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PS-CONNECTEDNESS OF L-SUBSETS

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ABSTRACT. It is known that connectedness is one of the important notions in topology. In this paper, a new notion of connectedness is introduced in L-topological spaces, which is called PS-connectedness. It contains some nice properties. Especially, the famous K. Fan's Theorem holds for PS-connectedness in L-topological spaces.

1. Introduction

It is known that connectedness always plays an important role in Topology. Connectedness has been generalized to fuzzy set theory in terms of many forms, such as connectedness, semi-connectedness, pre-connectedness, strong connectedness, I-type of strong connectedness in [1, 2, 6-11, 13]. Among them, [1, 2, 8, 9, 11] were defined in I-topological spaces, where $I = [0, 1]$. And [6, 7, 10, 13] were defined in L-topological spaces, where L is a fuzzy lattice.

There may well be another connectedness to be discovered which will teach us "good" thing. For this consideration, in this paper we introduce a new connectedness in L-topological spaces, which is called PS-connectedness. Every PS-connected set is I-type of strongly connected [7]; every I-type of strongly connected set is strongly connected [6] and every strongly connected set is connected [13]. Meanwhile, we prove that it preserves some nice properties of connected sets in general topological spaces, one of which, for the PS-connectedness, the famous K. Fan's Theorem holds in L-topological spaces.

2. Preliminaries

In this paper, L will denote a fuzzy lattice, i.e., completely distributive lattice with order-reversing involutions "$\prime$". 0 and 1 denote the smallest element and the largest element in L, respectively. Let X be a nonempty crisp set; L^X be the set of all L-subsets on X; $M(L)$ and $M^*(L^X)$ be the set of all

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nonzero irreducible elements in L and L^X, respectively. (L^X, δ) stands for an L-topological space, briefly $L - ts$.

Definition 2.1. ([12, 14]). Let L_1, and L_2 be fuzzy lattices. A mapping $f : L_1 \to L_2$ is called an order-homomorphism if the following conditions hold:

(1) $f(\bigvee A_i) = \bigvee f(A_i)$ for $\{A_i\} \subset L_1$.

(2) $f^{-1}(B') = (f^{-1}(B))'$, where $f^{-1}(B) = \bigvee\{A \in L_1 : f(A) \leq B\}$ for each $B \in L_2$.

Definition 2.2. ([4]). Let (L^X, δ) be an $L - ts$. Then $A \in L^X$ is called

(1) A pre-semiopen set if and only if $A \leq (A^-)_o$.

(2) A pre-semiclosed set if and only if $A \geq (A^o)_-$. Here A^o, A^-, A_o and A_- will denote the interior, closure, semi-interior and semiclosure of A, respectively.

It is clear that every semiopen set [3] is pre-semiopen and every preopen set [1, 4] is pre-semiopen in $L - ts$. That none of the converses need be true is shown by Example 2.3. The example also shows that the intersection of any two pre-semiopen sets need not be pre-semiopen. Even the intersection of a pre-semiopen set with a open set may fail to be pre-semiopen.

Example 2.3. Let $X = \{x, y, z\}, L = [0, 1], \forall a \in L, a' = 1 - a$, and $A, B, C \in L^X$ defined as follows:

$$A(x) = 0.2,\ A(y) = 0.4,\ A(z) = 0.5;$$
$$B(x) = 0.8,\ B(y) = 0.8,\ B(z) = 0.6;$$
$$C(x) = 0.3,\ C(y) = 0.2,\ C(z) = 0.4.$$

Then $\delta = \{0, A, B, 1\}$ is a topology on L^X. By easy computations it follows that

$$C \leq (C^-)_o = (A')_o = A',$$

hence, C is a pre-semiopen set. Clearly C is not a semiopen set neither a preopen set (in fact, because 0 is the only open set contained in C and $0^- = 0, C$ is not a semiopen set. And because $C \not\leq C^{-o} = A'^o = A, C$ is not a preopen set). Further, because $A \wedge C = B'$ and $B' \not\leq (B'^-)_o = (B')_o = 0$, $A \wedge C$ is not a pre-semiopen set.

