

## ON THE INFINITE PRODUCTS DERIVED FROM THETA SERIES I

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**ABSTRACT.** Let  $k$  be an imaginary quadratic field,  $\mathfrak{h}$  the complex upper half plane, and let  $\tau \in \mathfrak{h} \cap k$ ,  $q = e^{\pi i\tau}$ . In this article, we obtain algebraic numbers from the 130 identities of Rogers-Ramanujan continued fractions investigated in [28] and [29] by using Berndt's idea ([3]). Using this, we get special transcendental numbers. For example,  $\frac{q^{1/8}}{1} + \frac{-q}{1+q} + \frac{-q^2}{1+q^2} + \dots$  ([1]) is transcendental.

### 1. Introduction

Ramanujan [23] introduced the following functions:

$$P(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) z^n, \quad Q(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) z^n,$$

$$R(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) z^n,$$

with  $\sigma_k(n) = \sum_{d|n} d^k$  and  $z \in \mathbb{C}$ .

In 1995 Barré, Diaz, Gramain and Philibert proved the long-standing conjecture of Mahler and Manin, which reads that for any nonzero algebraic number  $\alpha$  in  $D = \{\alpha \in \mathbb{C}, |\alpha| < 1\}$ , the number  $J(\alpha) = \frac{1728Q(\alpha)^3}{Q(\alpha)^3 - R(\alpha)^2}$  is transcendental. We may also state it equivalently as follows:

Let  $\alpha$  and  $\beta \in \{2, 3, 4\}$  with  $\alpha \neq \beta$ . For any non-zero  $\alpha$  in  $D = \{\alpha \in \mathbb{C}, |\alpha| < 1\}$ , algebraic over  $\mathbb{Q}$ , the number  $\frac{\theta_\alpha(\alpha)}{\theta_\beta(\alpha)}$  is transcendental ([2]) where  $\theta_\alpha$  are the classical theta series.

Meanwhile, in the case  $\tau \in \mathfrak{h} \cap k$  we show in §2 that  $\frac{\theta_\alpha(q)}{\theta_\beta(q)}$  is an algebraic number.

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We showed in [17] that  $\frac{3\wp(\frac{\tau}{2})}{\pi^2\eta(\tau)^4}$  is an algebraic integer, where  $\wp(\cdot)$  is the Weierstrass  $\wp$ -function. Similarly, we derive that  $\frac{(\wp^{(2n)}(z))_{z=\alpha}}{\pi^{2n+2}}$  ( $\alpha = \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}$ ) are transcendental numbers and  $\frac{(\wp^{(n)}(z))_{z=\alpha}}{\pi^{n+2}\eta(\tau)^{2n+4}}$  ( $\alpha = \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}$ ) are algebraic numbers, where  $(n)$  means  $n$ -th derivative function and  $n \geq 1$  (Lemma 2.4).

Now let  $\mathfrak{G}_n = 2^{-\frac{1}{4}}q^{-\frac{1}{24}}\prod_{m=1}^{\infty}(1+q^{2m-1})$  and  $\mathfrak{g}_n = 2^{-\frac{1}{4}}q^{-\frac{1}{24}}\prod_{m=1}^{\infty}(1-q^{2m-1})$  be the class invariants, for  $n > 0$  and  $q = e^{-\pi\sqrt{n}}$  ([4, p. 21], [23, p. 23], [24]). In §3 we describe the functions  $j(\tau)$ ,  $\frac{\wp(\alpha)}{\eta(\tau)^4}$  ( $\alpha = \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}$ ),  $\frac{g_2(\tau)}{\eta(\tau)^8}$ ,  $\frac{g_3(\tau)}{\eta(\tau)^{12}}$  in terms of  $\mathfrak{G}_n$ ,  $\mathfrak{g}_n$  (Remark 3.1 (b)). And, under the assumption  $d \equiv 3 \pmod{4}$ , we prove that  $\sqrt{d}\frac{\sum_{m,n=-\infty}^{\infty}q^{m^2+mn+\frac{d+1}{4}n^2}}{\eta(\tau)^2}$  is an algebraic integer by making use of [11, (3.13)].

Around 1950, Lucy J. Slater produced a list of 130 identities of the Rogers-Ramanujan type as part of her Ph.D thesis and published them in [29]. In §4, we consider such Rogers-Ramanujan identities.

Ramanujan's general theta function is defined by

$$f(a, b) := \sum_{m=-\infty}^{\infty} a^{m(m+1)/2} b^{m(m-1)/2}, \quad |ab| < 1.$$

In particular, following Ramanujan's notation, we set

$$\begin{aligned} \phi(q) &:= f(q, q) = \sum_{m=-\infty}^{\infty} q^{m^2}, \quad \psi(q) := f(q, q^3) = \sum_{m=-\infty}^{\infty} q^{m(m+1)/2}, \\ f(-q) &:= f(-q, -q^2) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m-1)/2}. \end{aligned}$$

Berndt examined many properties of Ramanujan's general theta functions in [3]. By utilizing his idea, we are able to obtain various algebraic integers derived from the theta series (Theorem 4.1). In other words, we find that  $q^a \prod_{m=1}^{\infty}(1-q^{nm-t})(1-q^{nm-(n-t)})$  are algebraic numbers, when  $(a, n, t)$  takes on the following values:

$$\begin{aligned} &(-\frac{1}{12}, 2, 1), (-\frac{1}{12}, 3, 1), (-\frac{1}{24}, 4, 1), (\frac{1}{60}, 5, 1), (-\frac{11}{60}, 5, 2), \\ &(\frac{1}{12}, 6, 1), (\frac{13}{84}, 7, 1), (-\frac{11}{84}, 7, 2), (-\frac{23}{84}, 7, 3), \\ &(\frac{11}{48}, 8, 1), (-\frac{13}{48}, 8, 3), (\frac{11}{36}, 9, 1), (-\frac{1}{36}, 9, 2), (-\frac{13}{36}, 9, 4), \\ &(\frac{23}{60}, 10, 1), (-\frac{13}{60}, 10, 3), (\frac{13}{24}, 12, 1), (-\frac{11}{24}, 12, 5), (\frac{59}{84}, 14, 1), \\ &(-\frac{1}{84}, 14, 3), (-\frac{37}{84}, 14, 5), (\frac{37}{36}, 18, 1), (-\frac{11}{36}, 18, 5), (-\frac{23}{36}, 18, 7). \end{aligned}$$

From Theorem 4.1 we can find algebraic numbers for the Rogers-Ramanujan identities in [28] and [29] (Corollary 4.3).

On the other hand, Berndt and Yee also showed that

$$(0.1) \quad \begin{aligned} R(a, e^{-x}) &:= 1 + \frac{ae^{-x}}{1} + \frac{ae^{-2x^2}}{1} + \frac{ae^{-3x^3}}{1} + \dots \\ &= \frac{-1 + \sqrt{1 + 4a}}{2a} \exp \left( \frac{ax}{1 + 4a} - \frac{a(1 - a)x^2}{2(1 + 4a)^{5/2}} \right. \\ &\quad \left. + \frac{a(1 - a)(1 - 14a)x^3}{6(1 + 4a)^4} + \dots \right), \end{aligned}$$

as  $x \rightarrow 0^+$ . Here,  $R(a, e^{-x})$  is called the generalized Rogers-Ramanujan continued fraction with  $|a| < 1$  and  $a \in \mathbb{C}$  ([6]). Using this idea, we derive the fact that

$$\prod_{m=1}^{\infty} \frac{(1 - (e^{-\pi\sqrt{d}})^{5m-1})(1 - (e^{-\pi\sqrt{d}})^{5m-4})}{(1 - (e^{-\pi\sqrt{d}})^{5m-2})(1 - (e^{-\pi\sqrt{d}})^{5m-3})} = \frac{\sqrt{5} - 1}{2} e^{\frac{\pi\sqrt{d}}{5}},$$

as  $d \in \mathbb{Q}^+$  and  $d \rightarrow 0$  (Corollary 4.2(b)).

Let  $p(n)$  denote the number of unrestricted partitions of the non-negative integer  $n$ , and for two non-negative integers  $l$  and  $k$ , we set

$$P_{l,k} := P_{l,k}(a) := \sum_{n=0}^{\infty} p(ln + k)a^n, \quad |a| < 1.$$

We then find that  $q^{\frac{(i-j)}{7}} \frac{P_{7,i}}{P_{7,j}}$  are algebraic numbers, while  $q^{\frac{-1+24i}{168}} P_{7,i}$  are transcendental numbers for  $i, j = 0, \dots, 6$  (Corollary 4.2 (d)).

## 2. Imaginary quadratic case of theta series

We fix  $\tau \in k \cap \mathfrak{h}$ , where  $k$  is an imaginary quadratic field throughout this article, and hence we readily get that for  $\tau \in k \cap \mathfrak{h}$ ,  $|q| < 1$ . The main ingredient in our work is the analytic function  $\theta(z, \tau)$  in 2 variables defined by

$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z}$$

for  $z \in \mathbb{C}$  and  $\tau \in \mathfrak{h}$ .

Now, for simplicity we write

$$\begin{aligned} \theta_{00}(z, \tau) &:= \theta(z, \tau), \\ \theta_{01}(z, \tau) &:= \theta(z + \frac{1}{2}, \tau), \\ \theta_{10}(z, \tau) &:= e^{\frac{\pi i \tau}{4} + \pi i z} \theta(z + \frac{1}{2}\tau, \tau), \\ \theta_{11}(z, \tau) &:= e^{\frac{\pi i \tau}{4} + \pi i(z + \frac{1}{2})} \theta(z + \frac{1}{2}(1 + \tau), \tau), \end{aligned}$$

and  $\theta_2(q) := \theta_{10}(0, \tau)$ ,  $\theta_3(q) := \theta_{00}(0, \tau)$ ,  $\theta_4(q) := \theta_{01}(0, \tau)$  with  $q = e^{\pi i \tau}$ .

Then we are in need of the following well known proposition for later use.

**Proposition 2.1.** ([7], [8], [27]) *Let  $\alpha$  and  $\beta \in \{2, 3, 4\}$  with  $\alpha \neq \beta$ . For any non-zero  $a$  in  $D = \{a \in \mathbb{C}, |a| < 1\}$ , at least one of the numbers  $\theta_\alpha(a)$  and  $\theta_\beta(a)$  is transcendental.*

The product expansions

$$(2.0) \quad \begin{aligned} \theta_2(a) &= 2a^{1/4} \prod_{m=1}^{\infty} (1 - a^{2m})(1 + a^{2m})^2, \\ \theta_3(a) &= \prod_{m=1}^{\infty} (1 - a^{2m})(1 + a^{2m-1})^2, \\ \theta_4(a) &= \prod_{m=1}^{\infty} (1 - a^{2m})(1 - a^{2m-1})^2 \end{aligned}$$

assert that  $\theta_3(a)$  and  $\theta_4(a)$  never vanish, while  $\theta_2(a) = 0$  if and only if  $a = 0$ .

When  $\tau \in k \cap \mathfrak{h}$ , by [13, p. 15 (13.4) and (13.5)] and [17],

$$(2.1) \quad \begin{aligned} \sqrt{2}q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 + q^m) &= \sqrt{2}q^{\frac{1}{24}} \sum_{n=0}^{\infty} \frac{(-q)_n}{(q)_n} (1 + q^{2n+1})q^{\frac{3n^2+n}{2}}, \\ q^{\frac{-1}{24}} \prod_{m=1}^{\infty} (1 - q^{2m-1}) &= q^{-\frac{1}{24}} \sum_{n=0}^{\infty} \frac{(q; q^2)_n}{(q^2; q^2)_n} (-1)^n (1 - q^{4n+1})q^{3n^2}, \\ q^{\frac{-1}{24}} \prod_{m=1}^{\infty} (1 + q^{2m-1}) &= q^{-\frac{1}{24}} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n} (1 + q^{4n+1})q^{3n^2} \quad \text{and} \\ \frac{1}{q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 + q^m)} &= q^{-\frac{1}{24}} \sum_{n=0}^{\infty} \frac{(-q)_n}{(q)_n} \end{aligned}$$

are algebraic integers, where  $(c)_j := (c; q)_j := \prod_{m=0}^{j-1} (1 - cq^m)$ .

On the other hand we are also able to express  $\theta_\alpha (\alpha = 2, 3, 4)$  as infinite products

$$(2.2) \quad \theta_2(q) = \left( \sqrt{2}q^{1/12} \prod_{m=1}^{\infty} (1 + q^{2m}) \right)^2 \left( q^{1/12} \prod_{m=1}^{\infty} (1 - q^{2m}) \right),$$

$$(2.3) \quad \theta_3(q) = \left( q^{-1/24} \prod_{m=1}^{\infty} (1 + q^{2m-1}) \right)^2 \left( q^{1/12} \prod_{m=1}^{\infty} (1 - q^{2m}) \right),$$

$$(2.4) \quad \theta_4(q) = \left( q^{-1/24} \prod_{m=1}^{\infty} (1 - q^{2m-1}) \right)^2 \left( q^{1/12} \prod_{m=1}^{\infty} (1 - q^{2m}) \right).$$

Thus, we derive from (2.1)~(2.4) and Proposition 2.1 that

$$(2.5) \quad \eta(\tau) = q^{1/12} \prod_{m=1}^{\infty} (1 - q^{2m}) \text{ is transcendental.}$$

We then conclude from (2.0)~(2.5) that

$$(2.6) \quad \theta_2(q), \theta_3(q), \text{ and } \theta_4(q)$$

are all transcendental numbers and

$$(2.7) \quad \frac{\theta_i(q)}{\eta(\tau)} (i = 2, 3, 4) \text{ are algebraic integers.}$$

Combining these results, we get that

$$(2.8) \quad \frac{\theta_i(q)}{\theta_j(q)} (i, j = 2, 3, 4) \text{ are algebraic numbers.}$$

*Remark 2.2.* (a) In [17] we showed the infinite product forms of the Eisenstein series

$$\begin{aligned} g_2(\tau) &= \frac{4\pi^4}{3} \prod_{n=1}^{\infty} (1 - q^{2n})^8 \left[ \prod_{n=1}^{\infty} (1 + q^{(2n-1)})^{16} \right. \\ &\quad \left. - 16q \prod_{n=1}^{\infty} (1 + q^n)^8 + 256q^2 \prod_{n=1}^{\infty} (1 + q^{2n})^{16} \right] \end{aligned}$$

and

$$\begin{aligned} g_3(\tau) &= \frac{8\pi^6}{27} \prod_{n=1}^{\infty} (1 - q^{2n})^{12} \left( \prod_{n=1}^{\infty} (1 + q^{2n-1})^{24} \right. \\ &\quad \left. - 24q \prod_{n=1}^{\infty} (1 + q^{2n-1})^{16} (1 + q^{2n})^8 \right. \\ &\quad \left. - 384q^2 \prod_{n=1}^{\infty} (1 + q^{2n-1})^8 (1 + q^{2n})^{16} \right. \\ &\quad \left. + 4096q^3 \prod_{n=1}^{\infty} (1 + q^{2n})^{24} \right). \end{aligned}$$

It then follows from (2.1) and (2.5) that  $\frac{3g_2(\tau)}{4\pi^4} = Q(q^2)$  and  $\frac{27g_3(\tau)}{\pi^6} = R(q^2)$  are transcendental numbers. Here,

$$Q(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) z^n, \quad R(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) z^n,$$

with  $\sigma_k(n) = \sum_{d|n} d^k$  and  $z \in \mathbb{C}$ .

