LOCALLY HOMOGENEOUS CRITICAL METRICS ON FOUR-DIMENSIONAL MANIFOLDS

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ABSTRACT. We classify complete, locally homogeneous metrics with finite volume on four-dimensional manifolds which are critical points for the squared L^2-norm functionals of either the full Riemannian curvature tensor or the Weyl curvature tensor defined on the space of Riemannian metrics.

1. Introduction

In Riemannian geometry many interesting metrics are critical points for some natural geometric functionals defined on a space of Riemannian metrics on a manifold. In this article, we consider critical metrics for the squared L^2-norm functionals of either the Riemannian curvature tensor or the Weyl curvature tensor defined on the space of Riemannian metrics on a four-dimensional manifold, and call them $\mathcal{R}$-critical or $\mathcal{W}$-critical metrics, respectively. Some basic properties of these critical metrics on compact manifolds were described in the chapter 4 of [2].

These two families of critical metrics contain a number of interesting Riemannian metrics in dimension four. Indeed, Einstein metrics and zero-scalar-curved half-conformally-flat metrics are $\mathcal{R}$-critical [9]. Actually these are the only known examples of $\mathcal{R}$-critical metrics as far as we know. Similarly, conformally Einstein metrics and half-conformally-flat metrics are $\mathcal{W}$-critical. Other source of interest for critical metrics comes from recent works on $\mathcal{W}$-critical metrics, [1] and [15], in which they were called Bach flat metrics.

A few partial characterizations of critical metrics on compact manifolds were studied; Kähler $\mathcal{W}$-critical metrics in [3], Kähler $\mathcal{R}$-critical metrics in [6] and $\mathcal{R}$-critical metrics with non-positive sectional curvature in [7]. In an effort to generalize Jensen's classification [5] of four-dimensional Einstein homogeneous

metrics to the $\mathcal{R}$-critical case, Lamontagne has initiated the study of homogeneous $\mathcal{R}$-critical metrics in his thesis [8] and obtained a partial result. Indeed, he proved that any left invariant $\mathcal{R}$-critical metric on a four-dimensional simply connected unimodular Lie group whose Lie algebra has a non-trivial center is flat. In general, the classification of $\mathcal{R}$-critical homogeneous metrics seems yet difficult, see the Remark 1 at the end of Section 4.

In this paper we complete the classification of complete, locally homogeneous $\mathcal{W}$-critical metrics on four-manifolds with finite volume as follows;

Theorem 1. *A complete locally homogeneous $\mathcal{W}$-critical metric with finite volume on a four-manifold is locally isometric to one of the following: an Einstein symmetric space, the product of the standard two-sphere with constant curvature k and the two-dimensional hyperbolic plane with curvature $-k$, and the product of any three-dimensional space with constant curvature and $\mathbb{R}$.*

The classification in the case of $\mathcal{R}$-critical metrics is as follows;

Theorem 2. *A complete locally homogeneous $\mathcal{R}$-critical metric with finite volume on a four-manifold is locally isometric to either an Einstein symmetric space or the product of the standard two-sphere with constant curvature k and the two-dimensional hyperbolic plane with curvature $-k$.*

In order to prove these two theorems, we view the universal cover $\tilde{M}$ of a complete locally homogeneous manifold (M, g) with finite volume, together with the identity component $Isom^0(\tilde{g})$ of the isometry group of the lifted metric $\tilde{g}$ on $\tilde{M}$ as a *geometry* $(\tilde{M}, Isom^0(\tilde{g}))$, see Section 2 for the definition of geometry and other related notions. Then $(\tilde{M}, Isom^0(\tilde{g}))$ is *contained* in one of the twenty four-dimensional maximal geometries in the Filipkiewicz's classification list [4, 18]. So $(\tilde{M}, Isom^0(\tilde{g}))$ is one of the maximal geometries or non-maximal geometries. Each Riemannian manifold of these maximal or non-maximal 4-dimensional geometries is either a Riemannian symmetric space or a Lie group with a left-invariant metric. So aside from the easy-to-handle Riemannian symmetric spaces, we have to consider left-invariant metrics on Lie groups of these geometries. We test the criticality of these metrics case by case.

The system of algebraic equations for critical metrics in each case involves a number of undetermined structure constants. One may be tempted to simply use computer software program to resolve all the cases. But in some cases such blind computation just fails. And we believe that even if it worked, one has to verify the result by concepts and hand computation which we endeavored in this article.

The paper is organized as follows. In Section 2 we explain critical metrics and four-dimensional geometries. In Section 3 we classify complete, locally homogeneous $\mathcal{W}$-critical metrics with finite volume on four-dimensional manifolds. In Section 4 we classify the $\mathcal{R}$-critical case.

2. Preliminaries

In this section we explain critical metrics on a manifold and review geometries in dimension four.

2.1. $\mathcal{R}$-critical and $\mathcal{W}$-critical metrics

Let (M, g) be a four-dimensional smooth Riemannian manifold with Levi-Civita connection D. For vector fields X, Y, Z, W on M and a g-orthonormal frame $\{e_i\}$ of the tangent space T_pM at a given point $p \in M$, we define the Riemannian curvature tensor $R(X, Y)Z = D_{[X,Y]}Z - [D_X, D_Y]Z$, $R(X, Y, Z, W) = g(R(X, Y, Z), W)$, the Ricci tensor $r(X, Y) = \sum_{i=1}^{4} R(X, e_i, Y, e_i)$ and the scalar curvature $s = \sum_{i=1}^{4} r(e_i, e_i)$. Let $z = r - \frac{1}{4}sg$, the traceless Ricci tensor. In Besse's book [2] the Weyl curvature tensor W is defined by

$$W = R - \frac{s}{24}g \odot g - \frac{1}{2}z \odot g,$$

where for 2-tensors α, β and x, y, z, t in T_pM, the Kulkarni-Nomizu product $\alpha \odot g$ is the 4-tensor defined by

$$\alpha \odot \beta(x, y, z, t)$$
$$= \alpha(x, z)\beta(y, t) + \alpha(y, t)\beta(x, z) - \alpha(x, t)\beta(y, z)$$
$$- \alpha(y, z)\beta(x, t).$$

Using the abstract index notation, the components of Weyl curvature tensor are written by

$$(2.1) \qquad \begin{aligned} W_{ijkl} &= R_{ijkl} - \frac{1}{2}(r_{ik}g_{jl} - r_{jk}g_{il} + r_{jl}g_{ik} - r_{il}g_{jk}) \\ &+ \frac{s}{6}(g_{ik}g_{jl} - g_{il}g_{jk}). \end{aligned}$$

Suppose that M is not necessarily compact, and let $\mathcal{M}$ denote the space of smooth Riemannian metrics on M. For a metric g_0 on M and a precompact open subset U of M, let $\mathcal{M}_U^{g_0} = \{ g \in \mathcal{M} | g \equiv g_0 \text{ on } U^c \}$ where U^c is the complement of U. We define g_0 to be $\mathcal{R}$-critical if for any precompact open subset U of M and every smooth curve g_t in $\mathcal{M}_U^{g_0}$ with $g_t|_{t=0} = g_0$, it satisfies $\frac{d}{dt}|_{t=0}\mathcal{R}_U^{g_0}(g_t) = 0$ where $\mathcal{R}_U^{g_0} : \mathcal{M}_U^{g_0} \to \mathbb{R}$ is the map defined by $\mathcal{R}_U^{g_0}(g) = \int_U |R_g|^2 dv_g$. Following the computation of [2, p.134] and using the fact that every element $h \in T_g\mathcal{M}_U^{g_0}$, the tangent space at g, vanishes along the boundary of U, we obtain

$$\frac{d}{dt}|_{t=0}\mathcal{R}_U^{g_0}(g_t) = \int_U (2\delta_{g_0}^D d_{g_0}^D r_{g_0} - 2\check{R}_{g_0} + \frac{1}{2}|R_{g_0}|^2 g_0, h)dv_{g_0}$$

for any smooth curve g_t in $\mathcal{R}_U^{g_0}$ with $g_t|_{t=0} = g_0$ and $\frac{dg_t}{dt}|_{t=0} = h$. Here for $x, y, z \in T_p M$ and $\{e_i\}$ a g_0-orthonormal basis for $T_p M$, $d_{g_0}^D r_{g_0}(x, y, z) = D_x r_{g_0}(y, z) - D_y r_{g_0}(x, z)$, $\delta_{g_0}^D$ its formal adjoint, $\check{R}_{g_0}(x, y) = \sum_{i,j,k} R_{g_o}(x, e_i, e_j, e_k) R_{g_o}(y, e_i, e_j, e_k)$, and $|R_{g_0}|^2 = \sum_{i,j,k,l} R_{g_0}(e_i, e_j, e_k, e_l) R_{g_0}(e_i, e_j, e_k, e_l)$. It follows that a metric g_0 is $\mathcal{R}$-critical if and only if it satisfies the equation $2\delta_{g_0}^D d_{g_0}^D r_{g_0} - 2\check{R}_{g_0} + \frac{1}{2}|R_{g_0}|^2 g_0 = 0$ on M. Therefore this $\mathcal{R}$-critical definition is just the extension of that of Besse [2, p.118] when the manifold M is compact.

Similarly, we define a $\mathcal{W}$-critical metric on any smooth four-dimensional manifold, so a metric is $\mathcal{W}$-critical if and only if it satisfies the equation $2\delta^D D^* W + 2\overset{\circ}{W} r = 0$ on M, where $\overset{\circ}{W} r \ (x, y) = \sum_{i,j} W(x, e_i, y, e_j) r(e_i, e_j)$ for $x, y \in T_p M$ and $\{e_i\}$ an orthonormal basis for $T_p M$ [2, p.135].

