extensions of k satisfying the same context as in Theorem 10. Then there is a homomorphism $\chi : R(L_0/k) \to R(L/k)$ that makes the following diagram commute.*

$$(5)\qquad \begin{array}{ccc} H^2(L_0/k, \Omega_0) & \overset{\chi_1}{\longrightarrow} & H^2(L/k, \Omega_0) \\ & & \downarrow \\ \downarrow & & H^2(L/k, \Omega) \\ & & \downarrow \\ H^2_\iota(L_0/k) & & H^2_\iota(L/k) \\ \downarrow\cong & & \cong\downarrow \\ R(L_0/k) & \overset{\chi}{\longrightarrow} & R(L/k) \end{array}$$

Proof. We first note a difference here from (4) that $H^2(L/k, \Omega) \to H^2(L/k, L^*)$ need not be one to one. Hence the vertical arrows $H^2(L_0/k, \Omega_0) \to H^2_\iota(L_0/k)$ and $H^2(L/k, \Omega) \to H^2_\iota(L/k)$ are only surjective homomorphisms.

The two vertical isomorphisms in the above diagram are due to Theorem 13. By Theorem 10, we have an isomorphism $\psi : H^2(L_0/k, \Omega_0/k^*) \to H^2(L/k, \Omega_0/k^*)$. Moreover since the surjection $\Omega_0 \to \Omega_0/k^*$ induces both homomorphisms

$$H^2(L_0/k, \Omega_0) \stackrel{\pi_1}{\to} H^2(L_0/k, \Omega_0/k^*)$$

and

$$H^2(L/k, \Omega_0) \stackrel{\pi_2}{\to} H^2(L/k, \Omega_0/k^*),$$

the homomorphism $H^2(L_0/k, \Omega_0) \stackrel{\chi_1}{\to} H^2(L/k, \Omega_0)$ makes the diagram commute:

$$\begin{array}{ccc} H^2(L_0/k, \Omega_0) & \stackrel{\chi_1}{\to} & H^2(L/k, \Omega_0) \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ H^2(L_0/k, \Omega_0/k^*) & \stackrel{\psi}{\cong} & H^2(L/k, \Omega_0/k^*) \end{array}$$

Hence there exists a homomorphism $\chi : R(L_0/k) \to R(L/k)$ which makes the diagram (5) commute. □

A characterization of $R(L/k)$ by means of cohomology was given in Theorem 13 that there is an isomorphism $H^2_\iota(L/k) \cong R(L/k)$. An interesting cohomological description of radical group was proved in [2, Proposition 1.5] that if $L = k_{rad}$ is the maximal radical extension of k in an algebraic closure $\bar{k}$, then $\mu(\bar{k})$ is contained in L and there is a surjective homomorphism $H^2(L/k, \mu) \to R(k)$. One may also refer to the cohomological characterization of $PNil(k)$ in Proposition 1.6 [2] where $PNil(k) < B(k)$ consist of classes that may be represented by a projective Schur algebras of nilpotent type. It would be interesting to discover any relationships between $H^2(L/k, \Omega_L/k^*)$ and radical k-algebras split by $L = k(\Omega_L)$.

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