(b) Let

$$E_2^\theta(\tau) = \sum_{m,n \in \mathbb{Z}} \left[ \frac{1}{(2m\tau + (2n+1))^2} - \frac{1}{((2m+1)\tau + 2n)^2} \right].$$

It is well-known([22, p. 81]) that

$$\begin{aligned}
 E_2^\theta(\tau) &= \frac{\pi^2}{4} \theta_{00}(0, \tau)^4 \\
 (2.9) \quad &= \frac{\pi^2}{4} \prod_{m=1}^{\infty} (1 - q^{2m})^4 (1 + q^{2m-1})^8 \\
 &= \frac{\pi^2}{4} \left( q^{1/12} \prod_{m=1}^{\infty} (1 - q^{2m}) \right)^4 \left( q^{-1/24} \prod_{m=1}^{\infty} (1 + q^{2m-1}) \right)^8.
 \end{aligned}$$

Then we can deduce from (2.1) and (2.9) that  $\frac{4E_2^\theta(\tau)}{\pi^2 \eta(\tau)^4}$  is an algebraic integer and  $\frac{4E_2^\theta(\tau)}{\pi^2}$  is a transcendental number for  $\tau \in k \cap \mathfrak{h}$ .

Now, let  $\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$  with  $b \bmod d$  and  $|\alpha|$  be the determinant of  $\alpha$ , and set

$$\phi_\alpha(\tau) := |\alpha|^{12} \frac{\Delta(\alpha(\frac{\tau}{1}))}{\Delta((\frac{\tau}{1}))} = |\alpha|^{12} d^{-12} \frac{\Delta(\alpha\tau)}{\Delta(\tau)}$$

with  $\Delta(\tau) = (2\pi)^{12} q^2 \prod_{m=1}^{\infty} (1 - q^{2m})^{24}$ .

We recall the following fact.

**Proposition 2.3.** [20] *For any  $\tau \in k \cap \mathfrak{h}$ , the value  $\phi_\alpha(\tau)$  is an algebraic integer, which divides  $|\alpha|^{12}$ .*

Next, we recall that

$$\begin{aligned}
 \wp(z) &:= \wp(z; \Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2}, \\
 \wp'(z)^2 &= 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau), \\
 \wp^{(2)}(z) &:= \wp''(z) = 6\wp(z)^2 + g_2(\tau), \\
 \wp^{(3)}(z) &= 12\wp(z)\wp'(z) \quad \text{with } \Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau.
 \end{aligned}$$

By differentiation, we get that

$$\begin{aligned}
 \wp^{(4)}(z) &= 84\wp(z)^3 - 12g_3(\tau), \quad \wp^{(5)}(z) = 252\wp'(z)\wp(z)^2, \\
 \wp^{(6)}(z) &= 2016\wp(z)^4 - 252g_2(\tau)\wp(z)^2 - 504g_3(\tau)\wp(z), \\
 \wp^{(7)}(z) &= \wp'(z)(8064\wp(z)^3 - 504g_2(\tau)\wp(z) - 504g_3(\tau)), \\
 \wp^{(8)}(z) &= 145152\wp(z)^5 - 21168g_2(\tau)\wp(z)^3 - 27216g_3(\tau)\wp(z)^2, \\
 \wp^{(9)}(z) &= \wp'(z)(725760\wp(z)^4 - 63504g_2(\tau)\wp(z)^2 - 54432g_3(\tau)\wp(z)), \\
 \wp^{(10)}(z) &= 7257600\wp(z)^6 - 2685312g_2(\tau)\wp(z)^4 + 63504g_2(\tau)^2\wp(z)^2 \\
 &\quad - 3284064g_3(\tau)\wp(z)^3 + 127008g_2(\tau)g_3(\tau)\wp(z) + 54432g_3(\tau)^2, \\
 &\quad \dots, \\
 (2.10) \quad \wp^{(2n)}(z) &= \alpha\wp(z)^{n+1} + \sum_{8\gamma_i+12\delta_i+4\epsilon_i=4n+4} \beta_i g_2(\tau)^{\gamma_i} g_3(\tau)^{\delta_i} \wp(z)^{\epsilon_i},
 \end{aligned}$$

(2.11)

$$\wp^{(2n+1)}(z) = \wp'(z) \left( \alpha' \wp(z)^n + \sum_{8\gamma'_i + 12\delta'_i + 4\epsilon'_i = 4n} \beta'_i g_2(\tau)^{\gamma'_i} g_3(\tau)^{\delta'_i} \wp(z)^{\epsilon'_i} \right)$$

with  $n \geq 1$ ,  $\alpha, \alpha', \beta, \beta', \delta_i, \delta'_i, \epsilon_i, \epsilon'_i, \gamma_i, \gamma'_i \in \mathbb{Z}$ .

We showed in [17] that

$$(2.12) \quad \frac{3}{\pi^2} \frac{\wp(\frac{\tau}{2})}{\eta(\tau)^4}, \quad \frac{3}{\pi^2} \frac{\wp(\frac{\tau+1}{2})}{\eta(\tau)^4}, \quad \frac{3}{\pi^2} \frac{\wp(\frac{1}{2})}{\eta(\tau)^4}, \quad \frac{3}{4\pi^4} \frac{g_2(\tau)}{\eta(\tau)^8} \text{ and } \frac{27}{\pi^6} \frac{g_3(\tau)}{\eta(\tau)^{12}}$$

are algebraic integers.

Then by using (2.10), (2.11) and (2.12) we have the following lemma.

**Lemma 2.4.** *Notations being the same as above, assume that the values of all series are non-zero.*

(a)  $\frac{(\wp^{(n)}(z))_{z=\alpha}}{\pi^{n+2}\eta(\tau)^{2n+4}}$  is an algebraic number for  $\alpha = \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}$  and  $n \geq 1$ .

(b)  $\frac{(\wp^{(2n)}(z))_{z=\alpha}}{\pi^{2n+2}}$  is a transcendental number for  $\alpha = \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}$  and  $n \geq 1$ .

### 3. Algebraic number for Ramanujan

As usual, we set

$$\begin{aligned} (a; q)_n &:= \prod_{m=0}^{n-1} (1 - aq^m), \\ (a; q)_\infty &:= \prod_{m=0}^{\infty} (1 - aq^m), \\ (a, b, \dots, c; q)_\infty &:= (a; q)_\infty (b; q)_\infty \cdots (c; q)_\infty, \quad |q| < 1. \end{aligned}$$

Following Ramanujan, we define

$$f(-q^2) := (q^2; q^2)_\infty =: e^{-2\pi i \tau/24} \eta(\tau).$$

We see from [3, pp. 36–37] and famous result of Euler that

$$\begin{aligned} (3.1) \quad \psi(q) &= \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \\ &= \frac{\prod_{m=0}^{\infty} (1 - q^{2m} \cdot q^{2m})}{\prod_{m=0}^{\infty} (1 - q \cdot q^{2m})} = \frac{\prod_{m=1}^{\infty} (1 - q^{2m})}{\prod_{m=1}^{\infty} (1 - q^{2m-1})} \end{aligned}$$

and

$$(3.2) \quad \prod_{m=1}^{\infty} (1 - q^{2m})(1 - q^{2m-1})(1 + q^{2m-1})(1 + q^{2m}) = \prod_{m=1}^{\infty} (1 - q^{2m}).$$

By (3.2) we get that

$$\begin{aligned}
 & \psi(q) \\
 &= \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m-1})(1 + q^{2m}) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^m) \\
 (3.3) \quad &= \frac{1}{\sqrt{2}} q^{-1/12} q^{-1/24} \left( q^{1/12} \prod_{m=1}^{\infty} (1 - q^{2m}) \right) \left( \sqrt{2} q^{1/24} \prod_{m=1}^{\infty} (1 + q^m) \right) \\
 &= \frac{1}{\sqrt{2}} q^{-1/8} \left( q^{1/12} \prod_{m=1}^{\infty} (1 - q^{2m}) \right) \left( \sqrt{2} q^{1/24} \prod_{m=1}^{\infty} (1 + q^m) \right)
 \end{aligned}$$

and

$$\begin{aligned}
 \psi(-q) &= \prod_{m=1}^{\infty} (1 - q^m)(1 + q^{2m}) \\
 (3.4) \quad &= \frac{1}{\sqrt{2}} q^{-1/8} \left( q^{1/24} \prod_{m=1}^{\infty} (1 - q^m) \right) \left( \sqrt{2} q^{1/12} \prod_{m=1}^{\infty} (1 + q^{2m}) \right).
 \end{aligned}$$

Then, using (3.2) we obtain that

$$\begin{aligned}
 & \prod_{m=1}^{\infty} \frac{\left( \sqrt{2} q^{\frac{1}{12}} (1 + q^{2m}) \right) \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right)}{1} \\
 &= \prod_{m=1}^{\infty} \frac{\left( \sqrt{2} q^{\frac{1}{12}} (1 + q^{2m}) \right) \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right)}{(1 + q^{2m})(1 + q^{2m-1})(1 - q^{2m-1})} \\
 &= \prod_{m=1}^{\infty} \frac{\sqrt{2} q^{\frac{1}{24}}}{(1 - q^{2m-1})}
 \end{aligned}$$

and

$$\left( \sqrt{2} q^{\frac{1}{12}} \prod_{m=1}^{\infty} (1 + q^{2m}) \right) \left( q^{\frac{-1}{24}} \prod_{m=1}^{\infty} (1 - q^{2m-1}) \right) = \prod_{m=1}^{\infty} \frac{\sqrt{2} q^{\frac{1}{24}}}{(1 + q^{2m-1})}.$$

Thus by (2.1) we assert that

$$(3.5) \quad \frac{\sqrt{2} q^{\frac{1}{24}}}{\prod_{m=1}^{\infty} (1 - q^{2m-1})} \quad \text{and} \quad \frac{\sqrt{2} q^{\frac{1}{24}}}{\prod_{m=1}^{\infty} (1 + q^{2m-1})}$$

are algebraic integers.

Therefore, we derive from (2.1), (2.5) and (3.3)~(3.5) that

$$(3.6) \quad \frac{\sqrt{2} q^{1/8} \psi(\pm q)}{\eta(\tau)} \quad \text{and} \quad \frac{\eta(\tau)}{q^{1/8} \psi(\pm q)}$$

are algebraic integers and

$$(3.7) \quad q^{1/8} \psi(\pm q)$$

is a transcendental number.

Now we use the results from [3, p. 36], which says that

$$(3.8) \quad \begin{aligned} \phi(q) &= \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (-q^2; q^2)_\infty} = \frac{\prod_{m=1}^{\infty} (1 + q^{2m-1})(1 - q^{2m})}{\prod_{m=1}^{\infty} (1 - q^{2m-1})(1 + q^{2m})} \\ &= \prod_{m=1}^{\infty} (1 + q^{2m-1})^2 (1 - q^{2m}) = \theta_3(q) \end{aligned}$$

and

$$(3.9) \quad \phi(-q) = \prod_{m=1}^{\infty} (1 - q^{2m-1})^2 (1 - q^{2m}) = \theta_4(q).$$

Then, by (2.1) and (3.5) we conclude that

$$(3.10) \quad \frac{\phi(\pm q)}{\eta(\tau)} \quad \text{and} \quad \frac{2\eta(\tau)}{\phi(\pm q)}$$

are algebraic integers.

*Remark 3.1.* (a) It is well known that  $\frac{\theta_3(q^m)}{\theta_3(q^n)}$  is algebraic for  $m, n \in \mathbb{Q}$  ([5], [17]). And, in [3, pp. 103–104] we see that

$$\begin{aligned} \theta_3(e^{-\pi}) &= \frac{\pi^{1/4}}{\Gamma(\frac{3}{4})}, \quad \theta_3(e^{-\pi\sqrt{2}}) = \frac{\Gamma(\frac{9}{8})}{\Gamma(\frac{5}{4})} \sqrt{\frac{\Gamma(\frac{1}{4})}{2^{1/4}\pi}}, \\ \theta_3(e^{\pi i 2i}) &= \frac{\sqrt{2+\sqrt{2}}}{2} \frac{\pi^{1/4}}{\Gamma(\frac{3}{4})}, \end{aligned}$$

where  $\Gamma(\cdot)$  is the Gamma function. Here we can find examples for Remark 3.3([18]) and Theorem 4.4([5]), i.e.,

$$\left( \frac{\theta_3(e^{-\pi})}{\theta_3(e^{-2\pi})} \right)^4 - 8 \left( \frac{\theta_3(e^{-\pi})}{\theta_3(e^{-2\pi})} \right)^2 + 8 = 0$$

and

$$\left( 2 \frac{\theta_3(e^{-2\pi})}{\theta_3(e^{-\pi})} \right)^4 - 4 \left( 2 \frac{\theta_3(e^{-2\pi})}{\theta_3(e^{-\pi})} \right)^2 + 2 = 0.$$

Next, a theorem of Hurwitz([15], [26]) says that

$$g_2(i) = \frac{4\pi^4}{3} Q(e^{-2\pi}) = \frac{1}{16\pi^2} \Gamma(\frac{1}{4})^8.$$

From this we can also find an example for (2.7) such as

$$\eta(i)^{24} = \frac{1}{(2\pi)^{12}} \left( \frac{1}{16\pi^2} \Gamma(\frac{1}{4})^8 \right)^3 = \frac{1}{2^{24}\pi^{18}} \Gamma(\frac{1}{4})^{24},$$

and hence

$$\frac{\theta_3(e^{-\pi})^{24}}{\eta(i)^{24}} = \frac{2^{24}\pi^{24}}{\Gamma(\frac{1}{4})^{24}\Gamma(\frac{3}{4})^{24}} = \frac{2^{24}\pi^{24}}{(\sqrt{2}\pi)^{24}} = 2^{12}$$

because  $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$  ([30, p. 239]). So,  $\prod_{m=1}^{\infty} (1 + e^{-\pi(2m-1)}) = 2^{\frac{1}{4}} \zeta e^{\frac{-\pi}{24}}$  with  $\zeta^{48} = 1$ .

(b) Following Ramanujan [23, p. 23], [24], we define two class invariants  $\mathfrak{G}_n = 2^{-\frac{1}{4}} q^{-\frac{1}{24}} (-q; q^2)_\infty$  and  $\mathfrak{g}_n = 2^{-\frac{1}{4}} q^{-\frac{1}{24}} (q; q^2)_\infty$  for  $n > 0$  and  $q = e^{-\pi\sqrt{n}}$ .