Definition 2.4. ([4]). The pre-semiclosure of the L-subset A is the intersection of all pre-semiclosed sets, each containing A. It will be denoted by $A_\sim$.

Definition 2.5. ([5]). Let (L_1^X, δ) and (L_2^Y, τ) be two $L - ts's$ and $f : (L_1^X, \delta) \to (L_2^Y, \tau)$ an order-homomorphism. f is called:

(1) pre-semicontinuous if $f^{-1}(B)$ is a pre-semiopen set of L_1^X for each $B \in \tau$.

(2) pre-semi-irresolute if $f^{-1}(B)$ is a pre-semiopen set of L_1^X for each pre-semiopen set B of L_2^Y.

Clearly, the pre-semi-irresolute is pre-semicontinuous. The converse need not be true [5]. Also, the semicontinuous [3] (or precontinuous [5]) is pre-semicontinuous. That none of the converses need be true is shown by Example 2.6.

Example 2.6. Consider the $L-ts$ (L^X, δ) as described in Example 2.3 and take $\tau = \{0, C, 1\}$. Then (L^X, τ) is $L-ts$. Let $f : (L^X, \delta) \to (L^X, \tau)$ be an identity mapping. By Example 2.3 f is pre-semicontinuous, but not semicontinuous neither is precontinuous.

Theorem 2.7. ([5]). *Let $f : (L_1^X, \delta) \to (L_2^Y, \tau)$ is an order-homomorphism. Then*

(1) *f is pre-semicontinuous if and only if $(f^{-1}(B))_\sim \leq f^{-1}(B^-)$ for each $B \in L_2^Y$.*

(2) *f is pre-semi-irresolute if and only if $(f^{-1}(B))_\sim \leq f^{-1}(B_\sim)$ for each $B \in L_2^Y$.*

Definition 2.8. ([13], [6]). Let (L^X, δ) be an $L-ts$ and $A, B \in L^X$. Then A and B are said to be separated (I-type of weakly separated) if $A^- \wedge B = A \wedge B^- = 0$ $(A_- \wedge B = A \wedge B_- = 0)$.

Definition 2.9. ([13], [6]). Let (L^X, δ) be an $L-ts$ and $A \in L^X$. A is called connected (I-type of strongly connected) if A cannot be represented as a union of two separated (I-type of weakly separated) non-null sets. If $A = 1$ is connected (I-type of strongly connected), we call (L^X, δ) a connected (I-type of strongly connected) space.

3. PS-connectedness of L-subsets

Definition 3.1. Let (L^X, δ) be an $L-ts$ and $A, B \in L^X$. Then A and B are said to be PS-separated if $A_\sim \wedge B = A \wedge B_\sim = 0$.

Lemma 3.2. *Let (L^X, δ) be an $L-ts$ and $A, B \in L^X$. If A and B PS-separated and $C \leq A, D \leq B$, then C and D are also PS-separated.*

Proof. This is easy. □

Definition 3.3. Let (L^X, δ) be an $L-ts$ and $A \in L^X$. A is called a PS-connected set if A cannot be represented as a union of two PS-separated non-null sets. Specifically, when $A = 1$ is PS-connected, we call (L^X, δ) a PS-connected space.

Example 3.4. Let $X = \{x, y\}, L = \{0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1\}$. For any $a \in L, a' = 1 - a$ and $A, C, D \in L^X$ defined as follows:

$$A(x) = 4/7, \ A(y) = 2/7;$$
$$B(x) = 4/7, \ B(y) = 3/7;$$
$$C(x) = 3/7, \ C(y) = 0;$$
$$D(x) = 0, \ D(y) = 3/7.$$

Then $\delta = \{0, C, D, C \vee D, 1\}$ is a topology on L^X. We show that A is a PS-connected set in $L-ts$ (L^X, δ). In fact, A can only be expressed as the union of two disjoint non-null L-subsets, i.e., $A = P \vee Q, P \wedge Q = 0, P \neq 0, Q \neq 0$, where $P(x) = 4/7, P(y) = 0; Q(x) = 0, Q(y) = 2/7$. By easy computations it follows that $P_{\sim} = B$. Then $P_{\sim} \wedge Q \neq 0$, i.e., P and Q are not PS-separated. Hence A is PS-connected.