On a four-dimensional homogeneous manifold the above $\mathcal{R}$-critical and $\mathcal{W}$-critical equations can be written as follows, respectively [2, p.134];

$$(2.2) \qquad 4D^* Dr + 4r \circ r - 4\overset{\circ}{R} r - 2\check{R} + \frac{1}{2}|R|^2 g = 0,$$

$$(2.3) \qquad 2D^* Dr + 2r \circ r - 2\overset{\circ}{R} r - 2\overset{\circ}{W} r = 0,$$

where for $x, y \in T_p M$ and $\{e_i\}$ an orthonormal basis for $T_p M$, $\overset{\circ}{R} r(x, y) = \sum_{i,j} R(x, e_i, y, e_j) r(e_i, e_j)$ and $r \circ r(x, y) = \sum_i r(x, e_i) r(e_i, y)$.

In Section 3 and 4 we calculate the critical equation of a left invariant metric on a Lie group, so we need some curvature formulas of a left invariant metric with respect to an orthonormal basis on its Lie algebra. Let G be a Lie group with a left-invariant metric g and let $\{X_i\}$ be an orthonormal basis on its Lie algebra. Let $[X_i, X_j] = C_{ij}^k X_k$ and $D_{X_i} X_j = \Gamma_{ij}^k X_k$. The constants C_{ij}^k are called the structure constants and the constants Γ_{ij}^k are called Christoffel symbols with respect to this basis. Then Christoffel symbols are given by

$$\Gamma_{ij}^k = \frac{1}{2}\{g([X_i, X_j], X_k) - g([X_i, X_k], X_j) - g([X_j, X_k], X_i)\}$$
$$= \frac{1}{2}(C_{ij}^k - C_{ik}^j - C_{jk}^i)$$

and the Riemannian curvature tensor $R_{ijkl} = R(X_i, X_j, X_k, X_l)$ is given by

$$R_{ijkl} = g(R(X_i, X_j) X_k, X_l)$$
$$= g(D_{[X_i, X_j]} X_k - D_{X_i} D_{X_j} X_k + D_{X_j} D_{X_i} X_k, X_l)$$
$$= \sum_s (C_{ij}^s \Gamma_{sk}^l - \Gamma_{jk}^s \Gamma_{is}^l + \Gamma_{ik}^s \Gamma_{js}^l).$$

Therefore the left-hand sides of (2.2) and (2.3) are polynomials of C_{ij}^k's. In particular $D^*Dr(X_i, X_j)$'s are calculated as follows:

$$
\begin{aligned}
D^*Dr(X_i, X_j) &= - \sum_{a,b} r_{bj} X_i^*(D_{X_a} D_{X_a} X_b) \\
&\quad - 2 \sum_{a,b,k} r_{bk} X_i^*(D_{X_a} X_b) X_j^*(D_{X_a} X_k) \\
&\quad - \sum_{a,b} r_{bi} X_j^*(D_{X_a} D_{X_a} X_b) \\
&= - \sum_{a,b,k} r_{bj} \Gamma_{ab}^k \Gamma_{ak}^i - 2 \sum_{a,b,k} r_{bk} \Gamma_{ab}^i \Gamma_{ak}^j - \sum_{a,b,k} r_{bi} \Gamma_{ab}^k \Gamma_{ak}^j
\end{aligned}
$$

(2.4)

where $\{X_i^*\}$ is the dual basis of $\{X_i\}$.

2.2. Four-dimensional geometries

We review the classification of geometries in dimension 4. For the definition of geometry see Wall [17], [18], Scott [12] and Thurston [14]. A geometry is a pair (X, G) where;

1) X is a connected simply connected manifold,
2) G is a Lie group of diffeomorphisms of X acting transitively on X,
3) For each $x \in X$, the stabilizer subgroup G_x is compact,
4) G contains a discrete subgroup Γ such that the quotient space X/Γ contains a finite invariant measure in the sense of Ragunathan [11].

Two geometries (X, G) and (X', G') are regarded as the same geometry if there is a diffeomorphism of X onto X' throwing the action of G isomorphically onto G'. We say that a geometry (X, G) is *contained* in another geometry (X', G') if there is a diffeomorphism of X onto X' throwing the action of G into a subgroup of G'.

The four-dimensional geometries with maximal connected Lie groups in the above contained relation were classified by Filipkiewicz as follows ([4], [18]).

Theorem 3. *Any four-dimensional geometry (X, G) with a maximal connected group G is the same as (X', G') where G' is the identity component of the isometry group of X' and X' is one of the four-dimensional Riemannian symmetric spaces or the following Lie groups with some left invariant metrics; $\widetilde{SL(2)} \times E^1$, $Nil^3 \times E^1$, $Sol^3 \times E^1$, Nil^4, the solvable Lie groups $Sol_{m,n}^4$, Sol_0^4, Sol_1^4, F^4 with isometry group $\mathbb{R}^2 \ltimes SL(2)$.*

Here and below E^n is the n-dimensional Euclidean space, S^n is the n-dimensional standard sphere, H^n is the n-dimensional hyperbolic space, $\widetilde{SL(2)}$ is the universal covering space of the special linear group $SL(2)$, $SO(n)$ is the special orthogonal group, $SU(n)$ is the special unitary group, $U(n)$ is the unitary group, Nil^3 is the nilpotent Heisenberg group. The groups Sol_1^4 and Nil^4

are certain solvable and nilpotent groups respectively, whose Lie algebras have a non-trivial center. $Sol^4_{m,n}$, Sol^4_0 and F^4 are explained in Section 3.

We also list the non-maximal geometries with connected isometry group in dimension 4 as follows:

1. $(E^4, E^4 \ltimes K)$; $K = U(2), SU(2), SO(3), SO(2) \times SO(2), SO(2), (S^1)_{m,n}$, $\{1\}$, which are contained in the Rimannian symmetric space $(E^4, E^4 \ltimes SO(4))$. Any metric associated to one of these geometries is flat.

2. $(S^2 \times E^2, SO(3) \times E^2)$, $(H^2 \times E^2, PSL(2) \times E^2)$, which are contained in the Rimannian symmetric spaces $(S^2 \times E^2, S^2 \times E^2 \ltimes SO(2))$ and $(H^2 \times E^2, PSL(2) \times E^2 \ltimes SO(2))$ respectively. Here $PSL(2)$ is the isometry group of H^2. Any metric associated to one of these geometries yields the respective Riemannian symmetric space.

3. $(S^3 \times E^1, SU(2) \times E^1)$, $(S^3 \times E^1, U(2) \times E^1)$, which are contained in the Riemannian symmetric space $(S^3 \times E^1, SO(4) \times E^1)$. Here S^3 is identical to the Lie group $SU(2)$.

4. $(\widetilde{SL(2)} \times E^1, \widetilde{SL(2)} \times E^1)$, $(Nil^3 \times E^1, Nil^3 \times E^1)$, (Sol^4_0, H^4_λ).

Note that on each of $\widetilde{SL(2)} \times E^1$, $Nil^3 \times E^1$ and Sol^4_0 there exists a left-invariant metric with a five-dimensional isometry group which yields a maximal geometry, and that Lie groups $Sol^3 \times E^1$, Nil^4, $Sol^4_{m,n}$, Sol^4_1 and F^4 have no non-maximal geometries. Refer to [18] for details and notations.

The aim of this paper is the classification of $\mathcal{R}$-critical or $\mathcal{W}$-critical complete, locally homogeneous four-manifolds which have finite volume. Let (M, g) be a complete, locally homogeneous four-manifold, i.e., g is a complete Riemannian metric on M such that given $x, y \in M$, there are neighborhoods U and V of x and y, respectively, and an isometry $(U, x) \to (V, y)$. Suppose that (M, g) has a finite volume. Let $\widetilde{M}$ be the universal covering space of M and $\tilde{g}$ be the lifted metric of g. Then the pair $(\widetilde{M}, Isom^0(\tilde{g}))$ becomes a geometry where $Isom^0(\tilde{g})$ is the identity component of the isometry group of $\widetilde{M}$. It is either a maximal geometry or a non-maximal geometry. From Theorem 3 and the list above, it is either a Riemannian symmetric space or a Lie group with a left invariant metric.

3. $\mathcal{W}$-critical locally homogeneous manifolds

As explained in the last paragraph of the previous section, $(\widetilde{M}, Isom^0(\tilde{g}))$ is either a Riemannian symmetric space or a Lie group with a left invariant metric. Among the Riemannian symmetric spaces, E^4, S^4, H^4, $P^2(\mathbb{C})$, $H^2(\mathbb{C})$, $S^2 \times S^2$, $S^2 \times H^2$, $H^2 \times H^2$, $S^3 \times E^1$ and $H^3 \times E^1$ are $\mathcal{W}$-critical while $S^2 \times E^2$ and $H^2 \times E^2$ are not. For the left invariant metrics on Lie groups, it remains to classify left invariant $\mathcal{W}$-critical metrics on Lie groups in Theorem 3 or in the list of non-maximal geometries. Following Lamomtagne's suggestion [8] we first classify $\mathcal{W}$-critical left invariant metrics on unimodular Lie groups whose

Lie algebras have non-trivial center. A Lie group G is called unimodular if its left invariant Harr measure is also right invariant. The associated Lie algebra is called unimodular if the Lie group is unimodular. Note that the Lie groups in Theorem 3 or in the list of non-maximal geometries are unimodular because each of them admits a discrete subgroup with quotient of finite volume [10].