Berndt showed in [4, pp. 21-22] that

$$\mathfrak{g}_{4n} = 2^{\frac{1}{4}} \mathfrak{g}_n \mathfrak{G}_n, \quad (\mathfrak{g}_n \mathfrak{G}_n)^8 (\mathfrak{G}_n^8 - \mathfrak{g}_n^8) = \frac{1}{4} \quad \text{and} \quad \mathfrak{G}_{25} = \frac{1 + \sqrt{5}}{2}.$$

And, Jacobi ([30, p. 470]) showed that

$$\prod_{m=1}^{\infty} (1 + q^{2m-1})^8 - \prod_{m=1}^{\infty} (1 - q^{2m-1})^8 = 16q \prod_{m=1}^{\infty} (1 + q^{2m})^8.$$

Then by using [17], Jacobi's identity and class invariants we have that

$$\begin{aligned} \frac{-3\wp(\frac{\tau}{2})}{\pi^2 \eta(\tau)^4} &= q^{-\frac{1}{3}} \prod_{m=1}^{\infty} (1 + q^{2m-1})^8 + 16q^{\frac{2}{3}} \prod_{m=1}^{\infty} (1 + q^{2m})^8 \\ &= 8\mathfrak{G}_n^8 - 4\mathfrak{g}_n^8, \\ \frac{-3\wp(\frac{\tau+1}{2})}{\pi^2 \eta(\tau)^4} &= q^{-\frac{1}{3}} \prod_{m=1}^{\infty} (1 + q^{2m-1})^8 - 32q^{\frac{2}{3}} \prod_{m=1}^{\infty} (1 + q^{2m})^8 \\ &= -4\mathfrak{G}_n^8 + 8\mathfrak{g}_n^8, \\ \frac{-3\wp(\frac{1}{2})}{\pi^2 \eta(\tau)^4} &= -2q^{-\frac{1}{3}} \prod_{m=1}^{\infty} (1 + q^{2m-1})^8 + 16q^{\frac{2}{3}} \prod_{m=1}^{\infty} (1 + q^{2m})^8 \\ &= -4\mathfrak{G}_n^8 - 4\mathfrak{g}_n^8, \\ j(\tau)^{\frac{1}{3}} &= \frac{3g_2(\tau)}{4\pi^4 \eta(\tau)^8} \\ &= q^{-\frac{2}{3}} \prod_{m=1}^{\infty} (1 + q^{2m-1})^{16} - 16 \prod_{m=1}^{\infty} \left( q^{-\frac{1}{3}} (1 + q^{2m-1})^8 \right) \\ &\quad \times \left( q^{\frac{2}{3}} (1 + q^{2m})^8 \right) + 256q^{\frac{4}{3}} \prod_{m=1}^{\infty} (1 + q^{2m})^{16} \\ &= 16(\mathfrak{G}_n^{16} - \mathfrak{G}_n^8 \mathfrak{g}_n^8 + \mathfrak{g}_n^{16}), \end{aligned}$$

$$\begin{aligned}
\frac{27g_3(\tau)}{8\pi^6\eta(\tau)^{12}} &= q^{-1} \prod_{m=1}^{\infty} (1+q^{2m-1})^{24} - 24 \prod_{m=1}^{\infty} \left( q^{-\frac{2}{3}} (1+q^{2m-1})^{16} \right) \\
&\quad \times \left( q^{\frac{2}{3}} (1+q^{2m})^8 \right) - 384 \prod_{m=1}^{\infty} \left( q^{-\frac{1}{3}} (1+q^{2m-1})^8 \right) \\
&\quad \times \left( q^{\frac{4}{3}} (1+q^{2m})^{16} \right) + 4096q^2 \prod_{m=1}^{\infty} (1+q^{2m})^{24} \\
&= -32(2\mathfrak{G}_n^{24} - 3\mathfrak{G}_n^{16}\mathfrak{g}_n^8 - 3\mathfrak{G}_n^8\mathfrak{g}_n^{16} + 2\mathfrak{g}_n^{24}).
\end{aligned}$$

As usual, if  $a, b \in K$ , then  $E : y^2 = 4x^3 - ax - b$  is said to be defined over  $K$ . We then readily check that

$$\begin{aligned}
\left( \frac{27\wp'(z)}{8\pi^3\eta^6(\tau)} \right)^2 &= 4 \left( \frac{9\wp(z)}{4\pi^2\eta^4(\tau)} \right)^3 - \frac{27}{4} \left( \frac{9\wp(z)}{4\pi^2\eta^4(\tau)} \right) \left( \frac{3g_2(\tau)}{4\pi^4\eta(\tau)^8} \right) \\
&\quad - \frac{27}{8} \left( \frac{27g_3(\tau)}{8\pi^6\eta(\tau)^{12}} \right).
\end{aligned}$$

And we derive that

$$\begin{aligned}
Y^2 &= X^3 - \frac{27}{16}G_2X - \frac{27}{32}G_3 \\
&= X^3 - 27(\mathfrak{G}_n^{16} - \mathfrak{G}_n^8\mathfrak{g}_n^8 + \mathfrak{g}_n^{16})X \\
&\quad + 27(2\mathfrak{G}_n^{24} - 3\mathfrak{G}_n^{16}\mathfrak{g}_n^8 - 3\mathfrak{G}_n^8\mathfrak{g}_n^{16} + 2\mathfrak{g}_n^{24})
\end{aligned}$$

is an elliptic curve defined over  $\overline{\mathbb{Q}}$ , where

$$G_2 = \frac{3g_2(\tau)}{4\pi^4\eta(\tau)^8}, G_3 = \frac{27g_3(\tau)}{8\pi^6\eta(\tau)^{12}}.$$

Now, suppose  $d \equiv 3 \pmod{4}$ . Then we see from [9, p. 1738] that

$$(3.11) \quad \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+\frac{d+1}{4}n^2} = \phi(q)\phi(q^d) + 4q^{\frac{d+1}{4}}\psi(q^2)\psi(q^{2d}).$$

Substituting (3.3) and (3.8) into (3.11), we obtain that

$$\begin{aligned}
\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+\frac{d+1}{4}n^2} &= \phi(q)\phi(q^d) + \left( 2q^{1/4}\psi(q^2) \right) \left( 2q^{d/4}\psi(q^{2d}) \right) \\
&= \prod_{m=1}^{\infty} \left( q^{\frac{-1}{24}} (1+q^{2m-1}) \right)^2 \left( q^{\frac{1}{12}} (1-q^{2m}) \right) \\
&\quad \times \left( q^{-\frac{d}{24}} (1+q^{d(2m-1)}) \right)^2 \left( q^{\frac{d}{12}} (1-q^{2dm}) \right)
\end{aligned}$$

$$\begin{aligned}
& + \prod_{m=1}^{\infty} \left[ \left( 2q^{\frac{1}{4}} \right) \left( \frac{1}{\sqrt{2}q^{\frac{1}{4}}} \right) \left( q^{\frac{1}{6}}(1 - q^{4m}) \right) \left( \sqrt{2}q^{\frac{1}{12}}(1 + q^{2m}) \right) \right] \\
& \cdot \prod_{m=1}^{\infty} \left[ \left( 2q^{\frac{d}{4}} \right) \left( \frac{1}{\sqrt{2}q^{\frac{2d}{8}}} \right) \left( q^{\frac{2d}{12}}(1 - q^{4dm}) \right) \left( \sqrt{2}q^{\frac{d}{12}}(1 + q^{2dm}) \right) \right] \\
(3.12) \quad & = \prod_{m=1}^{\infty} \left( q^{\frac{-1}{24}}(1 + q^{2m-1}) \right)^2 \left( q^{-\frac{d}{24}}(1 + q^{d(2m-1)}) \right)^2 \eta(\tau)\eta(d\tau) \\
& + \prod_{m=1}^{\infty} 2 \left( \sqrt{2}q^{\frac{1}{12}}(1 + q^{2m}) \right) \left( \sqrt{2}q^{\frac{d}{12}}(1 + q^{2dm}) \right) \eta(2\tau)\eta(2d\tau).
\end{aligned}$$

Here, one can easily check that  $\sqrt{d}\frac{\eta(d\tau)}{\eta(\tau)}$  and  $\frac{\eta(\tau)}{\eta(d\tau)}$  are algebraic integers for  $d \in \mathbb{Z}^+$ . Hence, it follows from (2.1) and Proposition 2.3 that

$$(3.13) \quad \frac{\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+\frac{d+1}{4}n^2}}{\eta(\tau)\eta(d\tau)} \text{ and } \sqrt{d} \frac{\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+\frac{d+1}{4}n^2}}{\eta(\tau)^2}$$

are algebraic integers, and so

$$(3.14) \quad \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+\frac{d+1}{4}n^2}$$

is transcendental as desired.

#### 4. Approach to R. Slater's identities ([28],[29])

For convenience, we set  $\mathfrak{T} = \{t|t \text{ is transcendental}\}$ ,  $\mathfrak{A} = \{a|a \text{ is an algebraic integer}\}$  and  $\overline{\mathbb{Q}} = \{q|q \text{ is an algebraic number}\}$ .

By (2.1) and Proposition 2.3 we see that

$$\begin{aligned}
& q^{-\frac{1}{12}} \prod_{m=1}^{\infty} (1 \pm q^{2m-1})(1 \pm q^{2m-1}), \\
& q^{-\frac{1}{12}} \prod_{m=1}^{\infty} (1 \pm q^{3m-1})(1 \pm q^{3m-2}) = \prod_{m=1}^{\infty} \frac{q^{\frac{1}{24}}(1 \pm q^m)}{q^{\frac{3}{24}}(1 \pm q^{3m})}, \\
& q^{-\frac{1}{24}} \prod_{m=1}^{\infty} (1 \pm q^{4m-1})(1 \pm q^{4m-3}) = q^{-\frac{1}{24}} \prod_{m=1}^{\infty} (1 \pm q^{2m-1})
\end{aligned}$$

and

$$q^{\frac{1}{12}} \prod_{m=1}^{\infty} (1 \pm q^{6m-1})(1 \pm q^{6m-5}) = \prod_{m=1}^{\infty} \frac{q^{-\frac{1}{24}}(1 \pm q^{2m-1})}{q^{-\frac{3}{24}}(1 \pm q^{6m-3})}$$

are algebraic numbers with double signs in the same order.

Now, we apply Berndt's relations [3, p. 51] to theta functions, namely,

$$\phi(-q) + \phi(q^2) = 2 \frac{f^2(q^3, q^5)}{\psi(q)} \quad \text{and} \quad \phi(-q) - \phi(q^2) = -2q \frac{f^2(q, q^7)}{\psi(q)}$$

with  $f(a, b) = (-a; ab)_\infty(-b; ab)_\infty(ab; ab)_\infty$ . Then by routine calculations we can show that

$$\begin{aligned} 2\sqrt{2}q^{-\frac{13}{24}} \prod_{m=1}^{\infty} (1 + q^{8m-3})^2 (1 + q^{8m-5})^2 &= \left( \frac{\phi(-q)}{\eta(4\tau)} \right) \left( \sqrt{2}q^{\frac{1}{8}} \frac{\psi(q)}{\eta(4\tau)} \right) \\ &\quad + \left( \frac{\phi(q^2)}{\eta(4\tau)} \right) \left( \sqrt{2}q^{\frac{1}{8}} \frac{\psi(q)}{\eta(4\tau)} \right) \end{aligned}$$

and

$$\begin{aligned} -2\sqrt{2}q^{\frac{11}{24}} \prod_{m=1}^{\infty} (1 + q^{8m-1})^2 (1 + q^{8m-7})^2 &= \left( \frac{\phi(-q)}{\eta(4\tau)} \right) \left( \sqrt{2}q^{\frac{1}{8}} \frac{\psi(q)}{\eta(4\tau)} \right) \\ &\quad - \left( \frac{\phi(q^2)}{\eta(4\tau)} \right) \left( \sqrt{2}q^{\frac{1}{8}} \frac{\psi(q)}{\eta(4\tau)} \right). \end{aligned}$$

Thus, we deduce from [3, p. 51], (3.6) and (3.10) that

$$\sqrt{2\sqrt{2}q^{-\frac{13}{48}}} \prod_{m=1}^{\infty} (1 + q^{8m-3})(1 + q^{8m-5}) \in \mathfrak{A}$$

and

$$\sqrt{2\sqrt{2}q^{\frac{11}{48}}} \prod_{m=1}^{\infty} (1 + q^{8m-1})(1 + q^{8m-7}) \in \mathfrak{A}.$$

We now replace  $q$  by  $-q$  in [3, p. 51]. Then we obtain that

$$\sqrt{2\sqrt{2}q^{-\frac{13}{48}}} \prod_{m=1}^{\infty} (1 - q^{8m-3})(1 - q^{8m-5}) \in \mathfrak{A}$$

and

$$\sqrt{2\sqrt{2}q^{\frac{11}{48}}} \prod_{m=1}^{\infty} (1 - q^{8m-1})(1 - q^{8m-7}) \in \mathfrak{A}.$$

Berndt showed in [3, p. 262] that

$$(4.0) \quad f(-q)\{f(-q^{1/5}) + q^{1/5}f(-q^5)\} = f^2(-q^2, -q^3) - q^{2/5}f^2(-q, -q^4),$$

in other words,

$$\begin{aligned}
 & \prod_{m=1}^{\infty} (1 - q^{5m-2})^2 (1 - q^{5m-3})^2 (1 - q^{5m})^2 - q^{2/5} \\
 (4.1) \quad & \times \prod_{m=1}^{\infty} (1 - q^{5m-1})^2 (1 - q^{5m-4})^2 (1 - q^{5m})^2 \\
 & = \prod_{m=1}^{\infty} (1 - q^m) \left\{ \prod_{m=1}^{\infty} (1 - q^{\frac{m}{5}}) + q^{1/5} \prod_{m=1}^{\infty} (1 - q^{5m}) \right\}.
 \end{aligned}$$

Thus, we deduce from (4.0) and Entry 9(vii) of [3, p. 258] that

$$\begin{aligned}
 & q^{-\frac{11}{30}} \prod_{m=1}^{\infty} (1 - q^{5m-2})^2 (1 - q^{5m-3})^2 \\
 & - q^{\frac{1}{30}} \prod_{m=1}^{\infty} (1 - q^{5m-1})^2 (1 - q^{5m-4})^2 \\
 & = \left( \frac{q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)}{q^{\frac{5}{24}} \prod_{m=1}^{\infty} (1 - q^{5m})} \right) \left[ \frac{q^{\frac{1}{120}} \prod_{m=1}^{\infty} (1 - q^{\frac{m}{5}})}{q^{\frac{5}{24}} \prod_{m=1}^{\infty} (1 - q^{5m})} + 1 \right] \in \mathfrak{A}
 \end{aligned}$$

and

$$\begin{aligned}
 & \left( q^{-\frac{11}{30}} \prod_{m=1}^{\infty} (1 - q^{5m-2})^2 (1 - q^{5m-3})^2 \right) \\
 & \times \left( -q^{\frac{1}{30}} \prod_{m=1}^{\infty} (1 - q^{5m-1})^2 (1 - q^{5m-4})^2 \right) \\
 & = - \left( \frac{q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)}{q^{\frac{5}{24}} \prod_{m=1}^{\infty} (1 - q^{5m})} \right)^2 \in \mathfrak{A}.
 \end{aligned}$$

It is not difficult to check that

$$-q^{\frac{1}{30}} \prod_{m=1}^{\infty} (1 - q^{5m-1})^2 (1 - q^{5m-4})^2$$

and

$$q^{-\frac{11}{30}} \prod_{m=1}^{\infty} (1 - q^{5m-2})^2 (1 - q^{5m-3})^2$$

are roots of the polynomial equation

$$\begin{aligned}
 t^2 - \prod_{m=1}^{\infty} \left( \frac{q^{\frac{1}{24}} (1 - q^m)}{q^{\frac{5}{24}} (1 - q^{5m})} \right) \left( \frac{q^{\frac{1}{120}} (1 - q^{\frac{m}{5}})}{q^{\frac{5}{24}} (1 - q^{5m})} + 1 \right) t \\
 + \prod_{m=1}^{\infty} \left( - \frac{q^{\frac{1}{24}} (1 - q^m)}{q^{\frac{5}{24}} (1 - q^{5m})} \right) = 0.
 \end{aligned}$$

Here we recall the fact that if  $a_0, \dots, a_{n-1}$  are algebraic integers then any solution to  $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$  is also an algebraic integer([16, p. 77]).