Remark. Clearly, if L-subsets A and B in $L - ts$ (L^X, δ) are I-type of weakly separated then they are PS-separated. Every PS-connected set is I-type of strongly connected in $L - ts$. That the converses of them need not be true is shown by the following Example 3.5.

Example 3.5. Let $X = \{x, y\}$, $L = \{0, a, b, 1\}$, where $0 < a < 1, 0 < b < 1, 0' = 1, 1' = 0, a' = a, b' = b$, a and b are incomparable. Put $A, B, C \in L^X$ defined as follows:

$$A(x) = a, \ A(y) = b;$$

$$B(x) = 1, \ B(y) = 0;$$

$$C(x) = a, \ C(y) = 0;$$

$$D(x) = 0, \ D(y) = b.$$

Then $\delta = \{0, B, 1\}$ is a topology on L^X. We can easily show that A is I-type of strongly connected sets in $L - ts$ (L^X, δ). In fact, A can only be expressed as the union of two disjoint non-null L-subsets C and D, i.e. $A = C \vee D, C \wedge D = 0, C \neq 0, D \neq 0$. Simple computations give $C_{-} = 1$, and so $C_{-} \wedge D \neq 0$, i.e., C and D are not I-type of weakly separated. Hence A is I-type of strongly connected. Again, since

$$C \geq (C^o)_{-} = 0_{-} = 0, D \geq (D^o)_{-} = 0_{-} = 0.$$

C and D are pre-semiclosed sets, i.e., $C_{\sim} = C$ and $D_{\sim} = D$. Hence, $C_{\sim} \wedge D = C \wedge D_{\sim} = 0$, i.e., C and D are PS-separated. Thus, A is not PS-connected.

Theorem 3.6. *Let (L^X, δ) be an $L - ts$. Then the following conditions are equivalent:*

(1) *(L^X, δ) is not PS-connected.*

(2) *There exist two non-null pre-semiclosed sets A and B such that $A \vee B = 1$ and $A \wedge B = 0$.*

(3) *There exist two non-null pre-semiopen sets A and B such that $A \vee B = 1$ and $A \wedge B = 0$.*

Proof. (1) $\Rightarrow$ (2) : Let (L^X, δ) be not PS-connected. Then there exist two non-null L-subsets A and B such that $A_{\sim} \wedge B = A \wedge B_{\sim} = 0$ and $A \vee B = 1$, it follows that

$$A_{\sim} = A_{\sim} \wedge (A \vee B) = (A_{\sim} \wedge A) \vee (A_{\sim} \wedge B) = A.$$

Hence A is pre-semiclosed. Similarly, we can prove that B is pre-semiclosed. Thus (2) is held.

(2) $\Rightarrow$ (1), (2) $\Rightarrow$ (3) : Obvious. □

Corollary 3.7. *Let (L^X, δ) be an $L-ts$. Then the following conditions are equivalent:*

(1) *(L^X, δ) is PS-connected.*

(2) *If A and B are pre-semiopen sets, $A \vee B = 1$ and $A \wedge B = 0$, then $0 \in \{A, B\}$.*

(3) *If A and B are pre-semiclosed sets, $A \vee B = 1$ and $A \wedge B = 0$, then $0 \in \{A, B\}$.*

Theorem 3.8. *Let (L^X, δ) be an $L-ts$ and $A \in L^X$. Then the following conditions are equivalent:*

(1) *A is PS-connected.*

(2) *If $C, D \in L^X$ are PS-separated and $A \leq C \vee D$, then $A \wedge C = 0$ or $A \wedge D = 0$.*