Proposition 1. *Let (M, g) be a four-dimensional, unimodular, simply connected Lie group whose Lie algebra has a non-trivial center and g be a left invariant, $\mathcal{W}$-critical metric. Then (M, g) is one of the following: $\mathbb{R}^4$ with a flat metric, $\widetilde{E(2)} \times \mathbb{R}$ with a flat metric and $SU(2) \times \mathbb{R}$ with a product metric of any constant curvature metric on $SU(2)$ and a flat metric on $\mathbb{R}$. Here $\widetilde{E(2)}$ is $\mathbb{R}^2 \ltimes \mathbb{R}$, the universal cover of the two-dimensional Euclidean group $E(2) = \mathbb{R}^2 \ltimes SO(2)$.*

Proof. Let M be a 4-dimensional, unimodular Lie group having a non-trivial center and g a left invariant metric on M and $\mathcal{A}$ its Lie algebra. Take X_4 to be a non-trivial element in the center of $\mathcal{A}$. Let $\mathcal{I}$ be the orthogonal complement of X_4. Then the quotient algebra $\hat{\mathcal{I}} = \mathcal{A}/<X_4>$ has the induced inner product $\hat{g}$ by restricting g to $\mathcal{I}$, after identifying $\mathcal{I}$ with $\hat{\mathcal{I}}$ via the quotient map. For each element $\hat{V} \in \hat{\mathcal{I}}$, the linear transformation $ad(\hat{V}) : \hat{\mathcal{I}} \to \hat{\mathcal{I}}$, $ad(\hat{V})(\hat{W}) = [\hat{V}, \hat{W}]$ for any $\hat{W} \in \hat{\mathcal{I}}$, has trace zero. Therefore $\hat{\mathcal{I}}$ is unimodular [10] and has an orthonormal basis $\hat{X}_1 = X_1 + <X_4>$, $\hat{X}_2 = X_2 + <X_4>$, $\hat{X}_3 = X_3 + <X_4>$ such that $X_1, X_2, X_3 \in \mathcal{I}$ and $[\hat{X}_1, \hat{X}_2] = \hat{u}_3 \hat{X}_3, [\hat{X}_2, \hat{X}_3] = \hat{u}_1 \hat{X}_1, [\hat{X}_3, \hat{X}_1] = \hat{u}_2 \hat{X}_2$, for some constants $\hat{u}_i$'s. Therefore $\mathcal{A}$ has an orthonormal basis X_1, X_2, X_3, X_4 with the following commutator relations

$$(3.1) \quad \begin{aligned} [X_i, X_4] &= 0, \\ [X_1, X_2] &= u_3 X_3 + v_3 X_4, \\ [X_2, X_3] &= u_1 X_1 + v_1 X_4, \\ [X_3, X_1] &= u_2 X_2 + v_2 X_4. \end{aligned}$$

So we have the structure constants:

$$(3.2) \quad \begin{aligned} C_{12}^3 &= -C_{21}^3 = u_3, C_{12}^4 = -C_{21}^4 = v_3, C_{23}^1 = -C_{32}^1 = u_1, \\ C_{23}^4 &= -C_{32}^4 = v_1, C_{31}^2 = -C_{13}^2 = u_2, C_{31}^4 = -C_{13}^4 = v_2, \end{aligned}$$

and $C_{ij}^k = 0$ otherwise. We denote the left hand side of equation (2.3) by $\text{grad}\mathcal{W} = 2D^*Dr + 2r\circ r - 2\overset{\circ}{R}r - 2\overset{\circ}{W}r$ and let $\text{grad}\mathcal{W}(X_i, X_j) = W_{ij}$. Using (3.2) we can calculate the Riemannian curvature tensor $R_{ijkl} = R(X_i, X_j, X_k, X_l)$ as follows:

$$(3.3) \quad \begin{aligned} R_{1212} &= -\tfrac{3}{4}u_3^2 - \tfrac{1}{2}u_1 u_2 + \tfrac{1}{4}u_1^2 + \tfrac{1}{4}u_2^2 - \tfrac{3}{4}v_3^2 + \tfrac{1}{2}u_3 u_2 + \tfrac{1}{2}u_3 u_1, \\ R_{1213} &= \tfrac{3}{4}v_3 v_2, \\ R_{1214} &= \tfrac{1}{4}u_3 v_2 - \tfrac{1}{4}u_1 v_2 + \tfrac{1}{4}v_2 u_2, \\ R_{1223} &= -\tfrac{3}{4}v_1 v_3, \end{aligned}$$

$$R_{1224} = -\tfrac{1}{4}u_3v_1 + \tfrac{1}{4}u_2v_1 - \tfrac{1}{4}v_1u_1,$$

$$R_{1312} = \tfrac{3}{4}v_3v_2,$$

$$R_{1313} = -\tfrac{3}{4}u_2^2 - \tfrac{1}{2}u_3u_1 + \tfrac{1}{4}u_1^2 + \tfrac{1}{4}u_3^2 - \tfrac{3}{4}v_2^2 + \tfrac{1}{2}u_3u_2 + \tfrac{1}{2}u_1u_2,$$

$$R_{1314} = \tfrac{1}{4}u_2v_3 - \tfrac{1}{4}u_1v_3 + \tfrac{1}{4}v_3u_3,$$

$$R_{1323} = \tfrac{3}{4}v_1v_2,$$

$$R_{1334} = -\tfrac{1}{4}u_2v_1 + \tfrac{1}{4}u_3v_1 - \tfrac{1}{4}v_1u_1,$$

$$R_{1414} = \tfrac{v_2^2}{4} + \tfrac{v_3^2}{4},$$

$$R_{1424} = -\tfrac{1}{4}v_1v_2,$$

$$R_{1434} = -\tfrac{1}{4}v_1v_3,$$

$$R_{2323} = -\tfrac{3}{4}u_1^2 - \tfrac{1}{2}u_3u_2 + \tfrac{1}{4}u_2^2 + \tfrac{1}{4}u_3^2 - \tfrac{3}{4}v_1^2 + \tfrac{1}{2}u_3u_1 + \tfrac{1}{2}u_1u_2,$$

$$R_{2324} = \tfrac{1}{4}u_1v_3 - \tfrac{1}{4}u_2v_3 + \tfrac{1}{4}v_3u_3,$$

$$R_{2334} = -\tfrac{1}{4}u_1v_2 + \tfrac{1}{4}u_3v_2 - \tfrac{1}{4}v_2u_2,$$

$$R_{2424} = \tfrac{1}{4}v_3^2 + \tfrac{1}{4}v_1^2,$$

$$R_{2434} = -\tfrac{1}{4}v_3v_2,$$

$$R_{3434} = \tfrac{1}{4}v_2^2 + \tfrac{1}{4}v_1^2, \text{ and } R_{ijkl} = 0 \text{ otherwise.}$$

The Ricci curvature tensor $r_{ij} = r(X_i, X_j) = \sum_{k=1}^4 R_{ikjk}$ is computed as follows:

$$(3.4)\qquad\begin{aligned}
r_{11} &= -\tfrac{1}{2}u_3^2 - \tfrac{1}{2}u_2^2 + \tfrac{1}{2}u_1^2 - \tfrac{1}{2}v_3^2 + u_3u_2 - \tfrac{1}{2}v_2^2,\\
r_{22} &= -\tfrac{1}{2}u_3^2 - \tfrac{1}{2}u_1^2 + \tfrac{1}{2}u_2^2 - \tfrac{1}{2}v_3^2 + u_3u_1 - \tfrac{1}{2}v_1^2,\\
r_{33} &= -\tfrac{1}{2}u_2^2 - \tfrac{1}{2}u_1^2 + \tfrac{1}{2}u_3^2 - \tfrac{1}{2}v_2^2 + u_2u_1 - \tfrac{1}{2}v_1^2,\\
r_{44} &= \tfrac{1}{2}v_3^2 + \tfrac{1}{2}v_2^2 + \tfrac{1}{2}v_1^2,\\
r_{12} &= \tfrac{1}{2}v_2v_1, \quad r_{13} = \tfrac{1}{2}v_1v_3, \quad r_{14} = \tfrac{1}{2}v_1u_1,\\
r_{23} &= \tfrac{1}{2}v_3v_2, \quad r_{24} = \tfrac{1}{2}v_2u_2, \quad r_{34} = \tfrac{1}{2}v_3u_3.
\end{aligned}$$

We can also calculate from (2.4), (3.2) and (3.4)

$$\begin{aligned}
(D^*Dr)_{12} &= (D^*Dr)(X_1, X_2)\\
&= \tfrac{1}{2}v_2v_1^3 + \tfrac{1}{2}v_2^3v_1 + 2v_2v_1u_1u_2 - \tfrac{7}{4}v_2v_1u_3u_2 - \tfrac{7}{4}v_2v_1u_3u_1\\
&\quad + \tfrac{1}{2}v_3^2v_2v_1 + \tfrac{3}{2}v_2v_1u_3^2 + \tfrac{3}{4}v_2v_1u_1^2 + \tfrac{3}{4}v_2v_1u_2^2.
\end{aligned}$$

The rest of $(D^*Dr)_{ij}$ can be computed similarly. Using these, (2.1), (3.3) and (3.4) we first compute the non-diagonal entries, i.e., W_{ij}, $i \neq j$, as follows:
(3.5)

$$\begin{aligned}
W_{12} &= (D^*Dr)_{12} + 2\sum_{i=1}^4 r_{1i}r_{i2} - 2\sum_{i,j=1}^4 R_{1i2j}r_{ij} - 2\sum_{i,j=1}^4 W_{1i2j}r_{ij}\\
&= \tfrac{1}{3}v_2v_1(8v_3^2 + 8u_1^2 + 8u_2^2 - 4u_2u_3 - u_3^2 - 4u_1u_3 + 8v_2^2 + 8v_1^2 + 5u_2u_1),\\
W_{13} &= \tfrac{1}{3}v_3v_1(8u_1^2 + 8u_3^2 - u_2^2 - 4u_2u_3 - 4u_2u_1 + 5u_1u_3 + 8v_2^2 + 8v_3^2 + 8v_1^2),\\
W_{14} &= \tfrac{1}{3}u_1v_1(8v_2^2 - u_3^2 - 4u_1u_3 - u_2^2 + 8v_3^2 - 4u_2u_1 + 8v_1^2 + 8u_1^2 + 2u_2u_3),\\
W_{23} &= \tfrac{1}{3}v_3v_2(-4u_1u_3 - 4u_2u_1 + 8v_1^2 + 8v_3^2 + 8u_3^2 + 8v_2^2 + 5u_2u_3 + 8u_2^2 - u_1^2),\\
W_{24} &= \tfrac{1}{3}u_2v_2(8v_3^2 - u_1^2 - 4u_2u_3 - 4u_2u_1 - u_3^2 + 8v_1^2 + 8u_2^2 + 8v_2^2 + 2u_1u_3),\\
W_{34} &= \tfrac{1}{3}v_3u_3(8u_3^2 + 8v_1^2 - u_1^2 - u_2^2 - 4u_2u_3 - 4u_1u_3 + 8v_3^2 + 2u_2u_1 + 8v_2^2).
\end{aligned}$$

Note that there are symmetries among the indexes 1, 2 and 3 in (3.1) and so in (3.5).