Hence, it follows that

$$(4.2) \quad \sqrt{2}q^{\frac{1}{60}} \prod_{m=1}^{\infty} (1 - q^{5m-1})(1 - q^{5m-4}) \in \mathfrak{A}$$

and

$$(4.3) \quad \sqrt{2}q^{-\frac{11}{60}} \prod_{m=1}^{\infty} (1 - q^{5m-2})(1 - q^{5m-3}) \in \mathfrak{A}.$$

On the other hand, we know that

$$\begin{aligned} & q^{\frac{1}{60}} \prod_{m=1}^{\infty} (1 + q^{5m-1})(1 + q^{5m-4}) \\ &= \frac{q^{\frac{2}{60}} \prod_{m=1}^{\infty} (1 - q^{10m-2})(1 - q^{10m-8})}{q^{\frac{1}{60}} \prod_{m=1}^{\infty} (1 - q^{5m-1})(1 - q^{5m-4})} \in \overline{\mathbb{Q}} \end{aligned}$$

and

$$\begin{aligned} & q^{-\frac{11}{60}} \prod_{m=1}^{\infty} (1 + q^{5m-2})(1 + q^{5m-3}) \\ &= \frac{q^{-\frac{22}{60}} \prod_{m=1}^{\infty} (1 - q^{10m-4})(1 - q^{10m-6})}{q^{-\frac{11}{60}} \prod_{m=1}^{\infty} (1 - q^{5m-2})(1 - q^{5m-3})} \in \overline{\mathbb{Q}}. \end{aligned}$$

From the Entry 10.(ii) ([3, p. 262]) we see that

$$\begin{aligned} & -2q^{\frac{-13}{12}} \prod_{m=1}^{\infty} (1 - q^{50m-15})(1 - q^{50m-35}) \\ &+ 2q^{\frac{23}{12}} \prod_{m=1}^{\infty} (1 - q^{50m-5})(1 - q^{50m-45}) \\ &= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{2}{24}}(1 - q^{2m})}{q^{\frac{50}{24}}(1 - q^{50m})} \right) \left( q^{\frac{-1}{24}}(1 - q^{2m-1}) \right)^2 \\ &- \left( q^{-\frac{25}{24}} \prod_{m=1}^{\infty} (1 - q^{50m-25}) \right)^2 \in \mathfrak{A} \end{aligned}$$

and

$$\begin{aligned} & \left( -2q^{\frac{-13}{12}} \prod_{m=1}^{\infty} (1 - q^{50m-15})(1 - q^{50m-35}) \right) \\ & \times \left( 2q^{\frac{23}{12}} \prod_{m=1}^{\infty} (1 - q^{50m-5})(1 - q^{50m-45}) \right) \\ & = 2\sqrt{2} \left( \frac{q^{\frac{5}{24}} \prod_{m=1}^{\infty} (1 - q^{5m})}{q^{\frac{10}{24}} \prod_{m=1}^{\infty} (1 - q^{10m})} \right) \left( \sqrt{2} \frac{q^{\frac{25}{24}}}{\prod_{m=1}^{\infty} (1 - q^{50m-25})} \right) \in \mathfrak{A}. \end{aligned}$$

Therefore, we get that

$$\begin{aligned} & 2q^{\frac{-13}{60}} \prod_{m=1}^{\infty} (1 - q^{10m-3})(1 - q^{10m-7}) \in \mathfrak{A}, \\ & 2q^{\frac{23}{60}} \prod_{m=1}^{\infty} (1 - q^{10m-1})(1 - q^{10m-9}) \in \mathfrak{A}. \end{aligned}$$

Moreover, replacing  $q$  by  $-q$  leads us to the fact that

$$\begin{aligned} & 2q^{\frac{-13}{60}} \prod_{m=1}^{\infty} (1 + q^{10m-3})(1 + q^{10m-7}) \in \mathfrak{A}, \\ & 2q^{\frac{23}{60}} \prod_{m=1}^{\infty} (1 + q^{10m-1})(1 + q^{10m-9}) \in \mathfrak{A}. \end{aligned}$$

Let us define

$$\begin{aligned} s &= -q^{\frac{22}{24}} \prod_{m=1}^{\infty} (1 - q^{27m-3})(1 - q^{27m-24}), \\ t &= -q^{\frac{-2}{24}} \prod_{m=1}^{\infty} (1 - q^{27m-6})(1 - q^{27m-21}), \\ u &= q^{\frac{-26}{24}} \prod_{m=1}^{\infty} (1 - q^{27m-12})(1 - q^{27m-15}), \end{aligned}$$

and set  $a = -q$ ,  $b = -q^2$  and  $n = 3$  in the Entry 31 of Chapter 16 ([3]). Then,

$$\begin{aligned} A &:= s + t + u = q^{-\frac{26}{24}} \prod_{m=1}^{\infty} \frac{(1 - q^{27m})}{(1 - q^{27m})} \in \mathfrak{A} \quad \text{and} \\ B &:= stu = q^{-\frac{6}{24}} \prod_{m=1}^{\infty} \frac{(1 - q^{3m})}{(1 - q^{9m})} \in \mathfrak{A}. \end{aligned}$$

Then, by the Entry 2 ([3, p. 349]) we have

$$-\frac{u}{t} - \frac{s}{u} = \frac{t}{s}.$$

From these results we find that

$$\begin{aligned} & -B - \frac{1}{2}A \left( -At + t^2 + \sqrt{(At - t^2)^2 - 4tB} \right) \\ & + \frac{1}{2}t \left( -At + t^2 + \sqrt{(At - t^2)^2 - 4tB} \right) \\ & - \frac{\left( -At + t^2 + \sqrt{(At - t^2)^2 - 4tB} \right)^2}{4b} = 0. \end{aligned}$$

Thus, we conclude that

$$\begin{aligned} q^{-\frac{13}{36}} \prod_{m=1}^{\infty} (1 - q^{9m-4})(1 - q^{9m-5}) & \in \overline{\mathbb{Q}}, \\ q^{-\frac{1}{36}} \prod_{m=1}^{\infty} (1 - q^{9m-2})(1 - q^{9m-7}) & \in \overline{\mathbb{Q}}, \\ q^{\frac{11}{36}} \prod_{m=1}^{\infty} (1 - q^{9m-1})(1 - q^{9m-8}) & \in \overline{\mathbb{Q}}. \end{aligned}$$

Now, we consider the case of  $q^{7m}$ .

Put

$$\begin{aligned} a &= q^{\frac{13}{84}} \prod_{m=1}^{\infty} (1 - q^{7m-1})(1 - q^{7m-6}), \\ b &= q^{-\frac{11}{84}} \prod_{m=1}^{\infty} (1 - q^{7m-2})(1 - q^{7m-5}), \\ c &= q^{-\frac{23}{84}} \prod_{m=1}^{\infty} (1 - q^{7m-3})(1 - q^{7m-4}). \end{aligned}$$

Then we see from Entry 17(v) ([3, p. 303]), Entry 18 ([3, p. 306]) and [3, p. 307 (18.8)] that

$$b^2c + ac^2 + a^2b = A \in \mathfrak{A}, \quad abc = B \in \mathfrak{A}, \quad a^3b^2 - b^4c + abc^3 = 0.$$

Similarly, we can deduce that

$$\begin{aligned} q^{\frac{13}{84}} \prod_{m=1}^{\infty} (1 - q^{7m-1})(1 - q^{7m-6}) & \in \overline{\mathbb{Q}}, \\ q^{-\frac{11}{84}} \prod_{m=1}^{\infty} (1 - q^{7m-2})(1 - q^{7m-5}) & \in \overline{\mathbb{Q}}, \\ q^{-\frac{23}{84}} \prod_{m=1}^{\infty} (1 - q^{7m-3})(1 - q^{7m-4}) & \in \overline{\mathbb{Q}}. \end{aligned}$$

On the other hand, we note that

$$\begin{aligned}
& q^{\frac{13}{84}} \prod_{m=1}^{\infty} (1 + q^{7m-1})(1 + q^{7m-6}) \\
& = \frac{q^{\frac{26}{84}} \prod_{m=1}^{\infty} (1 - q^{14m-2})(1 - q^{14m-12})}{q^{\frac{13}{84}} \prod_{m=1}^{\infty} (1 - q^{7m-1})(1 - q^{7m-6})} \in \overline{\mathbb{Q}}, \\
& q^{-\frac{11}{84}} \prod_{m=1}^{\infty} (1 + q^{7m-2})(1 + q^{7m-5}) \\
& = \frac{q^{-\frac{22}{84}} \prod_{m=1}^{\infty} (1 - q^{14m-4})(1 - q^{14m-10})}{q^{-\frac{11}{84}} \prod_{m=1}^{\infty} (1 - q^{7m-2})(1 - q^{7m-5})} \in \overline{\mathbb{Q}}, \\
& q^{-\frac{23}{84}} \prod_{m=1}^{\infty} (1 + q^{7m-3})(1 + q^{7m-4}) \\
& = \frac{q^{-\frac{46}{84}} \prod_{m=1}^{\infty} (1 - q^{14m-6})(1 - q^{14m-8})}{q^{-\frac{23}{84}} \prod_{m=1}^{\infty} (1 - q^{7m-3})(1 - q^{7m-4})} \in \overline{\mathbb{Q}}.
\end{aligned}$$

Next, by [3, p. 49 Corollary (ii)] we have that

$$\begin{aligned}
& \sqrt{2}q^{\frac{-33}{24}} \prod_{m=1}^{\infty} (1 - q^{36m-15})(1 - q^{36m-21}) \\
& - \sqrt{2}q^{\frac{39}{24}} \prod_{m=1}^{\infty} (1 - q^{36m-3})(1 - q^{36m-33})
\end{aligned}$$

and

$$\begin{aligned}
& \left( \sqrt{2}q^{-\frac{33}{24}} \prod_{m=1}^{\infty} (1 - q^{36m-15})(1 - q^{36m-21}) \right) \\
& \times \left( -\sqrt{2}q^{\frac{39}{24}} \prod_{m=1}^{\infty} (1 - q^{36m-3})(1 - q^{36m-33}) \right)
\end{aligned}$$

are algebraic integers. Then, we can claim that

$$\begin{aligned}
& \sqrt{2}q^{\frac{-33}{24}} \prod_{m=1}^{\infty} (1 - q^{36m-15})(1 - q^{36m-21}) \in \mathfrak{A}, \\
& \sqrt{2}q^{\frac{39}{24}} \prod_{m=1}^{\infty} (1 - q^{36m-3})(1 - q^{36m-33}) \in \mathfrak{A},
\end{aligned}$$

and

$$\begin{aligned}
& \sqrt{2}q^{\frac{-11}{24}} \prod_{m=1}^{\infty} (1 - q^{12m-5})(1 - q^{12m-7}) \in \mathfrak{A}, \\
& \sqrt{2}q^{\frac{13}{24}} \prod_{m=1}^{\infty} (1 - q^{12m-1})(1 - q^{12m-11}) \in \mathfrak{A}.
\end{aligned}$$

Replacing  $q$  by  $-q$  we get that

$$\begin{aligned} \sqrt{2}q^{\frac{-11}{24}} \prod_{m=1}^{\infty} (1 + q^{12m-5})(1 + q^{12m-7}) &\in \mathfrak{A}, \\ \sqrt{2}q^{\frac{13}{24}} \prod_{m=1}^{\infty} (1 + q^{12m-1})(1 + q^{12m-11}) &\in \mathfrak{A}. \end{aligned}$$

By [3, p. 303] and the Entry 29(i) ([3, p. 45]) we see that

$$\begin{aligned} &\phi(-q) - \phi(-q^{49}) \\ &= -2qf(-q^{35}, -q^{63}) + 2q^4f(-q^{21}, -q^{77}) - 2q^9f(-q^7, -q^{91}), \\ &\quad \prod_{m=1}^{\infty} (1 - q^{98m-7})(1 - q^{98m-91})(1 - q^{98m-21})(1 - q^{98m-77}) \\ &\quad \times (1 - q^{98m-35})(1 - q^{98m-63}) \\ &= \prod_{m=1}^{\infty} \frac{(1 - q^{7m})}{(1 - q^{14m})(1 - q^{98m-49})}, \text{ and} \\ &\quad f(q, q^6)f(-q^4, -q^3) + f(-q, -q^6)f(q^4, q^3) \\ &= 2f(-q^5, -q^9)f(-q^4, -q^{10}). \end{aligned}$$

In a similar way as in the proof of the Entry 18 ([3, p. 305]), we can deduce that

$$\begin{aligned} q^{\frac{59}{84}} \prod_{m=1}^{\infty} (1 - q^{14m-1})(1 - q^{14m-13}) &\in \overline{\mathbb{Q}}, \\ q^{\frac{-1}{84}} \prod_{m=1}^{\infty} (1 - q^{14m-3})(1 - q^{14m-11}) &\in \overline{\mathbb{Q}}, \\ q^{\frac{-37}{84}} \prod_{m=1}^{\infty} (1 - q^{14m-5})(1 - q^{14m-9}) &\in \overline{\mathbb{Q}}. \end{aligned}$$

In fact, if we use only the Entry 29(i) ([3, p. 45]), we will get the same result.

Similarly, using the Entry 2(ix) ([3, p. 349]) and the Entry 29(i) ([3, p. 45]) we obtain that

$$\begin{aligned} q^{\frac{-23}{36}} \prod_{m=1}^{\infty} (1 \pm q^{18m-7})(1 \pm q^{18m-11}) &\in \overline{\mathbb{Q}}, \\ q^{\frac{-11}{36}} \prod_{m=1}^{\infty} (1 \pm q^{18m-5})(1 \pm q^{18m-13}) &\in \overline{\mathbb{Q}}, \\ q^{\frac{37}{36}} \prod_{m=1}^{\infty} (1 \pm q^{18m-1})(1 \pm q^{14m-17}) &\in \overline{\mathbb{Q}} \end{aligned}$$

with double signs in the same order.

Proceeding in the same manner as in the above we find with the aid of [3, p. 447] and the Entry 29(i) ([3, p. 45]) that

$$\begin{aligned} q^{\frac{-47}{48}} \prod_{m=1}^{\infty} (1 + q^{24m-11})(1 + q^{24m-13}) &\in \overline{\mathbb{Q}}, \\ q^{\frac{1}{48}} \prod_{m=1}^{\infty} (1 + q^{24m-5})(1 + q^{24m-19}) &\in \overline{\mathbb{Q}}, \\ q^{\frac{-23}{48}} \prod_{m=1}^{\infty} (1 + q^{24m-7})(1 + q^{24m-17}) &\in \overline{\mathbb{Q}}, \\ q^{\frac{73}{48}} \prod_{m=1}^{\infty} (1 + q^{24m-1})(1 + q^{24m-23}) &\in \overline{\mathbb{Q}}. \end{aligned}$$

Berndt ([3, p. 49]) proved that

$$\phi(q) = \phi(q^{n^2}) + \sum_{r=1}^{n-1} q^{r^2} f(q^{n(n-2r)}, q^{n(n+2r)}).$$

If we take  $n = 32$  in the above, we can show that

$$\begin{aligned} q^{\frac{-61}{96}} \prod_{m=1}^{\infty} (1 + q^{16m-7})(1 + q^{16m-9}) &\in \overline{\mathbb{Q}}, \\ q^{\frac{-37}{96}} \prod_{m=1}^{\infty} (1 + q^{16m-5})(1 + q^{16m-11}) &\in \overline{\mathbb{Q}}, \\ q^{\frac{11}{96}} \prod_{m=1}^{\infty} (1 + q^{16m-3})(1 + q^{16m-13}) &\in \overline{\mathbb{Q}}, \\ q^{\frac{83}{96}} \prod_{m=1}^{\infty} (1 + q^{16m-1})(1 + q^{16m-15}) &\in \overline{\mathbb{Q}}. \end{aligned}$$

Summarizing all the results mentioned in this section, we get the following theorem.