(3) *If $C, D \in L^X$ are PS-separated and $A \leq C \vee D$, then $A \leq C$ or $A \leq D$.*

(4) *There do not exist two pre-semiclosed L-sets C and D such that*

$$C \wedge A \neq 0, D \wedge A \neq 0, A \leq C \vee D \text{ and } C \wedge D \wedge A = 0.$$

(5) *There do not exist two pre-semiclosed L-sets C and D such that*

$$A \not\leq C, A \not\leq D, A \leq C \vee D \text{ and } C \wedge D \wedge A = 0.$$

Proof. (1) $\Rightarrow$ (2) : If $C, D \in L^X$ are PS-separated and $A \leq C \vee D$, then by Lemma 3.2 we know that $A \wedge C$ and $A \wedge D$ are PS-separated. Since A is PS-connected and $A = A \wedge (C \vee D) = (A \wedge C) \vee (A \wedge D)$, one of $A \wedge C$ and $A \wedge D$ equals to 0.

(2) $\Rightarrow$ (3) : Suppose that $A \wedge C = 0$ then $A = A \wedge (C \vee D) = (A \wedge C) \vee (A \wedge D) = A \wedge D$. So $A \leq D$. Similarly $A \wedge D = 0$ implies $A \leq C$.

(3) $\Rightarrow$ (1) : Suppose that C, D are PS-separated and $A = C \vee D$. by (3) we know that $A \leq C$ or $A \leq D$. If $A \leq C$, then $D = D \wedge A \leq D \wedge C \leq D \wedge C_{\sim} = 0$ since C, D are PS-separated. Similarly if $A \leq D$, then $A = 0$. So A can not be represented as a union of two PS-separated non-null L-subsets. Therefore A is PS-connected.

(1) $\Rightarrow$ (4) : Suppose that A is PS-connected and There exist two pre-semiclosed L-sets C and D such that

$$C \wedge A \neq 0, D \wedge A \neq 0, A \leq C \vee D \text{ and } C \wedge D \wedge A = 0.$$

Then obviously $(C \wedge A) \vee (D \wedge A) = (C \vee D) \wedge A = A$. We can prove $(C \wedge A)_{\sim} \wedge (D \wedge A) = 0$ from the following fact:

$$(C \wedge A)_{\sim} \wedge (D \wedge A) \leq C_{\sim} \wedge (D \wedge A) = C \wedge D \wedge A = 0.$$

Similarly we have $(C \wedge A) \wedge (D \wedge A)_{\sim} = 0$. This shows that A is not PS-connected, which is a contradiction.

(4) $\Rightarrow$ (5) : Suppose that there exist two pre-semiclosed L-sets C and D such that

$$A \not\leq C, A \not\leq D, A \leq C \vee D \text{ and } C \wedge D \wedge A = 0.$$

We easily prove that $C \wedge A \neq 0$ and $D \wedge A \neq 0$. This is a contradiction.

(5) $\Rightarrow$ (1) : Suppose that (5) is true and A is not PS-connected. Then there are $E \neq 0$ and $F \neq 0$ such that $A = E \vee F$ and $E_\sim \wedge F = E \wedge F_\sim = 0$. Let $C = E_\sim$ and $D = F_\sim$. Then $A = E \vee F \leq E_\sim \vee F_\sim = C \vee D$ and by

$$\begin{aligned} E_\sim \wedge F_\sim \wedge A = E_\sim \wedge F_\sim \wedge (E \vee F) &= (E_\sim \wedge F_\sim \wedge E) \vee (E_\sim \wedge F_\sim \wedge F) \\ &= (F_\sim \wedge E) \vee (E_\sim \wedge F) = 0 \vee 0 = 0, \end{aligned}$$

we know $C \wedge D \wedge A = 0$. Moreover we have that $A \not\leq C$ and $A \not\leq D$. In fact, if $A \leq C$, then $D \wedge A = D \wedge (A \wedge C) = 0$, i.e. $F_\sim \wedge A = 0$. Therefore $F = F \wedge A \leq F_\sim \wedge A = 0$. This is a contradiction. Analogously we have that $A \not\leq D$. This contradicts (5). □