If we assume that none of the v_i's vanish, then taking the sum of the equations $\frac{W_{12}}{v_1 v_2} = 0$, $\frac{W_{13}}{v_1 v_3} = 0$ and $\frac{W_{23}}{v_2 v_3} = 0$, we obtain $13u_1^2 + \frac{47}{4}u_2^2 + \frac{42}{4}u_3^2 + 24(v_1^2 + v_2^2 + v_3^2) + (u_1 - \frac{3}{2}u_2)^2 + (u_2 - \frac{3}{2}u_3)^2 + (u_1 - \frac{3}{2}u_3)^2 = 0$, which holds only if all the variables must vanish, a contradiction. Hence one of the v_i's must vanish.

From now on we may assume that $v_1 = 0$ without loss of generality, because of the symmetry of v_1, v_2, and v_3 in the commutator relations.

We consider two cases.

Case 1. $u_2 \neq 0$ and $u_3 \neq 0$. We subdivide this case;
Case 1-i) $v_2 = 0$ or $v_3 = 0$: by symmetry we may suppose that $v_2 = 0$.

The diagonal entries, i.e., W_{ij}, $i = j$, are computed as follows:
(3.6)
$$
\begin{aligned}
W_{11} = \ & -2v_3^4 + (2u_3u_2 + \tfrac{1}{2}u_2^2 - \tfrac{1}{6}u_1^2 + \tfrac{2}{3}u_3u_1 - \tfrac{1}{3}u_2u_1 - 4u_3^2)v_3^2 + 2u_3u_2^3 \\
& - \tfrac{2}{3}u_3^2u_2u_1 + \tfrac{2}{3}u_2^3u_1 - 2u_3^4 - 2u_2^4 - \tfrac{2}{3}u_2^2u_3u_1 + \tfrac{10}{3}u_1^4 + \tfrac{2}{3}u_3u_2u_1^2 \\
& + \tfrac{2}{3}u_3^3u_1 - 2u_1^3u_2 - 2u_1^3u_3 + 2u_3^3u_2,
\end{aligned}
$$
$$
\begin{aligned}
W_{22} = \ & -2v_3^4 + (\tfrac{2}{3}u_3u_2 - \tfrac{1}{6}u_2^2 + \tfrac{1}{2}u_1^2 + 2u_3u_1 - \tfrac{1}{3}u_2u_1 - 4u_3^2)v_3^2 - 2u_3u_2^3 \\
& - \tfrac{2}{3}u_3^2u_2u_1 - 2u_2^3u_1 - 2u_3^4 + \tfrac{10}{3}u_2^4 + \tfrac{2}{3}u_2^2u_3u_1 - 2u_1^4 - \tfrac{2}{3}u_3u_2u_1^2 \\
& + 2u_3^3u_1 + \tfrac{2}{3}u_1^3u_2 + 2u_1^3u_3 + \tfrac{2}{3}u_3^3u_2,
\end{aligned}
$$
$$
\begin{aligned}
W_{33} = \ & \tfrac{2}{3}v_3^4 + (-\tfrac{2}{3}u_3u_2 + \tfrac{1}{6}u_2^2 + \tfrac{1}{6}u_1^2 - \tfrac{2}{3}u_3u_1 - \tfrac{1}{3}u_2u_1 + 4u_3^2)v_3^2 + \tfrac{2}{3}u_3u_2^3 \\
& + \tfrac{2}{3}u_3^2u_2u_1 + 2u_2^3u_1 + \tfrac{10}{3}u_3^4 - 2u_2^4 - \tfrac{2}{3}u_2^2u_3u_1 - 2u_1^4 - \tfrac{2}{3}u_3u_2u_1^2 \\
& - 2u_3^3u_1 + 2u_1^3u_2 + \tfrac{2}{3}u_1^3u_3 - 2u_3^3u_2,
\end{aligned}
$$
$$
\begin{aligned}
W_{44} = \ & \tfrac{10}{3}v_3^4 + (-2u_3u_2 - \tfrac{1}{2}u_2^2 - \tfrac{1}{2}u_1^2 - 2u_3u_1 + u_2u_1 + 4u_3^2)v_3^2 - \tfrac{2}{3}u_3u_2^3 \\
& + \tfrac{2}{3}u_3^2u_2u_1 - \tfrac{2}{3}u_2^3u_1 + \tfrac{2}{3}u_3^4 + \tfrac{2}{3}u_2^4 + \tfrac{2}{3}u_2^2u_3u_1 + \tfrac{2}{3}u_1^4 + \tfrac{2}{3}u_3u_2u_1^2 \\
& - \tfrac{2}{3}u_3^3u_1 - \tfrac{2}{3}u_1^3u_2 - \tfrac{2}{3}u_1^3u_3 - \tfrac{2}{3}u_3^3u_2.
\end{aligned}
$$

We subdivide further:
Case 1-i-a) $v_3 \neq 0$.

To analyze (3.6) we note that each of (3.6) is quadratic polynomials with respect to v_3^2 in decreasing order. From the equation $W_{34} = 0$ in (3.5), we have
$$
v_3^2 = \frac{u_1^2 + u_2^2 - 8u_3^2 - 2u_1u_2 + 4u_2u_3 + 4u_1u_3}{8}.
$$
Substituting this to the diagonal entries in (3.6) we have
$$
\begin{aligned}
W_{11} = \ & \tfrac{3}{4}u_3u_2u_1^2 - \tfrac{3}{4}u_2^2u_3u_1 + \tfrac{9}{4}u_3u_2^3 - \tfrac{9}{4}u_1^3u_3 - \tfrac{15}{8}u_1^3u_2 + \tfrac{5}{8}u_2^3u_1 \\
& - \tfrac{1}{16}u_1^2u_2^2 - \tfrac{63}{32}u_2^4 + \tfrac{105}{32}u_1^4, \\
W_{22} = \ & -\tfrac{3}{4}u_3u_2u_1^2 + \tfrac{3}{4}u_2^2u_3u_1 - \tfrac{9}{4}u_3u_2^3 + \tfrac{9}{4}u_1^3u_3 + \tfrac{5}{8}u_1^3u_2 - \tfrac{15}{8}u_2^3u_1 \\
& - \tfrac{1}{16}u_1^2u_2^2 + \tfrac{105}{32}u_2^4 - \tfrac{63}{32}u_1^4.
\end{aligned}
$$
From the equation $16 \times (W_{11} + W_{22}) = 5(u_1 - u_2)^4 + 16(u_1^2 - u_2^2)^2 = 0$ we have $u_1 = u_2$. Conversely if $u_1 = u_2$, $v_1 = v_2 = 0$ and $v_3^2 = u_1u_3 - u_3^2$ then

all the W_{ij}'s vanish from (3.5) and (3.6). Therefore we have the solutions $\{u_1 = u_2 \neq 0, u_3 \neq 0, v_1 = v_2 = 0, v_3^2 = u_1 u_3 - u_3^2 \neq 0\}$. For these we calculate the curvature tensors $r_{11} = r_{22} = \frac{1}{2} u_1 u_3$, $r_{33} = \frac{1}{2} u_3^2$, $r_{44} = \frac{1}{2} u_1 u_3 - \frac{1}{2} u_3^2$, $r_{12} = r_{13} = r_{14} = r_{23} = r_{24} = 0$, $r_{34} = \frac{1}{2} \sqrt{(u_1 u_3 - u_3^2)} u_3$, $s = \frac{3}{2} u_1 u_3$ and all the W_{ijkl}'s vanish.

We will show that any metric g of these is isometric to a product metric on $SU(2) \times \mathbb{R}$.

Let $e_1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $e_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $e_3 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $e_4 = (\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1)$ be the standard basis for $su(2) \times \mathbb{R}$. Then for an orthonormal basis of g we may set $X_1 = \sqrt{u_1 u_3} e_1$, $X_2 = \sqrt{u_1 u_3} e_2$, $X_3 = u_1 e_3 - \frac{\sqrt{(u_1 u_3 - u_3^2)}}{u_3} e_4$ and $X_4 = e_4$ so that they satisfy (3.1). The following g-orthonormal change of basis $\widetilde{X_1} = X_1$, $\widetilde{X_2} = X_2$, $\widetilde{X_3} = \sqrt{\frac{u_3}{u_1}} X_3 + \sqrt{1 - \frac{u_3}{u_1}} X_4$ and $\widetilde{X_4} = -\sqrt{1 - \frac{u_3}{u_1}} X_3 + \sqrt{\frac{u_3}{u_1}} X_4$ diagonalizes Ricci curvature tensors, i.e., $r_{11} = r_{22} = r_{33} = \frac{u_1 u_3}{2}$, $r_{44} = 0$ and $r_{12} = r_{13} = r_{14} = r_{23} = r_{24} = r_{34} = 0$. Now the relations between e_i's and $\widetilde{X_i}$'s are as follows: $\widetilde{X_1} = \sqrt{u_1 u_3} e_1$, $\widetilde{X_2} = \sqrt{u_1 u_3} e_2$, $\widetilde{X_3} = \sqrt{u_1 u_3} e_3$ and $\widetilde{X_4} = -u_1 \sqrt{1 - \frac{u_3}{u_1}} e_3 + (\frac{u_1 - u_3}{\sqrt{u_1 u_3}} + \sqrt{\frac{u_3}{u_1}}) e_4$. We put $\alpha = \sqrt{u_1 u_3}$, $\beta = -u_1 \sqrt{1 - \frac{u_3}{u_1}}$ and $\gamma = \frac{u_1 - u_3}{\sqrt{u_1 u_3}} + \sqrt{\frac{u_3}{u_1}}$.