**Theorem 4.1.** *Let  $\tau \in \mathfrak{h} \cap k$ .*

- (a) *The values of  $q^a \prod_{m=1}^{\infty} (1 - q^{nm-t})(1 - q^{nm-(n-t)})$  are algebraic numbers, where  $(a, n, t)$  is the following:*

$$\begin{aligned} (-\frac{1}{12}, 2, 1), (-\frac{1}{12}, 3, 1), (-\frac{1}{24}, 4, 1), (\frac{1}{60}, 5, 1), (-\frac{11}{60}, 5, 2), \\ (\frac{1}{12}, 6, 1), (\frac{13}{84}, 7, 1), (-\frac{11}{84}, 7, 2), (-\frac{23}{84}, 7, 3), \end{aligned}$$

$$\begin{aligned} & \left(\frac{11}{48}, 8, 1\right), \left(-\frac{13}{48}, 8, 3\right), \left(\frac{11}{36}, 9, 1\right), \left(-\frac{1}{36}, 9, 2\right), \left(-\frac{13}{36}, 9, 4\right), \\ & \left(\frac{23}{60}, 10, 1\right), \left(-\frac{13}{60}, 10, 3\right), \left(\frac{13}{24}, 12, 1\right), \left(-\frac{11}{24}, 12, 5\right), \left(\frac{59}{84}, 14, 1\right), \\ & \left(-\frac{1}{84}, 14, 3\right), \left(-\frac{37}{84}, 14, 5\right), \left(\frac{37}{36}, 18, 1\right), \left(-\frac{11}{36}, 18, 5\right), \left(-\frac{23}{36}, 18, 7\right). \end{aligned}$$

- (b) The values of  $q^a \prod_{m=1}^{\infty} (1 + q^{nm-t})(1 + q^{nm-(n-t)})$  are algebraic numbers, when  $(a, n, t)$  runs over the cases

$$\begin{aligned} & \left(\frac{1}{12}, 1, 0\right), \left(-\frac{1}{12}, 2, 1\right), \left(-\frac{1}{12}, 3, 1\right), \left(-\frac{1}{24}, 4, 1\right), \left(\frac{1}{60}, 5, 1\right), \\ & \left(-\frac{11}{60}, 5, 2\right), \left(\frac{1}{12}, 6, 1\right), \left(\frac{13}{84}, 7, 1\right), \left(-\frac{11}{84}, 7, 2\right), \left(-\frac{23}{84}, 7, 3\right), \\ & \left(\frac{11}{48}, 8, 1\right), \left(-\frac{13}{48}, 8, 3\right), \left(\frac{11}{36}, 9, 1\right), \left(-\frac{1}{36}, 9, 2\right), \left(-\frac{13}{36}, 9, 4\right), \\ & \left(\frac{23}{60}, 10, 1\right), \left(-\frac{13}{60}, 10, 3\right), \left(\frac{13}{24}, 12, 1\right), \left(-\frac{11}{24}, 12, 5\right), \left(\frac{59}{84}, 14, 1\right), \\ & \left(-\frac{1}{84}, 14, 3\right), \left(-\frac{37}{84}, 14, 5\right), \left(\frac{37}{36}, 18, 1\right), \left(-\frac{11}{36}, 18, 5\right), \left(-\frac{23}{36}, 18, 7\right), \\ & \left(\frac{83}{96}, 16, 1\right), \left(\frac{11}{96}, 16, 3\right), \left(-\frac{37}{96}, 16, 5\right), \left(-\frac{61}{96}, 17, 7\right), \\ & \left(\frac{73}{48}, 24, 1\right), \left(\frac{1}{48}, 24, 5\right), \left(-\frac{23}{48}, 24, 7\right), \left(-\frac{47}{48}, 24, 11\right). \end{aligned}$$

**Corollary 4.2.** (a) Hirschhorn [14] introduced that

$$(a; q)_{\infty}(a^2; q)_{\infty}(a^{-2}q; q)_{\infty}(a^{-1}q; q)_{\infty}(q; q)_{\infty} = \sum_{n=-\infty}^{\infty} a^n c_n(q),$$

where

$$\begin{aligned} c_{5n}(q) &= \frac{(-1)^n q^{(5n^2-3n)/2}}{(q; q^5)_{\infty}(q^4; q^5)_{\infty}}, \quad c_{5n+1}(q) = -\frac{(-1)^n q^{(5n^2-n)/2}}{(q^2; q^5)_{\infty}(q^3; q^5)_{\infty}}, \\ c_{5n+2}(q) &= -\frac{(-1)^n q^{(5n^2+n)/2}}{(q^2; q^5)_{\infty}(q^3; q^5)_{\infty}}, \quad c_{5n+3}(q) = \frac{(-1)^n q^{(5n^2+3n)/2}}{(q; q^5)_{\infty}(q^4; q^5)_{\infty}} \text{ and} \\ c_{5n+4}(q) &= 0. \quad \text{Since } (5n^2-3n)/2, (5n^2-n)/2, (5n^2+n)/2 \text{ and } (5n^2+3n)/2 \text{ can not be } \pm \frac{1}{60}, \pm \frac{11}{60} \text{ for all } n \in \mathbb{Z}. \quad \text{We deduce that } c_{5n+t}(q) \quad (t = 0, 1, 2, 3) \text{ are transcendental numbers by (4.2) and (4.3).} \end{aligned}$$

- (b) If  $a \in \overline{\mathbb{Q}}$  and  $0 < a < 1$ , then the Rogers-Ramanujan continued fraction

$$RR(a) = 1 + \frac{a}{1 + \frac{a^2}{1 + \frac{a^3}{1 + \dots}}}$$

is transcendental ([10]). It first appeared in a paper by Rogers [25] in 1894 and is, in fact, the special case

$$R(1, q) =: RR(q) = \frac{(q; q^5)_{\infty}(q^4; q^5)_{\infty}}{(q^2; q^5)_{\infty}(q^3; q^5)_{\infty}}$$

for  $R(a, e^{-x})$  defined as in (0.1). In the case of  $q$ , however,

$$q^{\frac{1}{5}} RR(q) = \prod_{m=1}^{\infty} \frac{q^{\frac{1}{60}}(1-q^{5m-1})(1-q^{5m-4})}{q^{-\frac{11}{60}}(1-q^{5m-2})(1-q^{5m-3})}$$

is an algebraic number by Theorem 4.1.

On the other hands, Berndt and Yee proved that the generalized Rogers-Ramanujan continued fraction yields

$$\begin{aligned} R(a, x) = & \frac{-1 + \sqrt{1+4a}}{2a} \exp \left( \frac{ax}{1+4a} - \frac{a(1-a)x^2}{2(1+4a)^{5/2}} \right. \\ & \left. + \frac{a(1-a)(1-14a)x^3}{6(1+4a)^4} + \dots \right), \end{aligned}$$

as  $x \rightarrow 0^+$ . Here we note that each term of the asymptotic expansion beginning with the second has a factor of  $a(1-a)$ . Thus, if we set  $x = \pi\sqrt{d}$  and  $d \in \mathbb{Q}^+$  then

$$e^{\frac{-\pi\sqrt{d}}{5}} \prod_{m=1}^{\infty} \frac{(1-(e^{-\pi\sqrt{d}})^{5m-1})(1-(e^{-\pi\sqrt{d}})^{5m-4})}{(1-(e^{-\pi\sqrt{d}})^{5m-2})(1-(e^{-\pi\sqrt{d}})^{5m-3})} = \frac{\sqrt{5}-1}{2},$$

as  $d \rightarrow 0$ . Since

$$\frac{(q; q^6)_\infty (q^5; q^6)_\infty}{(q^3; q^6)_\infty^2} = \frac{1}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \dots,$$

we can show that

$$q^{\frac{1}{3}} \left( \frac{1}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \dots \right)$$

is an algebraic number again by Theorem 4.1. In like manner, using result in [5] we derive that

$$e^{\frac{-\pi\sqrt{d}}{3}} \prod_{m=1}^{\infty} \frac{(1-(e^{-\pi\sqrt{d}})^{6m-1})(1-(e^{-\pi\sqrt{d}})^{6m-5})}{(1-(e^{-\pi\sqrt{d}})^{6m-3})^2} = \frac{1}{2},$$

as  $d \in \mathbb{Q}^+$  and  $d \rightarrow 0$ .

- (c) It would be convenient to define an analogue of  $(a; q)_n$  for negative integer  $n$ . For  $-\infty < n < \infty$ , we define  $(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}$ . Then for  $|b/a| < |z| < 1$

$$\sum_{m=-\infty}^{\infty} \frac{(a; q)_m}{(b; q)_m} z^m = \frac{(az; q)_\infty (q/(az); q)_\infty (q; q)_\infty (b/a; q)_\infty}{(z; q)_\infty (b/(az); q)_\infty (b; q)_\infty (q/a; q)_\infty}.$$

Here, this formula is called the Ramanujan's  ${}_1\psi_1$  summation. Set  $a = q^{\frac{1}{5}}$ ,  $b = q^{\frac{7}{5}}$  and  $z = q$ . Then we have

$$\sum_{m=-\infty}^{\infty} \frac{(q^{\frac{1}{5}}; q)_m}{(q^{\frac{7}{5}}; q)_m} q^m = -q^{-\frac{1}{5}} (1+q^{\frac{1}{5}}) \frac{(q^{\frac{1}{5}}; q)_\infty}{(q^{\frac{2}{5}}; q)_\infty}.$$

Similarly, setting  $a = q^{\frac{3}{5}}$ ,  $b = q^{\frac{9}{5}}$  and  $z = q$  we get that

$$\sum_{m=-\infty}^{\infty} \frac{(q^{\frac{3}{5}}; q)_m}{(q^{\frac{9}{5}}; q)_m} q^m = -q^{-\frac{3}{5}} \frac{(1-q^{\frac{4}{5}})(q^{\frac{3}{5}}; q)_{\infty}}{(1-q^{\frac{1}{5}})(q^{\frac{4}{5}}; q)_{\infty}}.$$

Thus, we can find the identities for  $RR(q)$ , i.e.,

$$\begin{aligned} RR(q) &= \frac{\sum_{m=0}^{\infty} \frac{q^{m(m+1)}}{(q;q)_m}}{\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q;q)_m}} = \prod_{m=1}^{\infty} \frac{(1-q^{5m-1})(1-q^{5m-4})}{(1-q^{5m-2})(1-q^{5m-3})} \\ &= (1+q^{-2}) \frac{\sum_{m=-\infty}^{\infty} \frac{(q;q^5)_m}{(q^5;q^5)_m} q^{5m}}{\sum_{m=-\infty}^{\infty} \frac{(q^3;q^5)_m}{(q^3;q^5)_m} q^{5m}}. \end{aligned}$$

The first and second equalities can be found in [6] and [3, p. 30]. So we obtain that

$$q^{-\frac{9}{25}}(1+q^{\frac{2}{5}}) \frac{\sum_{m=-\infty}^{\infty} \frac{(q^{\frac{1}{5}}; q)_m}{(q^{\frac{9}{5}}; q)_m} q^m}{\sum_{m=-\infty}^{\infty} \frac{(q^{\frac{3}{5}}; q)_m}{(q^{\frac{9}{5}}; q)_m} q^m} = \prod_{m=1}^{\infty} \frac{q^{\frac{1}{300}}(1-q^{m-\frac{1}{5}})(1-q^{m-\frac{4}{5}})}{q^{-\frac{11}{300}}(1-q^{m-\frac{2}{5}})(1-q^{m-\frac{3}{5}})}$$

is an algebraic number.

- (d) Let  $p(n)$  stand for the number of unrestricted partitions of the non-negative integer  $n$ . Let  $l$  and  $k$  be non-negative integers and set

$$P_{l,k} := P_{l,k}(q) := \sum_{n=0}^{\infty} p(ln+k)q^n, \quad |q| < 1.$$

Ekin ([11]) discovered that

$$(4.4) \quad \begin{aligned} \mathfrak{P}_{7,0}(q)W_{7,3} &= W_{7,3}^7 - W_{7,3}^5W_{7,1} + 17W_{7,3}^3W_{7,1}^2 \\ &\quad + 10W_{7,3}W_{7,1}^3 + 2W_{7,3}^{-1}W_{7,1}^4, \end{aligned}$$

where

$$\begin{aligned} W_{l,j} &= W_{l,j}(q) \\ &= q^{\frac{6j^2}{l}-j} \prod_{m=1}^{\infty} \frac{(1-q^{l(m-1)+4j})(1-q^{lm-4j})}{(1-q^{l(m-1)+2j})(1-q^{lm-2j})}, \quad 1 \leq j \leq \frac{l-1}{2} \end{aligned}$$

and

$$\begin{aligned} \mathfrak{P}_{l,k}(q) &= q^{(l^2+l-l^3+24k-1)/(24l)} \prod_{m=1}^{\infty} \frac{(1-q^m)^{l+1}}{(1-q^{lm})^l} P_{l,k}(q), \quad 0 \leq k \leq (l-1). \end{aligned}$$

Besides (4.4), Ekin also found the identities for  $\mathfrak{P}_{7,i}$  for  $0 \leq i \leq 6$ . We claim by Theorem 4.1 that

$$(4.5) \quad W_{l,j}(q) \text{ are algebraic numbers}$$

when  $l = 4, 5, 6, 7, 8, 9, 10, 12, 14, 18$ . It follows from (4.4) that

$$\left[ \left( q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m) \right) \left( q^{-\frac{1}{168}} P_{7,0}(q) \right) \right] = \prod_{m=1}^{\infty} \left( \frac{q^{\frac{7}{24}} (1 - q^{7m})}{q^{\frac{1}{24}} (1 - q^m)} \right)^7 \\ \cdot (W_{7,3}^7 - W_{7,3}^5 W_{7,1} + 17 W_{7,3}^3 W_{7,1}^2 + 10 W_{7,3} W_{7,1}^3 + 2 W_{7,3}^{-1} W_{7,1}^4).$$

And hence,  $\mathfrak{P}_{7,0}$  and  $q^{\frac{1}{28}} \prod_{m=1}^{\infty} (1 - q^m) P_{7,0}(q)$  are algebraic numbers and  $q^{-\frac{1}{168}} P_{7,0}(q) = q^{-\frac{1}{168}} \sum_{n=0}^{\infty} p(7n) q^n$  is transcendental by (2.5).

On the other hand, we can find the expressions for  $P_{7,i}$  ( $i = 0, \dots, 6$ ) in [11], [19], [21]. By make use of these expressions we will construct algebraic numbers for  $\frac{P_{7,i}}{P_{7,j}}$  as follows.

For brevity, we let

$$a_i := q^{\frac{\alpha_i}{84}} \prod_{m=1}^{\infty} (1 - q^{7m-i})(1 - q^{7m-7+i})$$

with  $\alpha_1 = 13$ ,  $\alpha_2 = -11$  and  $\alpha_3 = -23$ .

It then follows that

$$\begin{aligned} & \prod_{m=1}^{\infty} \frac{q^{\frac{8}{24}} (1 - q^m)^8}{q^{\frac{49}{24}} (1 - q^{7m})^7} (q^{-\frac{1}{168}} P_{7,0}) \\ &= \frac{a_2^6}{a_1^6} + \frac{a_2^4}{a_1^3 a_3} + 17 \frac{a_2^2}{a_3^2} - 10 \frac{a_1^3}{a_3^3} + 2 \frac{a_1^6}{a_2^2 a_3^4}, \\ & \prod_{m=1}^{\infty} \frac{q^{\frac{8}{24}} (1 - q^m)^8}{q^{\frac{49}{24}} (1 - q^{7m})^7} (q^{\frac{23}{168}} P_{7,1}) \\ &= \frac{a_2^7}{a_3 a_1^6} + 8 \frac{a_2^5}{a_1^3 a_3^2} - 18 \frac{a_2^3}{a_3^3} + 11 \frac{a_2 a_1^3}{a_3^4} - 5 \frac{a_1^6}{a_2 a_3^5}, \\ & \prod_{m=1}^{\infty} \frac{q^{\frac{8}{24}} (1 - q^m)^8}{q^{\frac{49}{24}} (1 - q^{7m})^7} (q^{\frac{47}{168}} P_{7,2}) \\ &= \frac{a_1^6}{a_3^6} - \frac{a_1^4}{a_2 a_3^3} + 17 \frac{a_1^2}{a_2^2} + 10 \frac{a_3^3}{a_2^3} + 2 \frac{a_3^6}{a_1^2 a_2^4}, \\ & \prod_{m=1}^{\infty} \frac{q^{\frac{8}{24}} (1 - q^m)^8}{q^{\frac{49}{24}} (1 - q^{7m})^7} (q^{\frac{71}{168}} P_{7,3}) \\ &= - \frac{a_3^7}{a_1 a_2^6} - 8 \frac{a_3^5}{a_1^2 a_2^3} + 18 \frac{a_3^3}{a_1^3} - 11 \frac{a_3 a_2^3}{a_1^4} + 5 \frac{a_2^6}{a_3 a_1^5}, \\ & \prod_{m=1}^{\infty} \frac{q^{\frac{8}{24}} (1 - q^m)^8}{q^{\frac{49}{24}} (1 - q^{7m})^7} (q^{\frac{95}{168}} P_{7,4}) \\ &= - \frac{a_1^7}{a_2 a_3^6} + 8 \frac{a_1^5}{a_2^2 a_3^3} + 18 \frac{a_1^3}{a_2^3} + 11 \frac{a_1 a_3^3}{a_2^4} + 5 \frac{a_3^6}{a_1 a_2^5}, \end{aligned}$$

$$\prod_{m=1}^{\infty} \frac{q^{\frac{8}{24}}(1-q^m)^8}{q^{\frac{49}{24}}(1-q^{7m})^7} (q^{\frac{17}{24}} P_{7,5}) = 7 \frac{q^{\frac{8}{24}}(1-q^m)^4}{q^{\frac{28}{24}}(1-q^{7m})^4} + 49,$$

$$\prod_{m=1}^{\infty} \frac{q^{\frac{8}{24}}(1-q^m)^8}{q^{\frac{49}{24}}(1-q^{7m})^7} (q^{\frac{143}{168}} P_{7,6})$$

$$= \frac{a_3^6}{a_2^6} + \frac{a_3^4}{a_1 a_2^3} + 17 \frac{a_3^2}{a_1^2} - 10 \frac{a_2^3}{a_1^3} + 2 \frac{a_2^6}{a_3^2 a_1^4}.$$

So,  $q^{\frac{i-j}{7}} \frac{P_{7,i}}{P_{7,j}}$  and  $\frac{q^{\frac{-1+3i}{21}} P_{7,i}}{\prod_{m=1}^{\infty} (1-q^m)}$  are algebraic numbers and  $q^{\frac{-1+24i}{168}} P_{7,i}$  ( $i = 0, \dots, 6$ ) are transcendental numbers by Proposition 2.3, Theorem 4.1 and (2.5).