Corollary 3.9. *Each element in $M^*(L^X)$ is PS-connected.*

Theorem 3.10. *Let (L^X, δ) be an $L-ts$ and $A \in L^X$. Then A is PS-connected if and only if for any two nonzero $\vee$-irreducible element a and b in A, there exists a PS-connected L-set B such that $a, b \leq B \leq A$.*

Proof. The necessity is obvious. Now we prove the sufficiency. Suppose that A is not PS-connected, by Theorem 3.8, there exist two pre-semiclosed L-sets C and D such that

$$A \not\leq C, A \not\leq D, A \leq C \vee D \text{ and } C \wedge D \wedge A = 0.$$

Take two nonzero $\vee$-irreducible elements $a, b \leq A$ such that $a \not\leq C$ and $b \not\leq D$. Then for each $B \in L^X$ satisfying $a, b \leq B \leq A$, we have that

$$B \not\leq C, B \not\leq D, B \leq C \vee D \text{ and } C \wedge D \wedge B = 0.$$

By Theorem 3.8, B is not PS-connected, a contradiction. □

Theorem 3.11. *Let A be a PS-connected set in an $L - ts$ (L^X, δ). If $A \leq B \leq A_\sim$, then B is also PS-connected in (L^X, δ).*

Proof. Suppose that B is not PS-connected in (L^X, δ). Then there exist PS-separated sets C and D in (L^X, δ) such that $B = C \vee D$. Let $P = A \wedge C$ and $Q = A \wedge D$. Then $A = P \vee Q$. Since $P \leq C$ and $Q \leq D$, by Lemma 3.2, P and Q are PS-separated, contradicting the PS-connectedness of A. Hence B is PS-connected. □

Theorem 3.12. *Let $\{A_t : t \in T\}$ be a family of PS-connected sets in an $L-ts$ (L^X, δ). Suppose there is an $s \in T$ such that A_t and A_s are not PS-separated for each $t \neq s$. Then $\bigvee_{t \in T} A_t$ is PS-connected.*

Proof. Let $\bigvee_{t \in T} A_t = B \vee C, B_\sim \wedge C = B \wedge C_\sim = 0$ and for each $t \in T, B_t = A_t \wedge B, C_t = A_t \wedge C$. Then $A_t = B_t \vee C_t$ and $(B_t)_\sim \wedge C_t = B_t \wedge (C_t)_\sim = 0$. Since A_t is PS-connected, $B_t = 0$ or $C_t = 0$, it follows that $A_t = C_t \leq C$ or $A_t = B_t \leq B$. Specifically, we have $A_s = C_s \leq C$ or $A_s = B_s \leq B$. We may assume that $A_s = C_s \leq C$. Then for each $t \neq s, A_t \leq C$. In fact, if $A_t \not\leq C$, then $A_t \leq B$ and so

$$A_t \wedge (A_s)_\sim = A_t \wedge (C_s)_\sim \leq B \wedge C_\sim = 0,$$

$$(A_t)_\sim \wedge A_s = (A_t)_\sim \wedge C_s \leq B_\sim \wedge C = 0.$$

This shows that A_t and A_s are PS-separated. This is a contradiction. Hence for each $t \in T, A_t \leq C$. It follows that $\bigvee_{t\in T} A_t \leq C$, and so $B = B \wedge (\bigvee_{t\in T} A_t) \leq B \wedge C = 0$. Thus $\bigvee_{t\in T} A_t$ is PS-connected. □

Corollary 3.13. *Let $\{A_t : t \in T\}$ be a family of PS-connected sets in an $L-ts$ (L^X, δ). If $\bigwedge_{t\in T} A_t \neq 0$, then $\bigvee_{t\in T} A_t$ is PS-connected.*