Consider the map $f : SU(2) \times \mathbb{R} \longrightarrow SU(2) \times \mathbb{R}$ defined by $f(a, t) = (a \cdot \exp(\beta t e_3), \gamma t)$ and for $(a, t) \in SU(2) \times \mathbb{R}$ the left translation $L_{(a,t)} : SU(2) \times \mathbb{R} \longrightarrow SU(2) \times \mathbb{R}$ defined by $L_{(a,t)}(b, s) = (a \cdot b, t + s)$. Then choosing the curve $\lambda(s) = (\exp(s \alpha e_1), 0)$ passing through the identity $(I, 0)$ of $SU(2) \times \mathbb{R}$ with $\frac{d\lambda}{ds}|_{s=0} = \alpha e_1$ and differentiating $(f \circ L_{(a,t)})(\lambda(s)) = (a \exp(s \alpha e_1) \exp(\beta t e_3), \gamma t)$ at $s = 0$, we have

$$df_{(a,t)}((dL_{(a,t)})_{(I,0)}(\alpha e_1))$$
$$= a \alpha e_1 \exp(\beta t e_3)$$
$$= a \alpha \exp(\beta t e_3) \exp(-\beta t e_3) e_1 \exp(\beta t e_3)$$
$$= a \exp(\beta t e_3)(\cos \beta t \cdot \alpha e_1 - \sin \beta t \cdot \alpha e_2)$$

because $\exp(-\beta t e_3) e_1 \exp(\beta t e_3) = \cos \beta t \cdot e_1 - \sin \beta t \cdot e_2$. Similarly we have $df_{(a,t)}((dL_{(a,t)})_{(I,0)}(\alpha e_2)) = a \exp(\beta t e_3)(\sin \beta t \cdot \alpha e_1 + \cos \beta t \cdot \alpha e_2)$.

Let g_0 be the product metric on $SU(2) \times \mathbb{R}$ with the orthonormal basis $f_1 = \alpha e_1$, $f_2 = \alpha e_2$, $f_3 = \alpha e_3$, and $f_4 = e_4$ on its Lie algebra and let $Y_1 = \cos \beta t \cdot \alpha e_1 - \sin \beta t \cdot \alpha e_2$, $Y_2 = \sin \beta t \cdot \alpha e_1 + \cos \beta t \cdot \alpha e_2$, $Y_3 = \alpha e_3$ and $Y_4 = \beta e_3 + \gamma e_4$, which form an orthonormal basis for g. Then differentiating the curve $L_{f(a,t)}(\exp(s Y_1), 0) = (a \exp(\beta t e_3) \exp(s Y_1), \gamma t)$ at $s = 0$, we obtain $(dL_{f(a,t)})_{(I,0)}(Y_1) = a \exp(\beta t e_3) Y_1$. So $df_{(a,t)}((dL_{(a,t)})_{(I,0)}(f_1)) = (dL_{f(a,t)})_{(I,0)}(Y_1)$. Similarly for $i = 2, 3, 4$, we have $df_{(a,t)}((dL_{(a,t)})_{(I,0)} (f_i)) =$

$(dL_{f(a,t)})_{(I,0)}(Y_i)$. This means that

$$(f^*g)((dL_{(a,t)})_{(I,0)}(f_i), (dL_{(a,t)})_{(I,0)}(f_j))$$
$$= g(df_{(a,t)}((dL_{(a,t)})_{(I,0)}(f_i)), df_{(a,t)}((dL_{(a,t)})_{(I,0)}(f_j)))$$
$$= g((dL_{f(a,t)})_{(I,0)}(Y_i), (dL_{f(a,t)})_{(I,0)}(Y_j))$$
$$= (L^*_{f(a,t)}g)(Y_i, Y_j) = g(Y_i, Y_j) = \delta_{ij},$$

i.e., an orthonormal basis $\{(dL_{(a,t)})_{(I,0)}f_i\}$ for g_0 in $T_{(a,t)}(SU(2) \times \mathbb{R})$ is also an orthonormal basis for f^*g in $T_{(a,t)}(SU(2) \times \mathbb{R})$. So $f^*g = g_0$ and f is an isometry. Therefore any of the solution metrics of $\{u_1 = u_2 \neq 0, u_3 \neq 0, v_1 = v_2 = 0, v_3^2 = u_1 u_3 - u_3^2 \neq 0\}$ is isomeric to a product metric g_0 on $SU(2) \times \mathbb{R}$.

Case 1-i-b) $v_3 = 0$.

In this case M is a Riemannian product $G \times \mathbb{R}$, where G is a simply connected three-dimensional unimodular Lie group. By the classification of three-dimensional unimodular Lie group [10], G must be one of $SU(2)$, $\widetilde{SL(2,\mathbb{R})}$, $\widetilde{E(2)}$, $E(1,1)$, Nil^3 and E^3. Among them, $SU(2)$, $\widetilde{SL(2,\mathbb{R})}$, $E(1,1)$, Nil^3 and E^3 have compact quotients. For these Lie groups with compact quotient one may consider the $\mathcal{W}$-criticality on a compact quotient $\overline{G} \times S^1$ of $G \times \mathbb{R}$. By the following claim, the product metric on $\overline{M} = \overline{G} \times S^1$ is $\mathcal{W}$-critical if and only if it is conformally flat.

claim. Any product metric of a metric on a compact manifold $\overline{G}$ with a flat metric on a circle S^1 is $\mathcal{W}$-critical if and only if it is conformally flat.

Proof of claim. We denote a metric on $\overline{G}$ by g_1 and a flat metric on S^1 by g_2 and their product metric by $g = g_1 + g_2$ on $\overline{M} = \overline{G} \times S^1$. Consider a perturbation g^t of g of the form $g^t = g_1 + tg_2$. Since $|W|^2_{g^t} = |W|^2_g$, $\frac{d}{dt}|_{t=1}\mathcal{W}(g^t) = \int_{\overline{M}} |W|^2_g dv'_{g^t} = \int_{\overline{G} \times S^1} |W|^2_g dv_{g_1} dv_{g_2}$. Therefore g is $\mathcal{W}$-critical if and only if it is conformally-flat. $\square$

It is well known that a product metric on $G \times \mathbb{R}$ is conformally-flat if and only if G has constant curvature.

In the case of $\widetilde{E(2)}$, one of u_i's must vanish, say u_1 [9, p. 307]. In this case, i.e., $v_1 = v_2 = v_3 = u_1 = 0$, the diagonal entries in (3.6) become as follows:

$$
\begin{aligned}
W_{11} &= 2u_3^3 u_2 + 2u_3 u_2^3 - 2u_3^4 - 2u_2^4, \\
W_{22} &= \tfrac{2}{3}u_3^3 u_2 - 2u_3 u_2^3 - 2u_3^4 + \tfrac{10}{3}u_2^4, \\
W_{33} &= -2u_3^3 u_2 + \tfrac{2}{3}u_3 u_2^3 + \tfrac{10}{3}u_3^4 - 2u_2^4, \\
W_{44} &= -\tfrac{2}{3}u_3^3 u_2 - \tfrac{2}{3}u_3 u_2^3 + \tfrac{2}{3}u_3^4 + \tfrac{2}{3}u_2^4.
\end{aligned}
$$

The solutions to the equations $W_{11} = W_{22} = W_{33} = W_{44} = 0$ are $u_2 = u_3$. Therefore when $M = \widetilde{E(2)} \times \mathbb{R}$, we have solutions $\{v_1 = v_2 = v_3 = 0, u_1 = 0, u_2 = u_3\}$, up to permutations of u_1, u_2 and u_3 and M has a flat metric from (3.3).

Case 1-ii) $v_2 \neq 0$ and $v_3 \neq 0$.

First we compute

(3.7)
$$
\begin{aligned}
W_{44} = \quad & -2v_2^2 u_2 u_1 - \tfrac{1}{2}v_2^2 u_3^2 - 2v_3^2 u_3 u_1 + \tfrac{2}{3}u_3^2 u_2 u_1 - \tfrac{1}{2}v_2^2 u_1^2 - 2v_2^2 u_2 u_3 \\
& -\tfrac{1}{2}u_1^2 v_3^2 + v_2^2 u_3 u_1 + \tfrac{2}{3}u_2^2 u_3 u_1 + \tfrac{2}{3}u_3 u_2 u_1^2 - \tfrac{1}{2}u_2^2 v_3^2 + v_3^2 u_2 u_1 \\
& -\tfrac{2}{3}u_3^3 u_1 + 4u_2^2 v_3^2 - 2u_3 u_2 v_3^2 + 4v_2^2 u_2^2 - \tfrac{2}{3}u_1^3 u_2 + \tfrac{20}{3}v_3^2 v_2^2 - \tfrac{2}{3}u_2^3 u_1 \\
& -\tfrac{2}{3}u_3^3 u_2 - \tfrac{2}{3}u_3 u_2^3 - \tfrac{2}{3}u_1^3 u_3 + \tfrac{10}{3}v_2^4 + \tfrac{10}{3}v_3^4 + \tfrac{2}{3}u_3^4 + \tfrac{2}{3}u_2^4 + \tfrac{2}{3}u_1^4.
\end{aligned}
$$

Subtracting two equations $\frac{W_{23}}{v_2 v_3} = 0$ and $\frac{W_{24}}{u_2 v_2} = 0$ in (3.5) we have $9u_3^2 + 9u_2 u_3 - 6u_1 u_3 = 0$. So $u_1 = \frac{3}{2}(u_2 + u_3)$. Substituting $u_1 = \frac{3}{2}(u_2 + u_3)$ to W_{44} in (3.7) we have $24 \times W_{44} = -138u_2 u_3(v_2^2 + v_3^2) - 3u_3^2(v_2^2 + v_3^2) - 3u_2^2(v_2^2 + v_3^2) + 80(v_2^2 + v_3^2)^2 + 128u_3^3 u_2 + 128u_3 u_2^3 + 282u_3^2 u_2^2 + 19u_3^4 + 19u_2^4$. If we substitute $u_1 = \frac{3}{2}(u_2 + u_3)$ to $W_{34} = 0$, we get $v_2^2 + v_3^2 = \frac{u_2^2 + 46u_2 u_3 + u_3^2}{32}$. Substituting this to $24 \times W_{44} = 0$ we obtain the equation $9u_2^4 + 60u_2^3 u_3 + 118u_2^2 u_3^2 + 60u_2 u_3^3 + 9u_3^4 = (u_2 + 3u_3)^2(3u_2 + u_3)^2 = 0$. The solution to this equation is $u_3 = -3u_2$ or $u_2 = -3u_3$. If $u_3 = -3u_2$ then $v_2^2 + v_3^2 = -4u_2^2$ and if $u_2 = -3u_3$ then $v_2^2 + v_3^2 = -4u_3^2$, so that we get no solution in this case.