(e) Adiga and T. Kim [1] wrote that

$$\begin{aligned} M(q) &= \exp \int \left( \frac{1}{8q} - \frac{1}{4q} \left[ \phi^2(-q) \phi^2(-q^3) - 1 - 12q^3 \frac{\psi'(q^3)}{\psi(q^3)} \right] \right) dq \\ &= q^{\frac{1}{8}} \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \\ &= \frac{q^{1/8}}{1 + 1 + q} \frac{-q}{1 + q^2} + \dots \end{aligned}$$

By (2.1) and (2.5),  $M(q)$  is transcendental.

**Corollary 4.3.** Lucy Slater's list of the 130 identities of the Rogers-Ramanujan type appeared in [28] and [29, pp. 152-167]. We use the same notations as in [28]. Then by Proposition 2.1, (2.1) and Theorem 4.1, we get the following facts:

Identity A.1 ([12, p. 274])

$$\begin{aligned} &q^{\frac{1}{24}} \sum_{n=-\infty} (-1)^n q^{n(3n+1)/2} \\ &= q^{\frac{1}{24}} \left( 1 + \sum_{n=-\infty} (-1)^n q^{n(3n-1)/2} (1 + q^n) \right) \\ &= q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m) \in \mathfrak{T} \end{aligned}$$

Identity A.2

$$\sqrt{2} q^{\frac{1}{24}} \sum_{n=0} \frac{q^{n(n+1)/2}}{(q; q)_n} = \sqrt{2} q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 + q^m) \in \mathfrak{A}$$

Identity A.3,23 ([12])

$$q^{\frac{-1}{24}} \sum_{n=0} \frac{(-1)^n q^{n^2}}{(q^2; q^2)_n} = q^{\frac{-1}{24}} \prod_{m=1}^{\infty} (1 - q^{2m-1}) \in \mathfrak{A}$$

Identity A.4

$$\begin{aligned} & q^{-\frac{1}{8}} \sum_{n=0}^{\infty} \frac{(-1)^n (-q; q^2)_n q^{n^2}}{(q^4; q^4)_n} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{-1}{24}} (1 - q^{2m-1}) \right) \left( q^{\frac{-2}{24}} (1 - q^{4m-2}) \right) \in \mathfrak{A} \end{aligned}$$

Identity A.5

$$\begin{aligned} & \sqrt{2} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(2n+1)+\frac{1}{24}}}{(q^2; q^2)_n (-q; q^2)_{n+1}} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2} q^{\frac{2}{24}} (1 + q^{2m}) \right) \left( q^{\frac{-1}{24}} (1 - q^{2m-1}) \right) \in \mathfrak{A} \end{aligned}$$

Identity A.6

$$\begin{aligned} & \sqrt{6} \sum_{n=0}^{\infty} \frac{(-1; q)_n q^{n^2}}{(q; q)_n^2} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2} q^{\frac{-2}{24}} (1 + q^{3m-1})(1 + q^{3m-2}) \right) \left( \sqrt{3} \frac{q^{\frac{3}{24}} (1 - q^{3m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.8

$$2q^{\frac{1}{8}} \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(n+1)/2}}{(q; q)_n} = \prod_{m=1}^{\infty} \left( 2 \frac{q^{\frac{4}{24}} (1 - q^{4m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \mathfrak{A}$$

Identity A.9,84

$$\sqrt{2} q^{\frac{1}{24}} \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q; q)_{2n+1}} = \prod_{m=1}^{\infty} \left( \sqrt{2} q^{\frac{1}{24}} (1 + q^m) \right) \in \mathfrak{A}$$

Identity A.10

$$\begin{aligned} & \sqrt{2} \sum_{n=0}^{\infty} \frac{(-1; q)_{2n} q^{n^2}}{(q^2; q^2)_n (q^2; q^4)_n} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2} (1 + q^{2m-1})(1 + q^m) \right) \in \mathfrak{A} \end{aligned}$$

Identity A.11,51,64

$$\begin{aligned} & 2q^{\frac{1}{8}} \sum_{n=0}^{\infty} \frac{(-q; q)_{2n} q^{n(n+1)}}{(q; q^2)_{n+1} (q^4; q^4)_n} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2} q^{\frac{1}{24}} (1 + q^m) \right) \left( \sqrt{2} q^{\frac{2}{24}} (1 + q^{2m}) \right) \in \mathfrak{A} \end{aligned}$$

Identity A.12

$$\begin{aligned} & \sqrt{2} \sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n(n+1)/2}}{(q;q)_n} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right) \left( \sqrt{2} q^{\frac{1}{24}} \frac{1}{(1 - q^{2m-1})} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.13

$$\begin{aligned} & \prod_{m=1}^{\infty} 2 \frac{q^{\frac{4}{24}} (1 - q^{4m})}{q^{\frac{1}{24}} (1 - q^m)} \in \mathfrak{A}, \quad \sqrt{2} \prod_{m=1}^{\infty} \frac{q^{-\frac{1}{24}} (1 + q^{2m-1})}{q^{-\frac{1}{24}} (1 - q^{2m-1})} \in \mathfrak{A}, \\ & \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n-1)/2}}{(q;q)_n} = \prod_{m=1}^{\infty} \frac{(1 - q^{4m})}{(1 - q^m)} + \prod_{m=1}^{\infty} \frac{(1 + q^{2m-1})}{(1 - q^{2m-1})} \in \mathfrak{T} \end{aligned}$$

Identity A.14

$$\begin{aligned} & \sqrt{5} q^{\frac{11}{60}} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{1}{60}} (1 - q^{5m-1})(1 - q^{5m-4}) \right) \left( \sqrt{5} \frac{q^{\frac{5}{24}} (1 - q^{5m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.15

$$\begin{aligned} & \sqrt{5} q^{\frac{17}{120}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n-2)}}{(-q;q^2)_n (q^4;q^4)_n} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{1}{60}} (1 - q^{5m-1})(1 - q^{5m-4}) \right) \left( \sqrt{5} \frac{q^{\frac{5}{24}} (1 - q^{5m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.16

$$\begin{aligned} & \sqrt{5} q^{\frac{1}{10}} \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^4;q^4)_n} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{-1}{12}} \frac{1}{(1 + q^{2m})} \right) \left( \sqrt{5} \frac{q^{\frac{11}{60}}}{(1 - q^{5m-2})(1 - q^{5m-3})} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.17

$$\begin{aligned} & 2\sqrt{5} q^{\frac{9}{40}} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n (-q;q^2)_{n+1}} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2} q^{\frac{1}{12}} (1 + q^{2m}) \right) \left( \sqrt{2} q^{\frac{1}{60}} (1 - q^{5m-1})(1 - q^{5m-4}) \right) \\ & \quad \times \left( \sqrt{5} \frac{q^{\frac{5}{24}} (1 - q^{5m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.18

$$\begin{aligned} & \sqrt{5}q^{\frac{-1}{60}} \sum_{n=0} \frac{q^{n^2}}{(q;q)_n} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{-11}{60}} (1 - q^{5m-2})(1 - q^{5m-3}) \right) \left( \sqrt{5} \frac{q^{\frac{5}{24}} (1 - q^{5m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.19

$$\begin{aligned} & \sqrt{5}q^{\frac{-7}{120}} \sum_{n=0} \frac{(-1)^n q^{3n^2}}{(-q;q^2)_n (q^4;q^4)_n} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{-11}{60}} (1 - q^{5m-2})(1 - q^{5m-3}) \right) \left( \sqrt{5} \frac{q^{\frac{5}{24}} (1 - q^{5m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.20

$$\begin{aligned} & \sqrt{5}q^{\frac{-1}{10}} \sum_{n=0} \frac{q^{n^2}}{(q^4;q^4)_n} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{-1}{12}} \frac{1}{(1 + q^{2m})} \right) \left( \sqrt{5} \frac{q^{\frac{-1}{60}}}{(1 - q^{5m-1})(1 - q^{5m-4})} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.21

$$\begin{aligned} & 2\sqrt{5}q^{\frac{-1}{10}} \sum_{n=0} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(-q;q^2)_n (q^4;q^4)_n} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{-1}{24}} (1 - q^{2m-1}) \right) \\ & \quad \times \left( \sqrt{5} \frac{q^{\frac{5}{24}} (1 - q^{5m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \left( 2q^{\frac{-11}{60}} (1 + q^{5m-2})(1 + q^{5m-3}) \right) \in \mathfrak{A} \end{aligned}$$

Identity A.22

$$\begin{aligned} & 2\sqrt{6}q^{\frac{1}{3}} \sum_{n=0} \frac{(-q;q)_n q^{n(n+1)}}{(q;q^2)_{n+1} (q;q)_n} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2} \frac{q^{\frac{-1}{24}} (1 - q^{2m-1})}{q^{\frac{-3}{24}} (1 - q^{6m-3})} \right) \left( \sqrt{2} q^{\frac{1}{24}} (1 + q^m) \right) \\ & \quad \times \left( \sqrt{6} \frac{q^{\frac{6}{24}} (1 - q^{6m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.24,26

$$\begin{aligned} 2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1;q)_{2n}q^n}{(q^2;q^2)_n} &= 2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{n^2}}{(q;q^2)_{n+1}(q;q)_n} \\ &= \prod_{m=1}^{\infty}\left(q^{\frac{-3}{24}}(1-q^{6m-3})\right)^2\left(\sqrt{2}q^{\frac{1}{24}}(1+q^m)\right)\left(\sqrt{6}\frac{q^{\frac{6}{24}}(1-q^{6m})}{q^{\frac{1}{24}}(1-q^m)}\right) \in \mathfrak{A} \end{aligned}$$

Identity A.25

$$\begin{aligned} \sqrt{3}q^{\frac{-1}{8}}\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{n^2}}{(q^4;q^4)_n} &= \\ = \prod_{m=1}^{\infty}\left(q^{\frac{-3}{24}}(1-q^{6m-3})\right)^2\left(q^{\frac{-1}{24}}(1+q^{2m-1})\right)\left(\sqrt{3}\frac{q^{\frac{6}{24}}(1-q^{6m})}{q^{\frac{2}{24}}(1-q^{2m})}\right) &\in \mathfrak{A} \end{aligned}$$

Identity A.27,87

$$\begin{aligned} \sqrt{6}q^{\frac{1}{4}}\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{2n(n+1)}}{(q;q^2)_{n+1}(q^4;q^4)_n} &= \\ = \prod_{m=1}^{\infty}\left(\sqrt{2}\frac{q^{\frac{-1}{24}}(1+q^{2m-1})}{q^{\frac{-3}{24}}(1+q^{6m-3})}\right)\left(\sqrt{3}\frac{q^{\frac{6}{24}}(1-q^{6m})}{q^{\frac{2}{24}}(1-q^{2m})}\right) &\in \mathfrak{A} \end{aligned}$$

Identity A.28

$$\begin{aligned} 2\sqrt{3}q^{\frac{1}{3}}\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_nq^{n(n+1)}}{(q;q)_{2n+1}} &= \\ = \prod_{m=1}^{\infty}\left(\sqrt{2}q^{\frac{2}{24}}(1+q^{6m-1})(1+q^{6m-5})\right)\left(\sqrt{3}\frac{q^{\frac{6}{24}}(1-q^{6m})}{q^{\frac{2}{24}}(1-q^{2m})}\right) & \\ \times \left(\sqrt{2}q^{\frac{2}{24}}(1+q^{2m})\right) &\in \mathfrak{A} \end{aligned}$$

Identity A.29

$$\begin{aligned} \sqrt{6}q^{\frac{-1}{24}}\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{n^2}}{(q;q)_{2n}} &= \\ = \prod_{m=1}^{\infty}\left(\sqrt{2}\frac{q^{\frac{2}{24}}(1+q^{2m})}{q^{\frac{6}{24}}(1+q^{6m})}\right)\left(\sqrt{3}\frac{q^{\frac{6}{24}}(1-q^{6m})}{q^{\frac{2}{24}}(1-q^{2m})}\right) & \\ \times \left(q^{\frac{-1}{24}}(1+q^{2m-1})\right) &\in \mathfrak{A} \end{aligned}$$

Identity A.30

$$\begin{aligned}
& 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n (-1; q^2)_n (q; q^2)_n q^n}{(q^2; q^2)_n} \\
&= \prod_{m=1}^{\infty} \left( q^{-\frac{3}{24}} (1 + q^{6m-3}) \right)^2 \left( \sqrt{2} q^{\frac{2}{24}} (1 + q^{2m}) \right) \left( \frac{\sqrt{2} q^{\frac{1}{24}}}{(1 + q^{2m-1})} \right) \\
&\quad \times \left( \sqrt{3} \frac{q^{\frac{1}{24}} (1 - q^{6m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \left( q^{\frac{-1}{24}} (1 - q^{2m-1}) \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.31

$$\begin{aligned}
& q^{\frac{61}{168}} \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2; q^2)_n (-q; q)_{2n+1}} \\
&= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{7}{24}} (1 - q^{7m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \left( q^{\frac{13}{84}} (1 - q^{7m-1})(1 - q^{7m-6}) \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.32

$$\begin{aligned}
& q^{\frac{13}{168}} \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2; q^2)_n (-q; q)_{2n}} \\
&= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{7}{24}} (1 - q^{7m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \left( q^{\frac{-11}{84}} (1 - q^{7m-2})(1 - q^{7m-5}) \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.33

$$\begin{aligned}
& q^{\frac{-11}{168}} \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2; q^2)_n (-q; q)_{2n}} \\
&= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{7}{24}} (1 - q^{7m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \left( q^{\frac{-23}{84}} (1 - q^{7m-3})(1 - q^{7m-4}) \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.34

$$\begin{aligned}
& 2\sqrt{2} q^{\frac{7}{16}} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n(n+2)}}{(q^2; q^2)_n} = \prod_{m=1}^{\infty} \left( \sqrt{2} q^{\frac{2}{24}} (1 + q^{2m}) \right) \\
&\quad \times \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right) \left( \sqrt{2\sqrt{2}} q^{\frac{11}{48}} (1 - q^{8m-1})(1 - q^{8m-7}) \right) \\
&\quad \times \left( \sqrt{2} \frac{q^{\frac{4}{24}}}{(1 - q^{8m-4})} \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.35,106

$$4\sqrt{2\sqrt{2}q^{\frac{9}{16}}}\sum_{n=0} \frac{(-q;q^2)_n(-q;q)_nq^{n(n+3)/2}}{(q;q)_{2n+1}} = \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{1}{24}}(1+q^m) \right) \\ \times \left( \sqrt{2\sqrt{2}q^{\frac{11}{48}}}(1-q^{8m-1})(1-q^{8m-7}) \right) \left( \sqrt{8}\frac{q^{\frac{8}{24}}(1-q^{8m})}{q^{\frac{1}{24}}(1-q^m)} \right) \in \mathfrak{A}$$