Theorem 3.14. *Let $f : L_1^X \to L_2^Y$ be a pre-semi-irresolute order-homomorphism. If A is PS-connected in L_1^X, then $f(A)$ is PS-connected in L_2^Y.*

Proof. Let $f(A) = B \vee C, B_\sim \wedge C = B \wedge C_\sim = 0$, and $P = f^{-1}(B), Q = f^{-1}(C)$. Then

$$A \leq f^{-1}f(A) = f^{-1}(B) \vee f^{-1}(C) = P \vee Q.$$

From Theorem 2.7, we have

$$P_\sim = (f^{-1}(B))_\sim \leq f^{-1}(B_\sim),$$
$$Q_\sim = (f^{-1}(C))_\sim \leq f^{-1}(C_\sim).$$

It follows that

$$P_\sim \wedge Q \leq f^{-1}(B_\sim) \wedge f^{-1}(C) = f^{-1}(B_\sim \wedge C) = f^{-1}(0) = 0,$$
$$P \wedge Q_\sim \leq f^{-1}(B) \wedge f^{-1}(C_\sim) = f^{-1}(B \wedge C_\sim) = f^{-1}(0) = 0.$$

Put $G = A \wedge P$ and $H = A \wedge Q$, then $A = G \vee H$ and $G_\sim \wedge H = G \wedge H_\sim = 0$. Since A is PS-connected, $G = 0$ or $H = 0$. We may assume that $G = 0$. Then $A = H \leq Q$, and so $f(A) \leq f(Q) = ff^{-1}(C) \leq C$. It follows that $B = B \wedge f(A) \leq B \wedge C = 0$. This shows that $f(A)$ is PS-connected in L_2^Y. □

Corollary 3.15. *Let $f : L_1^X \to L_2^Y$ be a pre-semi-irresolute order-homomorphism and onto. If L_1^X is a PS-connected space, then so is L_2^Y.*

Theorem 3.16. *Let $f : L_1^X \to L_2^Y$ be a pre-semicontinuous order-homomorphism. If A is PS-connected in L_1^X, then $f(A)$ is connected in L_2^Y.*

Proof. By using Definitions 2.5,2.9 and Theorem 2.7 this is similar to the proof of Theorem 3.14. □

Corollary 3.17. *Let $f : L_1^X \to L_2^Y$ be a pre-semicontinuous order-homomorphism and onto. If L_1^X is a PS-connected space, then L_2^Y is connected.*

We know in general topology there are different ways to describe connectedness of a subset. K. Fan's Theorem is supposed to be the most interesting one, which has clear geometrical characterization. Now, the famous K. Fan's theorem will be extended to the PS-connectedness of L-subsets in $L - ts$.

Definition 3.18. Let (L^X, δ) be an $L - ts$, $x_\lambda \in M^*(L^X)$ and P a pre-semiclosed set in (L^X, δ). P is called a pre-semiclosed remote-neighborhood, or briefly, $PSC - RN$ of x_λ, if $x_\lambda \notin P$. The set of all $PSC - RNs$ of x_λ will be denoted by $\zeta(x_\lambda)$.

Theorem 3.19. *Let (L^X, δ) be an $L-ts$ and $A \in L^X$.$M^*(A)$ denotes the set of all points of $A, \zeta(x)$ denotes the set of all $PSC-RNs$ of x for each $x \in M^*(A)$. Then A is PS-connected if and only if for each pair a, b of points of $M^*(A)$ and each mapping $P : M^*(A) \to \bigcup\{\zeta(x) : x \in M^*(A)\}$, where $P(x) \in \zeta(x)$ for each $x \in M^*(A)$, there exists in $M^*(A)$ a finite number of points $x_1 = a, x_2, \ldots, x_n = b$ such that $A \not\leq P(x_i) \vee P(x_{i+1})$, $i = 1, 2, \ldots, n-1$.*