Case 2. $u_2 = 0$ or $u_3 = 0$.

Case 2-i) $u_2 = 0$ and $u_3 = 0$.

From (3.7) we have $6 \times W_{44} = (u_1^2 - \frac{3}{2}v_2^2)^2 + (u_1^2 - \frac{3}{2}v_3^2)^2 + 2u_1^4 + \frac{71}{4}v_2^4 + \frac{71}{4}v_3^4 + 40v_2^2 v_3^2$. Hence we have the trivial solution: $u_1 = u_2 = u_3 = v_1 = v_2 = v_3 = 0$.

When the solution is trivial, M is $\mathbb{R}^4$ with flat metric.

Case 2-ii) $u_2 = 0$ and $u_3 \neq 0$.

We subdivide further:

Case 2-ii-a) $v_2 \neq 0$ and $v_3 \neq 0$.

From $W_{23} = 0$ in (3.5) we have $v_2^2 = \frac{u_1^2 - 8u_3^2 + 4u_1 u_3 - 8v_3^2}{8}$. With this substitution the diagonal entries are computed:

(3.8)
$$
\begin{aligned}
W_{11} &= v_3^2 u_3 u_1 + \tfrac{13}{4}u_3^3 u_1 - \tfrac{9}{2}u_3^2 v_3^2 - \tfrac{19}{8}u_1^3 u_3 - \tfrac{9}{2}u_3^4 + \tfrac{105}{32}u_1^4 + \tfrac{1}{16}u_1^2 u_3^2, \\
W_{22} &= v_3^2 u_3 u_1 + \tfrac{7}{4}u_3^3 u_1 - \tfrac{3}{2}u_3^2 v_3^2 + \tfrac{17}{8}u_1^3 u_3 - \tfrac{3}{2}u_3^4 - \tfrac{63}{32}u_1^4 - \tfrac{5}{16}u_1^2 u_3^2, \\
W_{33} &= v_3^2 u_3 u_1 + \tfrac{1}{4}u_3^3 u_1 + \tfrac{3}{2}u_3^2 v_3^2 + \tfrac{5}{8}u_1^3 u_3 + \tfrac{3}{2}u_3^4 - \tfrac{63}{32}u_1^4 - \tfrac{11}{16}u_1^2 u_3^2, \\
W_{44} &= -3v_3^2 u_3 u_1 - \tfrac{21}{4}u_3^3 u_1 + \tfrac{9}{2}u_3^2 v_3^2 - \tfrac{3}{8}u_1^3 u_3 + \tfrac{9}{2}u_3^4 + \tfrac{21}{32}u_1^4 + \tfrac{15}{16}u_1^2 u_3^2.
\end{aligned}
$$

From the equation $W_{22} - W_{33} = \frac{3}{2}u_3^3 u_1 - 3u_3^2 v_3^2 + \frac{3}{2}u_1^3 u_3 - 3u_3^4 + \frac{3}{8}u_1^2 u_3^2 = 0$ we have $v_3^2 = \frac{4u_3^2 u_1 + 4u_1^3 - 8u_3^3 + u_1^2 u_3}{8u_3}$. Substituting this to (3.8) we have $W_{11} = -\frac{9}{2}u_1^3 u_3 + \frac{121}{32}u_1^4, W_{22} = W_{33} = \frac{3}{2}u_1^3 u_3 - \frac{47}{32}u_1^4, W_{44} = \frac{3}{2}u_1^3 u_3 - \frac{27}{32}u_1^4$ and so u_1 must vanish. Then we have $v_3^2 = -u_3^2$ and hence there is no solution in this case.

So v_2 or v_3 must vanish.

Case 2-ii-b) $v_2 = 0$ and $v_3 = 0$.

In this case we have treated already in Case 1-ii-b).

Case 2-ii-c) $v_2 = 0$ and $v_3 \neq 0$.

From the equation $W_{34} = 0$ in (3.5) we have $v_3^2 = \frac{u_1^2 + 4u_1 u_3 - 8u_3^2}{8}$. Substituting this to the diagonal entries in (3.6) we obtain $W_{11} = -\frac{9}{4}u_1^3 u_3 + \frac{105}{32}u_1^4$, $W_{22} = \frac{9}{4}u_1^3 u_3 - \frac{63}{32}u_1^4$, $W_{33} = \frac{3}{4}u_1^3 u_3 - \frac{63}{32}u_1^4$, $W_{44} = -\frac{3}{4}u_1^3 u_3 + \frac{21}{32}u_1^4$. So u_1 must vanish and hence $u_3 = v_3 = 0$. Therefore we have no solution in this case.

Case 2-ii-d) $v_2 \neq 0$ and $v_3 = 0$.

We can compute W_{22}, W_{33} and W_{44} as follows:

(3.9)
$$W_{22} = \tfrac{1}{6}v_2^2 u_3^2 + \tfrac{1}{6}v_2^2 u_1^2 - \tfrac{1}{3}v_2^2 u_3 u_1 + 2u_3^3 u_1 + 2u_1^3 u_3 + \tfrac{2}{3}v_2^4 - 2u_3^4 - 2u_1^4,$$
$$W_{33} = -\tfrac{1}{6}v_2^2 u_3^2 + \tfrac{1}{2}v_2^2 u_1^2 - \tfrac{1}{3}v_2^2 u_3 u_1 - 2u_3^3 u_1 + \tfrac{2}{3}u_1^3 u_3 - 2v_2^4 + \tfrac{10}{3}u_3^4 - 2u_1^4,$$
$$W_{44} = -\tfrac{1}{2}v_2^2 u_3^2 - \tfrac{1}{2}v_2^2 u_1^2 + v_2^2 u_3 u_1 - \tfrac{2}{3}u_3^3 u_1 - \tfrac{2}{3}u_1^3 u_3 + \tfrac{10}{3}v_2^4 + \tfrac{2}{3}u_3^4 + \tfrac{2}{3}u_1^4.$$

From the equation $W_{22} + 3W_{44} = \frac{4}{3}v_2^2(-u_3^2 - u_1^2 + 2u_3 u_1 + 8v_2^2) = 0$, we have $v_2^2 = \frac{u_1^2 + u_3^2 - 2u_1 u_3}{8}$. Substituting this to (3.9) we have
$$W_{22} = -\tfrac{3}{32}(21u_3^2 + 22u_3 u_1 + 21u_1^2)(-u_1 + u_3)^2,$$
$$W_{33} = \tfrac{1}{32}(-u_1 + u_3)(63u_1^3 + 43u_3 u_1^2 + 45u_3^2 u_1 + 105u_3^3).$$
If $u_1 = u_3$, then $v_2^2 = 0$, a contradiction. Therefore the solutions to the equations $W_{22} = 0$, $W_{33} = 0$, $W_{44} = 0$ satisfy $u_1 = u_3 = 0$. Therefore we have no solution in this case.

Case 2-iii) $u_2 \neq 0$ and $u_3 = 0$.

This case is symmetric to the Case 2-ii).

Summarizing Case 1 and Case 2, when the solution is trivial, M is $\mathbb{R}^4$ with a flat metric; when $v_1 = v_2 = v_3 = 0$, M is a product $G \times \mathbb{R}$ where $G = \mathbb{R}^3$, $\widetilde{E(2)}$ or $SU(2)$ with a metric of constant curvature; when the solutions are $\{u_1 = u_2 \neq 0, u_3 \neq 0, v_1 = v_2 = 0, v_3^2 = u_1 u_3 - u_3^2 \neq 0\}$, (M, g) is isometric to $SU(2) \times \mathbb{R}$ with a product metric of a constant curvature metric on $SU(2)$ and flat metric on $\mathbb{R}$. This finishes the proof of Proposition 1. $\qquad\Box$

In the list of maximal and non-maximal geometries in Section 2.2, every metric of the geometry $(E^4, E^4 \rtimes K)$ is flat and every metric of the geometries $(S^2 \times E^2, SO(3) \times E^2)$, $(H^2 \times E^2, PSL(2) \times E^2)$ is the respective Riemannian symmetric space. The Lie groups $S^3 \times E^1$, $\widetilde{SL(2)} \times E^1$, $Sol^3 \times E^1$, Nil^4, Sol_1^4 have non-trivial center. Hence it remains to consider only the three Lie groups $Sol_{m,n}^4$, Sol_0^4 and F^4, whose Lie algebras have trivial center.