Identity A.36

$$2\sqrt{2\sqrt{2}q^{\frac{-1}{16}}}\sum_{n=0} \frac{(-q;q^2)_nq^{n^2}}{(q^2;q^2)_n} = \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{2}{24}}(1+q^{2m}) \right) \\ \times \left( \sqrt{2\sqrt{2}q^{\frac{-13}{48}}}(1-q^{8m-3})(1-q^{8m-5}) \right) \left( q^{\frac{-1}{24}}(1+q^{2m-1}) \right) \\ \times \left( \sqrt{2}\frac{q^{\frac{4}{24}}}{(1-q^{8m-4})} \right) \in \mathfrak{A}$$

Identity A.37,105

$$4\sqrt{2\sqrt{2}q^{\frac{1}{16}}}\sum_{n=0} \frac{(-q;q^2)_n(-q;q)_nq^{n(n+1)/2}}{(q;q)_{2n+1}} = \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{1}{24}}(1+q^m) \right) \\ \times \left( \sqrt{2\sqrt{2}q^{\frac{-13}{48}}}(1-q^{8m-3})(1-q^{8m-5}) \right) \left( \sqrt{8}\frac{q^{\frac{8}{24}}(1-q^{8m})}{q^{\frac{1}{24}}(1-q^m)} \right) \in \mathfrak{A}$$

Identity A.38,86

$$8q^{\frac{23}{48}}\sum_{n=0} \frac{q^{2n(n+1)}}{(q;q)_{2n+1}} = \prod_{m=1}^{\infty} \left( \sqrt{2\sqrt{2}q^{\frac{-13}{48}}}(1-q^{8m-3})(1-q^{8m-5}) \right) \\ \times \left( \sqrt{2\sqrt{2}q^{\frac{11}{24}}}(1-q^{16m-2})(1-q^{16m-14}) \right) \left( \sqrt{8}\frac{q^{\frac{8}{24}}(1-q^{8m})}{q^{\frac{1}{24}}(1-q^m)} \right) \in \mathfrak{A}$$

Identity A.39, 83

$$8q^{\frac{-1}{48}}\sum_{n=0} \frac{q^{2n^2}}{(q;q)_{2n}} = \prod_{m=1}^{\infty} \left( \sqrt{2\sqrt{2}q^{\frac{11}{48}}}(1-q^{8m-1})(1-q^{8m-7}) \right) \\ \times \left( \sqrt{2\sqrt{2}q^{\frac{-13}{24}}}(1-q^{16m-6})(1-q^{16m-10}) \right) \left( \sqrt{8}\frac{q^{\frac{8}{24}}(1-q^{8m})}{q^{\frac{1}{24}}(1-q^m)} \right) \\ = 2^{5/4}\prod_{m=1}^{\infty} \left( \sqrt{2\sqrt{2}q^{-\frac{13}{48}}}(1+q^{8m-3})(1+q^{8m-5}) \right) \\ \times \left( 2\frac{q^{\frac{8}{24}}(1-q^{8m})}{q^{\frac{2}{24}}(1-q^{2m})} \right) \in \mathfrak{A}$$

Identity A.40

$$\begin{aligned} & q^{\frac{5}{9}} \sum_{n=0} \frac{(q; q)_{3n+1} q^{3n(n+1)}}{(q^3; q^3)_n (q^3; q^3)_{2n+1}} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{11}{36}} (1 - q^{9m-1}) (1 - q^{9m-8}) \right) \left( \frac{q^{\frac{9}{24}} (1 - q^{9m})}{q^{\frac{3}{24}} (1 - q^{3m})} \right) \in \overline{\mathbb{Q}}. \end{aligned}$$

Identity A.41

$$\begin{aligned} & q^{\frac{2}{9}} \sum_{n=0} \frac{(q; q)_{3n} (1 - q^{3n+2}) q^{3n(n+1)}}{(q^3; q^3)_n (q^3; q^3)_{2n+1}} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{-1}{36}} (1 - q^{9m-2}) (1 - q^{9m-7}) \right) \left( \frac{q^{\frac{9}{24}} (1 - q^{9m})}{q^{\frac{3}{24}} (1 - q^{3m})} \right) \in \overline{\mathbb{Q}}. \end{aligned}$$

Identity A.42

$$\begin{aligned} & q^{\frac{-1}{9}} \sum_{n=0} \frac{(q; q)_{3n} q^{3n^2}}{(q^3; q^3)_n (q^3; q^3)_{2n}} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{-13}{36}} (1 - q^{9m-4}) (1 - q^{9m-5}) \right) \left( \frac{q^{\frac{9}{24}} (1 - q^{9m})}{q^{\frac{3}{24}} (1 - q^{3m})} \right) \in \overline{\mathbb{Q}}. \end{aligned}$$

Identity A.43

$$\begin{aligned} & 4\sqrt{5}q^{\frac{4}{5}} \sum_{n=0} \frac{(-q; q)_n q^{n(n+3)/2}}{(q; q^2)_{n+1} (q; q)_n} = \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{1}{24}} (1 + q^m) \right) \\ & \times \left( 2q^{\frac{23}{60}} (1 - q^{10m-1}) (1 - q^{10m-9}) \right) \left( \sqrt{10} \frac{q^{\frac{10}{24}} (1 - q^{10m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.44,63

$$\begin{aligned} & 2\sqrt{5}q^{\frac{49}{120}} \sum_{n=0} \frac{q^{3n(n+1)/2}}{(q; q^2)_{n+1} (q; q)_n} = 2\sqrt{5}q^{\frac{49}{120}} \sum_{n=0} \frac{(-q; q)_n q^{3n(n+1)/2}}{(q; q)_{2n+1}} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{1}{30}} (1 - q^{10m-2}) (1 - q^{10m-8}) \right) \left( \sqrt{10} \frac{q^{\frac{10}{24}} (1 - q^{10m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.45

$$\begin{aligned} & 4\sqrt{5}q^{\frac{1}{5}} \sum_{n=0} \frac{(-q; q)_n q^{n(n+1)/2}}{(q; q^2)_{n+1} (q; q)_n} = \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{1}{24}} (1 + q^m) \right) \\ & \times \left( 2q^{\frac{-13}{60}} (1 - q^{10m-3}) (1 - q^{10m-7}) \right) \left( \sqrt{10} \frac{q^{\frac{10}{24}} (1 - q^{10m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.46,62

$$\begin{aligned}
& 2\sqrt{5}q^{\frac{1}{120}} \sum_{n=0} \frac{q^{n(3n-1)/2}}{(q;q)_n(q;q)_n} \\
&= 2\sqrt{5}q^{\frac{1}{120}} \sum_{n=0} \frac{(-q;q)_n q^{n(3n+1)/2}}{(q;q)_{2n+1}} \\
&= \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{-11}{30}} (1 - q^{10m-4})(1 - q^{10m-6}) \right) \left( \sqrt{10} \frac{q^{\frac{10}{24}} (1 - q^{10m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.47

$$\sqrt{2} \sum_{n=0} \frac{(-1;q^2)_n q^{n^2}}{(q;q)_{2n}} = \prod_{m=1}^{\infty} \left( \sqrt{2} \frac{(1 + q^{2m-1})}{(1 - q^{2m-1})} \right) \in \mathfrak{A}$$

Identity A.48

$$\begin{aligned}
& 2\sqrt{6} \sum_{n=0} \frac{(-1;q^2)_n q^{n(n+1)}}{(q;q)_{2n}} \\
&= \left( \sqrt{2}q^{\frac{-11}{24}} \prod_{m=1}^{\infty} (1 - q^{12m-5})(1 - q^{12m-7}) \right) \\
&\quad \times \left( \sqrt{12} \frac{q^{\frac{12}{24}} \prod_{m=1}^{\infty} (1 - q^{12m})}{q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)} \right) \\
&\quad - \left( \sqrt{2}q^{\frac{13}{24}} \prod_{m=1}^{\infty} (1 - q^{12m-1})(1 - q^{12m-11}) \right) \\
&\quad \times \left( \sqrt{12} \frac{q^{\frac{12}{24}} \prod_{m=1}^{\infty} (1 - q^{12m})}{q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)} \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.49

$$\begin{aligned}
& 2\sqrt{6}q \sum_{n=0} \frac{(-q^2;q^2)_n (1 - q^{n+1}) q^{n(n+2)}}{(q;q)_{2n+2}} \\
&= \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{13}{24}} (1 - q^{12m-1})(1 - q^{12m-11}) \right) \\
&\quad \times \left( \sqrt{12} \frac{q^{\frac{12}{24}} (1 - q^{12m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.50

$$\begin{aligned} & 2\sqrt{6}q^{\frac{5}{8}} \sum_{n=0} \frac{(-q; q^2)_n q^{n(n+2)}}{(q; q)_{2n+1}} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2} \frac{q^{\frac{6}{24}}}{(1 - q^{12m-6})} \right) \left( \frac{q^{\frac{2}{24}}(1 - q^{2m})}{q^{\frac{4}{24}}(1 - q^{4m})} \right) \\ &\quad \times \left( \sqrt{12} \frac{q^{\frac{12}{24}}(1 - q^{12m})}{q^{\frac{1}{24}}(1 - q^m)} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.52,85

$$\sqrt{2}q^{\frac{1}{24}} \sum_{n=0} \frac{q^{n(2n-1)}}{(q; q)_{2n}} = \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{1}{24}}(1 + q^m) \right) \in \mathfrak{A}$$

Identity A.53

$$\begin{aligned} & \sqrt{6}q^{\frac{-1}{8}} \sum_{n=0} \frac{(q; q^2)_{2n} q^{4n^2}}{(q^4; q^4)_{2n}} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{-11}{24}}(1 - q^{12m-5})(1 - q^{12m-7}) \right) \\ &\quad \times \left( \sqrt{3} \frac{q^{\frac{12}{24}}(1 - q^{12m})}{q^{\frac{4}{24}}(1 - q^{4m})} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.54

$$\begin{aligned} & 2\sqrt{6} \sum_{n=0} \frac{(-q^2; q^2)_{n-1}(1 + q^n)q^{n^2}}{(q; q)_{2n}} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{-11}{24}}(1 - q^{12m-5})(1 - q^{12m-7}) \right) \left( \sqrt{12} \frac{q^{\frac{12}{24}}(1 - q^{12m})}{q^{\frac{1}{24}}(1 - q^m)} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.55

$$\begin{aligned} & \sqrt{6}q^{\frac{7}{8}} \sum_{n=0} \frac{(q; q^2)_{2n+1} q^{4n(n+1)}}{(q^4; q^4)_{2n+1}} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{13}{24}}(1 - q^{12m-1})(1 - q^{12m-11}) \right) \left( \sqrt{3} \frac{q^{\frac{12}{24}}(1 - q^{12m})}{q^{\frac{4}{24}}(1 - q^{4m})} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.56

$$\begin{aligned} & 2\sqrt{6}q \sum_{n=0} \frac{(-q; q)_n q^{n(n+2)}}{(q; q^2)_{n+1}(q; q)_{n+1}} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{13}{24}}(1 + q^{12m-1})(1 + q^{12m-11}) \right) \left( \sqrt{12} \frac{q^{\frac{12}{24}}(1 - q^{12m})}{q^{\frac{1}{24}}(1 - q^m)} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.57

$$\begin{aligned} & \sqrt{6}q^{\frac{7}{8}} \sum_{n=0} \frac{(-q; q^2)_{2n+1} q^{4n(n+1)}}{(q^4; q^4)_{2n+1}} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{13}{24}}(1+q^{12m-1})(1+q^{12m-11}) \right) \left( \sqrt{3} \frac{q^{\frac{12}{24}}(1-q^{12m})}{q^{\frac{4}{24}}(1-q^{4m})} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.58

$$\begin{aligned} & 2\sqrt{6} \left( 1 + \sum_{n=1} \frac{(-q; q)_{n-1} q^{n^2}}{(q; q^2)_n (q; q)_n} \right) \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{-11}{24}}(1+q^{12m-5})(1+q^{12m-7}) \right) \\ & \quad \times \left( \sqrt{12} \frac{q^{\frac{12}{24}}(1-q^{12m})}{q^{\frac{1}{24}}(1-q^m)} \right) \in \mathfrak{A}, \\ & 2\sqrt{6} \sum_{n=1} \frac{(-q; q)_{n-1} q^{n^2}}{(q; q^2)_n (q; q)_n} \in \mathfrak{A} \end{aligned}$$

Identity A.59

$$\begin{aligned} & q^{\frac{143}{168}} \sum_{n=0} \frac{q^{n(n+2)}}{(q; q^2)_{n+1} (q; q)_n} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{13}{42}}(1-q^{14m-2})(1-q^{14m-12}) \right) \left( \frac{q^{\frac{14}{24}}(1-q^{14m})}{q^{\frac{1}{24}}(1-q^m)} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.60

$$\begin{aligned} & q^{\frac{47}{168}} \sum_{n=0} \frac{q^{n(n+1)}}{(q; q^2)_{n+1} (q; q)_n} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{-11}{42}}(1-q^{14m-4})(1-q^{14m-10}) \right) \left( \frac{q^{\frac{14}{24}}(1-q^{14m})}{q^{\frac{1}{24}}(1-q^m)} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.61

$$\begin{aligned} & q^{\frac{-1}{168}} \sum_{n=0} \frac{q^{n^2}}{(q; q^2)_n (q; q)_n} = \left( q^{\frac{-23}{42}} \prod_{m=1}^{\infty} (1-q^{14m-6})(1-q^{14m-8}) \right) \\ & \quad \times \left( \frac{q^{\frac{14}{24}} \prod_{m=1}^{\infty} (1-q^{14m})}{q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1-q^m)} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.65

$$\begin{aligned}
& 8q^{\frac{1}{8}} \sum_{n=0} \frac{(-q^2; q^4)_n (-q^2; q^2)_n q^{n(n+1)}}{(q^4; q^4)_n (-q; q^2)_n (q; q^2)_{n+1}} \\
&= \prod_{m=1}^{\infty} \left( \sqrt{2\sqrt{2}} q^{\frac{11}{48}} (1 + q^{8m-1}) (1 + q^{8m-7}) \right) \\
&\quad \times \left( \sqrt{2\sqrt{2}} q^{-\frac{13}{48}} (1 - q^{8m-3}) (1 - q^{8m-5}) \right) \left( q^{-\frac{4}{24}} (1 + q^{8m-4}) \right) \\
&\quad \times \left( \sqrt{2} q^{\frac{2}{24}} (1 + q^{2m}) \right) \left( 2 \frac{q^{\frac{8}{24}} (1 - q^{8m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.66

$$\begin{aligned}
& 2^{\frac{9}{4}} \sum_{n=0} \frac{(-1; q^4)_n (-q; q^2)_n q^{n^2}}{(q^2; q^2)_{2n}} \\
&= \left( \frac{\sqrt{2} q^{\frac{1}{24}}}{\prod_{m=1}^{\infty} (1 - q^{2m-1})} \right) \left( 2q^{\frac{12}{24}} \frac{(1 - q^{16m})}{(1 - q^{4m})} \right) (\sqrt{2\sqrt{2}}) \\
&\quad \times \left[ q^{-\frac{13}{24}} (1 - q^{16m-6}) (1 - q^{16m-10}) \right. \\
&\quad \left. + q^{\frac{11}{24}} (1 - q^{16m-2}) (1 - q^{16m-14}) \right] \in \mathfrak{A}
\end{aligned}$$