Proof. **Sufficiency.** Suppose that A is not PS-connected.Then there are $B, C \in L^X$ and $B \neq 0, C \neq 0$ such that $B_\sim \wedge C = B \wedge C_\sim = 0$ and $A = B \vee C$. Consider the mapping

$$P : M^*(A) \to \bigcup\{\zeta(x) : x \in M^*(A)\},$$

defined by

$$P(x) = \begin{cases} C_\sim, & if\ x \leq B, \\ B_\sim, & if\ x \leq C. \end{cases}$$

By $B_\sim \wedge C = B \wedge C_\sim = 0$, we have $x \not\leq P(x)$. Since $P(x)$ is a pre-semiclosed set, $P(x) \in \zeta(x)$ for each $x \in M^*(A)$. Take the point a out of B and take the point b out of C. Then $a, b \in M^*(A)$. Since for arbitrary finite points $x_1 = a, x_2, \ldots, x_n = b$, either $x_i \leq B$ or $x_i \leq C (i = 1, \ldots, n)$ must be held, $P(x_i) = C_\sim$ or $P(x_i) = B_\sim$. But $P(x_1) = C_\sim$ and $P(x_n) = B_\sim$, hence there exists $0 \leq j \leq n-1$ such that $P(x_j) = C_\sim$ and $P(x_{j+1}) = B_\sim$. This shows that $A = B \vee C \leq P(x_j) \vee P(x_{j+1})$, a contradiction. Thus sufficiency is proved.
Necessity. Suppose that condition of theorem is not held, i.e. there are points $a, b \in M^*(A), a \neq b$ and there is a mapping

$$P : M^*(A) \to \bigcup\{\zeta(x) : x \in M^*(A)\},$$

where $P(x) \in \zeta(x)$ for each $x \in M^*(A)$, such that

$$A \not\leq P(x_i) \vee P(x_{i+1}),\ i = 1, 2, \ldots, n-1$$

is not held for arbitrary finite points $x_1, \ldots, x_n \in M^*(A)$. For the sake of convenience, we follow the agreement that for arbitrary $a, b \in M^*(A), a$ and b are joined if there are finite points $x_1, \ldots, x_n \in M^*(A)$ such that

$$A \not\leq P(x_i) \vee P(x_{i+1}),\ i = 1, 2, \ldots, n-1.$$

Otherwise, a and b are not joined. Let

$$\mu = \{x \in M^*(A) : a \text{ and } x \text{ are joined}\},$$

$$\nu = \{x \in M^*(A) : a \text{ and } x \text{ are not joined}\},$$

$$B = \vee\mu,$$

$$C = \vee\nu.$$

Obviously, a and a are joined and so $a \in \mu$ and $a \leq B$. By hypothesis a and b are not joined, and so $b \in \nu$ and $b \leq C$. Hence $B \neq 0, C \neq 0$. Since for each $x \in M^*(A)$ or $x \in \mu$, or $x \in \nu, A = B \vee C$. Now we need only prove $B_\sim \wedge C = B \wedge C_\sim = 0$. Suppose that $B_\sim \wedge C \neq 0$, and for each $x \leq B_\sim \wedge C$. By $x \leq B_\sim$, we have $B \not\leq P(x)$, and so there is $y \in \mu$ such that $y \not\leq P(x)$. Hence

$y \not\leq P(x) \vee P(y)$ and $y \leq B \leq A$. Thus, $A \not\leq P(x) \vee P(y).y$ and a are joined so a and x are joined. On the other hand, by $x \leq C$, we have $C \not\leq P(x)$, and so there is $z \in \nu$ such that $z \not\leq P(x)$. Hence, $z \not\leq P(x) \vee P(z)$ and $z \leq C \leq A$. Thus, $A \not\leq P(x) \vee P(z)$. By x and a are joined, a and z are joined. This contradicts the $z \in \nu$. Thus, $B_{\sim} \wedge C = 0$. In a similar way we can prove the $B \wedge C_{\sim} = 0$. Thus necessity is proved. □

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