We first consider left invariant metrics on $Sol_{m,n}^4$ and calculate its $\mathcal{W}$-critical equation. The Lie algebra of $Sol_{m,n}^4$ has a basis $\{e_i\}$ such that

(3.10) $[e_1, e_2] = xe_2, \qquad [e_1, e_3] = ye_3, \qquad [e_1, e_4] = ze_4,$

and $[e_i, e_j] = 0$ otherwise ([18]). Here, $x > y > z$ are real numbers, $x+y+z = 0$, and e^x, e^y, e^z are the roots of $t^3 - mt^2 + nt - 1 = 0$ with m and n distinct positive integers. Let g be a left invariant metric on $Sol_{m,n}^4$. We can choose an orthonormal basis $\{X_i, i = 1, 2, 3, 4\}$ on the tangent space at the identity such

that, for some constants k_{ij}'s,

$$X_2 = k_{22}e_2 = \tilde{e}_2,$$
$$X_3 = k_{32}\tilde{e}_2 + k_{33}e_3 = k_{32}\tilde{e}_2 + \tilde{e}_3,$$
$$X_4 = k_{42}\tilde{e}_2 + k_{43}\tilde{e}_3 + k_{44}e_4 = k_{42}\tilde{e}_2 + k_{43}\tilde{e}_3 + \tilde{e}_4,$$
$$X_1 = k_{11}e_1 + k_{12}\tilde{e}_2 + k_{13}\tilde{e}_3 + k_{14}\tilde{e}_4.$$

Note that $e_1, \tilde{e}_2, \tilde{e}_3$ and $\tilde{e}_4$, defined above, still satisfy (3.10), so we may replace e_i's by $\tilde{e}_i$'s. The orthonormal basis X_i's have the following commutator relations:

$[X_1, X_2] = k_{11}xX_2,$

$[X_1, X_3] = k_{11}k_{32}(x-y)X_2 + k_{11}yX_3,$

$[X_1, X_4] = k_{11}(k_{43}k_{32}(z-y) - k_{42}z + k_{42}x)X_2 + k_{11}k_{43}(y-z)X_3 + k_{11}zX_4,$

the right hand side of which are homogeneous in k_{11}. Then each term of left hand side of the $\mathcal{W}$-critical equation (2.3) is homogeneous in k_{11} of degree 4. So we may assume that $k_{11} = 1$. Scaling the basis $X_i \longmapsto \frac{X_i}{x}$ and so the metric, we may assume that $x = 1$. Finally using the relation $z = -x - y$, we may have an orthonormal basis X_i's with the commutator relations

$$[X_1, X_2] = X_2,$$
$$[X_1, X_3] = bX_2 + yX_3,$$
$$[X_1, X_4] = cX_2 + uX_3 + (-1-y)X_4,$$

for some constants b, c, u and y. With these one can calculate the curvature tensors as follows:

$$R_{1212} = \tfrac{1}{4}b^2 + \tfrac{1}{4}c^2 - 1, \quad R_{1213} = -b + \tfrac{1}{4}cu,$$
$$R_{1214} = -c - \tfrac{1}{4}bu, \quad R_{1313} = -\tfrac{3}{4}b^2 + \tfrac{1}{4}u^2 - y^2,$$
$$R_{1314} = -\tfrac{3}{4}bc - yu, \quad R_{1414} = -\tfrac{3}{4}c^2 - \tfrac{3}{4}u^2 - (1+y)^2,$$
$$R_{2323} = \tfrac{1}{4}b^2 - y, \quad R_{2324} = \tfrac{1}{4}bc - \tfrac{1}{2}u,$$
$$R_{2334} = \tfrac{1}{2}yc - \tfrac{1}{4}bu, \quad R_{2424} = \tfrac{1}{4}c^2 + 1 + y,$$
$$R_{2434} = \tfrac{1}{4}cu + \tfrac{1}{2}b(1+y), \quad R_{3434} = \tfrac{1}{4}u^2 + y(1+y),$$
$$R_{ijkl} = 0 \text{ otherwise.}$$
$$r_{11} = -\tfrac{1}{2}b^2 - \tfrac{1}{2}c^2 - \tfrac{1}{2}u^2 - 2y^2 - 2 - 2y, \quad r_{22} = \tfrac{1}{2}b^2 + \tfrac{1}{2}c^2,$$
$$r_{33} = -\tfrac{1}{2}b^2 + \tfrac{1}{2}u^2, \quad r_{44} = -\tfrac{1}{2}c^2 - \tfrac{1}{2}u^2,$$
$$r_{12} = r_{13} = r_{14} = 0, \quad r_{23} = -\tfrac{1}{2}b + \tfrac{1}{2}cu + \tfrac{1}{2}by,$$
$$r_{24} = -c - \tfrac{1}{2}yc, \quad r_{34} = -\tfrac{1}{2}bc - yu - \tfrac{1}{2}u.$$

From (2.1),

$$W_{1212} = \tfrac{1}{6}b^2 + \tfrac{1}{6}c^2 - \tfrac{1}{3} + \tfrac{1}{6}u^2 + \tfrac{2}{3}y^2 + \tfrac{2}{3}y,$$
$$W_{1313} = -\tfrac{1}{4}b^2 + \tfrac{1}{4}u^2 + 1 + \tfrac{1}{4}c^2 + y,$$
$$W_{1414} = -\tfrac{1}{4}c^2 - \tfrac{1}{4}u^2 + \tfrac{1}{4}b^2 - y, W_{1213} = -\tfrac{3}{4}b - \tfrac{1}{4}by,$$
$$W_{1214} = -\tfrac{1}{2}c - \tfrac{1}{4}bu + \tfrac{1}{4}yc, \quad W_{1314} = -\tfrac{1}{2}bc - \tfrac{1}{2}yu + \tfrac{1}{4}u.$$

By the formula (2.4), we have

$$
\begin{aligned}
(D^*Dr)_{11} = \ & -8 - 16y - 4c^2 - 4u^2 - 24y^2 - 4b^2 - b^2c^2 - \tfrac{1}{2}b^4 \\
& -u^2b^2 - u^2c^2 - 4y^2b^2 - 4y^2c^2 - 4yb^2 - 4yc^2 - 4yu^2 \\
& -\tfrac{1}{2}u^4 - 8y^4 - 16y^3 - 4u^2y^2 - \tfrac{1}{2}c^4.
\end{aligned}
$$

Using these one can compute and rearrange W_{11} as follows:

$$
\begin{aligned}
& -2W_{11} \\
= \ & -2\{2(D^*Dr)_{11} + 2\sum_{i=1}^4 r_{1i}r_{i1} - 2\sum_{i,j=1}^4 R_{1i1j}r_{ij} - 2\sum_{i,j=1}^4 W_{1i1j}r_{ij}\} \\
= \ & 16 + 32y + 11b^2 + 48y^2 + 18cuby - b^2u^2 + 4u^4 + 16y^4 + 32y^3 + 4c^4 \\
& +11b^2y^2 + 4b^4 + 8b^2c^2 + 20u^2y + 20c^2y + 20u^2y^2 + 11c^2y^2 + 2b^2y \\
& +8c^2u^2 + 20c^2 + 11u^2 \\
= \ & 16(y^2 + y + 1)^2 + (11y^2 + 2y + 11)b^2 + (11y^2 + 20y + 20)c^2 + 3b^4 \\
& +8c^2u^2 + (b^2 - \tfrac{u^2}{2})^2 + \tfrac{15}{4}u^4 + 4c^4 + 8(bc + \tfrac{9}{8}uy)^2 \\
& +(\tfrac{79}{8}y^2 + 20y + 11)u^2 \\
> \ & 0.
\end{aligned}
$$

The Lie group Sol_0^4 is the case of equal roots in $Sol_{m,n}^4$, i.e., $x = y = -\tfrac{1}{2}z$, and hence the calculation of $grad\ \mathcal{W}$ for Sol_0^4 is contained in that of $Sol_{m,n}^4$.

Consequently, we have the following proposition.

Proposition 2. *Any left invariant metric on $Sol_{m,n}^4$ or Sol_0^4 is not $\mathcal{W}$-critical.*

F^4 is the only geometry without compact quotient among the geometries in Theorem 3. The geometry F^4 is described as the manifold $\mathbb{R}^2 \times B$ with a left invariant metric g whose isometry group is $\mathbb{R}^2 \ltimes SL_2(\mathbb{R})$. Here B is the subgroup of $SL_2(\mathbb{R})$ consisting of the upper triangular matrices with positive diagonal entries. Note that there is no other geometry contained in F^4 than itself [18, p.127]. So we only need to check the $\mathcal{W}$-critical equations for the left invariant metric g with isometry group $\mathbb{R}^2 \ltimes SL_2(\mathbb{R})$. For each $p = (u, v, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in \mathbb{R}^2 \ltimes SL_2(\mathbb{R})$, the isometry r_p from $\mathbb{R}^2 \times B$ onto itself is defined by

$$
\begin{aligned}
& r_p(x, y, \begin{pmatrix} \beta & \alpha \\ 0 & \beta^{-1} \end{pmatrix}) \\
= \ & (u + ax + by,\ v + cx + dy,\ \begin{pmatrix} \frac{\beta}{(c\alpha+d)^2+c^2\beta^2} & \frac{(a\alpha+b)(c\alpha+d)+ac\beta^2}{(c\alpha+d)^2+c^2\beta^2} \\ 0 & \frac{(c\alpha+d)^2+c^2\beta^2}{\beta} \end{pmatrix})
\end{aligned}
$$

where $(x, y, \begin{pmatrix} \beta & \alpha \\ 0 & \beta^{-1} \end{pmatrix}) \in \mathbb{R}^2 \times B$, $\beta > 0$ and $\alpha \in \mathbb{R}$. The Lie algebra of $\mathbb{R}^2 \times B$ has basis vectors $e_1 = (1, 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})$, $e_2 = (0, 1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})$, $e_3 = $

$(0, 0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}))$, $e_4 = (0, 0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}))$ with the following commutator relations:

$$[e_1, e_2] = 0, \quad [e_3, e_1] = e_1, \quad [e_3, e_2] = -e_2,$$

$$[e_3, e_4] = 2e_4, \quad [e_4, e_1] = 0, \quad [e_4, e_2] = e_1.$$

Using the fact that the isotropy group at the identity $e = (0, 0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}))$ is

$\{(0, 0, \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}))| \ 0 \leq \theta \leq 2\pi\}$, we will show that e_i's are orthogonal

with respect to g. For fixed $\theta := (0, 0, \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}))$, by the map $(r_\theta)_*$ we have the following:

$$(r_{\frac{\pi}{2}})_{*e}(e_1) = -e_2,$$

$$(r_{\frac{\pi}{4}})_{*e}(e_1) = \frac{1}{\sqrt{2}}e_1 - \frac{1}{\sqrt{2}}e_2, \quad (r_{\frac{\pi}{4}})_{*e}(e_2) = \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2,$$

$$(r_{\frac{\pi}{4}})_{*e}(e_3) = -e_4, \quad (r_{\frac{\pi}{4}})_{*e}(e_4) = e_3.$$

Since r_θ is an isometry, putting above into $g(e_i, e_j) = g(r_{\theta*}(e_i), r_{\theta*}(e_j))$ for all i, j, we deduce that e_1, e_2, e_3, e_4 are g-orthogonal each other with respect to g. Therefore we can choose an orthonormal basis X_1, X_2, X_3, X_4 such that $X_1 = k_1 e_1$, $X_2 = k_2 e_2$, $X_3 = k_3 e_3$, $X_4 = k_4 e_4$ with the following commutator relations:

$$[X_1, X_2] = 0, \ [X_1, X_3] = -k_3 X_3, \ [X_1, X_4] = 0,$$

$$[X_2, X_3] = k_3 X_2, \ [X_2, X_4] = -\frac{k_2}{k_1}k_4 X_1, \ [X_3, X_4] = 2k_3 X_4.$$

Since $1 = g(X_2, X_2) = k_2^2 g(e_2, e_2)$ and $1 = g(X_1, X_1) = k_1^2 g(e_1, e_1)$, $k_1^2 = k_2^2$. So we may choose an orthonormal basis X_1, X_2, X_3, X_4 such that

$$[X_1, X_2] = 0, \ [X_1, X_3] = -aX_1, \ [X_1, X_4] = 0,$$

$$[X_2, X_3] = aX_2, \ [X_2, X_4] = -bX_1, \ [X_3, X_4] = 2aX_4.$$

for some constants a and b. In this basis the curvature tensors are computed as follows: $R_{1212} = \frac{1}{4}b^2 + a^2, R_{1313} = -a^2, R_{1414} = \frac{1}{4}b^2 - 2a^2, R_{2323} = -a^2, R_{2424} = -\frac{3}{4}b^2 + 2a^2, R_{3434} = -4a^2, R_{ijkl} = 0$ otherwise, and $(D^*Dr)_{44} = -\frac{1}{2}b^4 - 8a^2b^2 + 16a^4$. Finally we have

$$W_{44} = -16a^4 + 6a^2b^2 - 2b^4,$$

which cannot be zero. Thus we have the following proposition.

Proposition 3. *Any left invariant metric on F^4 is not $\mathcal{W}$-critical.*

Proof of Theorem 1. For a complete, locally homogeneous space (M, g) the pair $(\widetilde{M}, Isom^0 \ \tilde{g})$ is a geometry. So $(\widetilde{M}, \tilde{g})$ is a Riemannian symmetric space or a Lie group with a left invariant metric in Theorem 3 or in the list of non-maximal

geometries. Among the Riemannian symmetric spaces, Einstein symmetric spaces and $S^2 \times H^2$, $S^3 \times E^1$ and $H^3 \times E^1$ are $\mathcal{W}$-critical. By Propositions 1, 2 and 3, any $\mathcal{W}$-critical left invariant metric on the Lie groups in Theorem 3 or in the list of non-maximal geometries is $\mathbb{R}^4$ with flat metric, $\widetilde{E(2)} \times \mathbb{R}$ with flat metric or $SU(2) \times \mathbb{R}$ with a product metric of any constant curvature metric on $SU(2)$ and a flat metric on $\mathbb{R}$. Among these $\widetilde{E(2)} \times \mathbb{R}$ doesn't have a quotient of finite volume [12, p.476]. The proof of Theorem 1 is finished.   □

4. $\mathcal{R}$-critical locally homogeneous manifolds

In this section we classify $\mathcal{R}$-critical complete, locally homogeneous manifolds of finite volume. As mentioned in previous section they are locally isometric to one of Riemannian symmetric spaces or Lie groups with left invariant metrics. The $\mathcal{R}$-critical Riemannian symmetric spaces are E^4, S^4, H^4, $P^2(\mathbb{C})$, $H^2(\mathbb{C})$, $S^2 \times S^2$, $S^2 \times H^2$, $H^2 \times H^2$, while $S^2 \times E^2$, $H^2 \times E^2$, $S^3 \times E^1$, $H^3 \times E^1$ are not $\mathcal{R}$-critical. In [8], Lamontagne classified $\mathcal{R}$-critical left invariant metrics on unimodular Lie groups whose Lie algebras have non-trivial center.

Theorem 4. (Lamontagne [8]) *Let (M, g) be a four-dimensional, unimodular, simply connected Lie group whose Lie algebra has a non-trivial center and g be a left invariant, $\mathcal{R}$-critical metric. Then (M, g) is one of the following: $\mathbb{R}^4$ or $\widetilde{E(2)} \times \mathbb{R}$ with a flat metric.*

We omit the proof of Theorem 4, which is similar to that of Proposition 1.

Following the same reasoning as the $\mathcal{W}$-critical case of Section 3, we are left to calculate only three Lie groups $Sol^4_{m,n}$, Sol^4_0 and F^4, whose Lie algebras have trivial center.

As explained in Section 3, the Lie algebra of $Sol^4_{m,n}$ has a basis X_i with the following commutator relations:

$$[X_1, X_2] = X_2,$$
$$[X_1, X_3] = bX_2 + yX_3,$$
$$[X_1, X_4] = cX_2 + uX_3 + (-1 - y)X_4,$$

Calculating $\mathrm{grad}\mathcal{R}$ using this, we can arrange $\frac{4}{3}R_{11} = \frac{4}{3}\mathrm{grad}\mathcal{R}(X_1, X_1)$ as follows:

$$\frac{4}{3}\{4(D^*Dr)_{11} + 4\sum_{i=1}^4 r_{1i}r_{i1} - 4\sum_{i,j=1}^4 R_{1i1j}r_{ij}$$
$$-2\sum_{i,j,k=1}^4 R_{1ijk}^2 + \frac{1}{2}\sum_{i,j,k,l=1}^4 R_{ijkl}^2\}$$
$$= -24 - 48y - 48y^3 - 24y^4 - 28u^2y^2 - 16b^2y^2 - \frac{11}{2}u^4 - \frac{11}{2}c^4 - 11c^2u^2$$
$$-16y^2c^2 - 16b^2 - 11b^2c^2 - \frac{11}{2}b^4 + u^2b^2 - 24cuby - 16u^2 - 28c^2y$$
$$-28u^2y - 4b^2y - 72y^2 - 28c^2$$
$$= -(16y^2 + 4y + 16)b^2 - \frac{11}{2}c^4 - (16y^2 + 28y + 16)c^2 - 11u^2c^2 - \frac{11}{2}u^4$$

$$-24(y^2 + y + 1)^2 - 11(cb + \tfrac{12}{11}uy)^2 - \tfrac{164}{11}(y^2 + \tfrac{77}{41}y + \tfrac{44}{41})u^2$$
$$-\tfrac{11}{2}\{(b^2 - \tfrac{1}{11}u^2)^2 + \tfrac{120}{121}u^4\}$$
$$< \quad 0.$$

Proposition 4. *Any left invariant metric on $Sol^4_{m,n}$ or Sol^4_0 is not $\mathcal{R}$-critical.*

For F^4, with respect to the basis chosen in Section 3 we have

$$R_{44} = \dot{-}42a^4 + 9a^2b^2 - \frac{33}{8}b^4,$$

which is negative because both a and b are nonzero.

Proposition 5. *Any left invariant metric on F^4 is not $\mathcal{R}$-critical.*

Proof of Theorem 2. As explained in the proof of Theorem 1, we need only to consider Riemannian symmetric spaces or left invariant metrics on Lie groups in Theorem 3 or in the list of non-maximal geometries. The $\mathcal{R}$-critical Riemannian symmetric spaces are Einstein symmetric spaces and $S^2 \times H^2$. By Theorem 4, Propositions 4 and 5, any $\mathcal{R}$-critical left invariant metric on the Lie groups in Theorem 3 or in the list of non-maximal geometries is $\mathbb{R}^4$ or $\widetilde{E(2)} \times \mathbb{R}$ with a flat metric. As $\widetilde{E(2)} \times \mathbb{R}$ with a flat metric does not admit a quotient of finite volume, we get the conclusion of Theorem 2. □

Remark 1. More generally, one may try to classify $\mathcal{R}$-critical homogeneous metrics. For this we need to compute on solvable Lie groups in the list of [16]. One can reduce the number of undetermined constants to 4 or 5. But since the $\mathcal{R}$-critical equations are too complicated, it is not easy to solve the system of equations. For instance, one can reduce the number of undetermined constants to 5 for $\mathfrak{g}_{4,9}(\alpha)$, but it is quite hard to analyze its $\mathcal{R}$-critical solutions.

Remark 2. In [13], Smedt and Salamon classified anti-self-dual (hence $\mathcal{W}$-critical) left invariant metrics on solvable Lie groups and found an example which is not a Riemannian symmetric space. But this example does not have a quotient of finite volume.

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