Identity A.67

$$\begin{aligned}
& 2^{\frac{9}{4}} \sum_{n=0} \frac{(-1; q^4)_n (-q; q^2)_n q^{n(n+2)}}{(q^2; q^2)_{2n}} \\
&= \left( \frac{\sqrt{2} q^{\frac{1}{24}}}{\prod_{m=1}^{\infty} (1 - q^{2m-1})} \right) \left( 2q^{\frac{12}{24}} \frac{(1 - q^{16m})}{(1 - q^{4m})} \right) (\sqrt{2\sqrt{2}}) \\
&\quad \times \left[ q^{-\frac{13}{24}} (1 - q^{16m-6}) (1 - q^{16m-10}) \right. \\
&\quad \left. - q^{\frac{11}{24}} (1 - q^{16m-2}) (1 - q^{16m-14}) \right] \in \mathfrak{A}
\end{aligned}$$

Identity A.68

$$\begin{aligned}
& 4\sqrt[4]{2} q \sum_{n=0} \frac{(-q; q^2)_n (-q^4; q^4)_n q^{n(n+2)}}{(-q^2; q^2)_{n+1} (q^2; q^2)_n (q^2; q^4)_{n+1}} \\
&= \prod_{m=1}^{\infty} \left( q^{-\frac{1}{24}} (1 + q^{2m-1}) \right) \left( \sqrt{2\sqrt{2}} q^{\frac{11}{24}} (1 - q^{16m-2}) (1 - q^{16m-14}) \right) \\
&\quad \times \left( 2\sqrt{2} \frac{q^{\frac{16}{24}} (1 - q^{16m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.69

$$\begin{aligned} & 2^{\frac{9}{4}} q \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_n q^{n(n+2)}}{(q; q)_{2n+2}} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2\sqrt{2}} q^{\frac{11}{24}} (1 - q^{16m-2})(1 - q^{16m-14}) \right) \\ &\quad \times \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right) \left( \sqrt{8} \frac{q^{\frac{8}{24}} (1 - q^{8m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.70

$$\begin{aligned} & 4q^{\frac{3}{8}} \sum_{n=0}^{\infty} \frac{(-q; q^2)_{n+1} (-q^2; q^4)_n q^{n(n+2)}}{(q^2; q^4)_{n+1} (q^4; q^4)_n} = \prod_{m=1}^{\infty} \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right) \\ &\quad \times \left( \frac{\sqrt{2} q^{\frac{8}{24}}}{(1 - q^{16m-8})} \right) \left( \frac{q^{\frac{4}{24}} (1 - q^{4m})}{q^{\frac{16}{24}} (1 - q^{16m})} \right) \left( \frac{\sqrt{8} q^{\frac{16}{24}} (1 - q^{16m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.71

$$\begin{aligned} & 4\sqrt[4]{2} \sum_{n=0}^{\infty} \frac{(-q^4; q^4)_{n-1} (-q; q^2)_n q^{n^2}}{(q^2; q^4)_n (q^2; q^2)_n (-q^2; q^2)_{n-1}} = \prod_{m=1}^{\infty} \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right) \\ &\quad \times \left( \sqrt{2\sqrt{2}} q^{\frac{-13}{24}} (1 - q^{16m-6})(1 - q^{16m-10}) \right) \left( \frac{\sqrt{8} q^{\frac{16}{24}} (1 - q^{16m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.72

$$\begin{aligned} & 4\sqrt[4]{2} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n (q^4; q^4)_{n-1} q^{n^2}}{(q^2; q^4)_n (q^2; q^2)_n (q^2; q^2)_{n-1}} = \prod_{m=1}^{\infty} \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right) \\ &\quad \times \left( \sqrt{2\sqrt{2}} q^{\frac{-13}{24}} (1 + q^{16m-6})(1 + q^{16m-10}) \right) \left( \frac{\sqrt{8} q^{\frac{16}{24}} (1 - q^{16m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \in \mathfrak{A} \end{aligned}$$

Identity A.72-a

$$\begin{aligned} & 2\sqrt{2} q^{\frac{-1}{8}} \sum_{n=0}^{\infty} \frac{(-q^4; q^4)_n (-q; q^2)_{n+1} (1 + q^{2n+1}) q^{n^2}}{(q^2; q^2)_{2n+2}} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{-8}{24}} (1 - q^{16m-8}) \right)^2 \left( \sqrt{8} \frac{q^{\frac{16}{24}} (1 - q^{16m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \\ &\quad \times \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right) \in \mathfrak{A} \end{aligned}$$

Identity A.73

$$\begin{aligned} & 2\sqrt{3}q^{\frac{1}{4}} \frac{(q^6, q^{12}, q^{18}; q^{18})_\infty (-q; q)_\infty}{(q; q)_\infty} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{6} \frac{q^{\frac{6}{24}}(1 - q^{6m})}{q^{\frac{1}{24}}(1 - q^m)} \right) \left( \sqrt{2}q^{\frac{1}{24}}(1 + q^m) \right) \in \mathfrak{A}, \end{aligned}$$

$$\begin{aligned} & 6 \frac{(q^9, q^9, q^{18}; q^{18})_\infty (-q; q)_\infty}{(q; q)_\infty} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{18} \frac{q^{\frac{18}{24}}(1 - q^{18m})}{q^{\frac{1}{24}}(1 - q^m)} \right) \left( q^{\frac{-9}{24}}(1 - q^{18m-9}) \right)^2 \\ &\quad \times \left( \sqrt{2}q^{\frac{1}{24}}(1 + q^m) \right) \in \mathfrak{A}, \end{aligned}$$

$$\begin{aligned} & \sum_{n=0} \frac{(q^3; q^3)_{n-1}(-q; q)_n q^{n(n-1)/2}}{(q; q)_{2n-1}(q; q)_n} = \frac{(q^6, q^{12}, q^{18}; q^{18})_\infty (-q; q)_\infty}{(q; q)_\infty} \\ &+ \frac{(q^9, q^9, q^{18}; q^{18})_\infty (-q; q)_\infty}{(q; q)_\infty} \in \mathfrak{T} \end{aligned}$$

Identity A.74

$$\begin{aligned} & 6\sqrt{2}q \frac{(q^3, q^{15}, q^{18}; q^{18})_\infty}{(q; q)_\infty (q; q^2)_\infty} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{3}{12}}(1 - q^{18m-3})(1 - q^{18m-15}) \right) \\ &\quad \times \left( \sqrt{18} \frac{q^{\frac{18}{24}}(1 - q^{18m})}{q^{\frac{1}{24}}(1 - q^m)} \right) \left( \frac{\sqrt{2}q^{\frac{1}{24}}}{(1 - q^{2m-1})} \right) \in \mathfrak{A}, \\ & \sum_{n=1} \frac{(q^3; q^3)_{n-1}(-q; q)_n q^{n(n-1)/2}}{(q; q)_{2n}(q; q)_{n-1}} = \frac{(q^6, q^{12}, q^{18}; q^{18})_\infty}{(q; q)_\infty (q; q^2)_\infty} \\ &+ \frac{(q^9, q^9, q^{18}; q^{18})_\infty}{(q; q)_\infty (q; q^2)_\infty} - q \frac{(q^3, q^{15}, q^{18}; q^{18})_\infty}{(q; q)_\infty (q; q^2)_\infty} - 1 \in \mathfrak{T} \end{aligned}$$

Identity A.75

$$\begin{aligned} & 6\sqrt{2} \sum_{n=1} \frac{(q^3; q^3)_{n-1}(-q; q)_n q^{n(n+1)/2}}{(q; q)_{2n}(q; q)_{n-1}} \\ &= 6\sqrt{2} \frac{(q^9, q^9, q^{18}; q^{18})_\infty}{(q; q)_\infty (q; q^2)_\infty} - 6\sqrt{2}q \frac{(q^3, q^{15}, q^{18}; q^{18})_\infty}{(q; q)_\infty (q; q^2)_\infty} - 6\sqrt{2} \in \mathfrak{A} \end{aligned}$$

Identity A.76

$$\begin{aligned}
& 6\sqrt{2}q \sum_{n=0} \frac{(q^3; q^3)_n (-q; q)_{n+1} q^{n(n+3)/2}}{(q; q)_{2n+2} (q; q)_n} \\
&= \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{6}{24}} (1 - q^{18m-3})(1 - q^{18m-15}) \right) \left( \sqrt{18} \frac{q^{\frac{18}{24}} (1 - q^{18m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \\
&\quad \times \left( \sqrt{2}q^{\frac{1}{24}} (1 + q^m) \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.77

$$\begin{aligned}
& 6q^{\frac{1}{4}} \sum_{n=0} \frac{(q^3; q^3)_n (-q; q)_{n+1} (1 - q^{n+1}) q^{n(n+1)/2}}{(q; q)_{2n+2} (q; q)_n} \\
&= \prod_{m=1}^{\infty} \left( q^{\frac{-12}{24}} (1 - q^{18m-6})(1 - q^{18m-12}) \right) \left( \sqrt{18} \frac{q^{\frac{18}{24}} (1 - q^{18m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \\
&\quad \times \left( \sqrt{2}q^{\frac{1}{24}} (1 + q^m) \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.78

$$\begin{aligned}
& 6 \left( 1 + \sum_{n=1} \frac{(q^3; q^3)_{n-1} (-1; q)_{n+1} q^{n(n+1)/2}}{(q; q)_{2n} (q; q)_{n-1}} \right) \\
&= \prod_{m=1}^{\infty} \left( q^{\frac{-18}{24}} (1 - q^{18m-9})^2 \right) \left( \sqrt{18} \frac{q^{\frac{18}{24}} (1 - q^{18m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \\
&\quad \times \left( \sqrt{2}q^{\frac{1}{24}} (1 + q^m) \right) \in \mathfrak{A}, \\
& 6 \sum_{n=1} \frac{(q^3; q^3)_{n-1} (-1; q)_{n+1} q^{n(n+1)/2}}{(q; q)_{2n} (q; q)_{n-1}} \in \mathfrak{A}
\end{aligned}$$

Identity A.79,98

$$\begin{aligned}
& \sqrt{10}q^{\frac{-1}{40}} \sum_{n=0} \frac{q^{n^2}}{(q; q)_{2n}} \\
&= \prod_{m=1}^{\infty} \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right) \left( \sqrt{10} \frac{q^{\frac{20}{24}} (1 - q^{20m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \\
&\quad \times \left( q^{\frac{-11}{15}} (1 - q^{20m-8})(1 - q^{20m-12}) \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.80

$$\begin{aligned} q^{\frac{25}{168}} \sum_{n=0} \frac{(-q; q)_n q^{n(n+1)/2}}{(q; q)_{2n+1}} &= \prod_{m=1}^{\infty} \left( q^{\frac{1}{24}} (1 + q^m) \right) \left( \frac{q^{\frac{7}{24}} (1 - q^{7m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \\ &\times \left( q^{\frac{-11}{84}} (1 - q^{7m-2})(1 - q^{7m-5}) \right) \\ &\times \left( q^{\frac{-1}{84}} (1 - q^{14m-3})(1 - q^{14m-11}) \right) \in \overline{\mathbb{Q}}. \end{aligned}$$

Identity A.81

$$\begin{aligned} q^{\frac{1}{168}} \sum_{n=0} \frac{(-q; q)_n q^{n(n+1)/2}}{(q; q)_{2n}} &= \prod_{m=1}^{\infty} \left( q^{\frac{1}{24}} (1 + q^m) \right) \left( \frac{q^{\frac{7}{24}} (1 - q^{7m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \\ &\times \left( q^{\frac{13}{84}} (1 - q^{7m-1})(1 - q^{7m-6}) \right) \\ &\times \left( q^{\frac{-37}{84}} (1 - q^{14m-5})(1 - q^{14m-9}) \right) \in \overline{\mathbb{Q}}. \end{aligned}$$

Identity A.82

$$\begin{aligned} q^{\frac{121}{168}} \sum_{n=0} \frac{(-q; q)_n q^{n(n+3)/2}}{(q; q)_{2n+1}} &= \prod_{m=1}^{\infty} \left( q^{\frac{1}{24}} (1 + q^m) \right) \left( \frac{q^{\frac{7}{24}} (1 - q^{7m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \\ &\times \left( q^{\frac{-23}{84}} (1 - q^{7m-3})(1 - q^{7m-4}) \right) \\ &\times \left( q^{\frac{59}{84}} (1 - q^{14m-1})(1 - q^{14m-13}) \right) \in \overline{\mathbb{Q}}. \end{aligned}$$

Identity A.88

$$\begin{aligned} &q \frac{(q^6, q^{21}, q^{27}; q^{27})_\infty}{(q; q)_\infty} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{-1}{12}} (1 - q^{27m-6})(1 - q^{27m-21}) \right) \left( \frac{q^{\frac{27}{24}} (1 - q^{27m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \overline{\mathbb{Q}}, \\ &q^2 \frac{(q^3, q^{24}, q^{27}; q^{27})_\infty}{(q; q)_\infty} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{11}{12}} (1 - q^{27m-3})(1 - q^{27m-24}) \right) \left( \frac{q^{\frac{27}{24}} (1 - q^{27m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \overline{\mathbb{Q}}, \\ &\sum_{n=1} \frac{(q^3; q^3)_{n-1} (1 - q^{n+1}) q^{n^2-1}}{(q; q)_{2n} (q; q)_{n-1}} \\ &= \frac{(q^6, q^{21}, q^{27}; q^{27})_\infty}{(q; q)_\infty} - q^2 \frac{(q^3, q^{24}, q^{27}; q^{27})_\infty}{(q; q)_\infty} \in \mathfrak{T} \end{aligned}$$

Identity A.89

$$\begin{aligned}
& \frac{(q^{12}, q^{15}, q^{27}; q^{27})_\infty}{(q; q)_\infty} \\
&= \prod_{m=1}^{\infty} \left( q^{\frac{-13}{12}} (1 - q^{27m-12})(1 - q^{27m-15}) \right) \left( \frac{q^{\frac{27}{24}} (1 - q^{27m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \overline{\mathbb{Q}}, \\
& \sum_{n=1}^{\infty} \frac{(q^3; q^3)_{n-1} q^{n(n+1)}}{(q; q)_{2n} (q; q)_{n-1}} \\
&= \frac{(q^{12}, q^{15}, q^{27}; q^{27})_\infty}{(q; q)_\infty} - q \frac{(q^6, q^{21}, q^{27}; q^{27})_\infty}{(q; q)_\infty} - 1 \in \overline{\mathbb{Q}}
\end{aligned}$$

Identity A.90

$$\begin{aligned}
& q^2 \sum_{n=0}^{\infty} \frac{(q^3; q^3)_n q^{n(n+3)}}{(q; q)_{2n+2} (q; q)_n} \\
&= \prod_{m=1}^{\infty} \left( q^{\frac{11}{12}} (1 - q^{27m-3})(1 - q^{27m-24}) \right) \left( \frac{q^{\frac{27}{24}} (1 - q^{27m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \overline{\mathbb{Q}}
\end{aligned}$$

Identity A.91

$$\begin{aligned}
& q \sum_{n=0}^{\infty} \frac{(q^3; q^3)_n q^{n(n+2)}}{(q; q)_{2n+2} (q; q)_n} \\
&= \prod_{m=1}^{\infty} \left( q^{\frac{-1}{12}} (1 - q^{27m-6})(1 - q^{27m-21}) \right) \left( \frac{q^{\frac{27}{24}} (1 - q^{27m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \overline{\mathbb{Q}}
\end{aligned}$$

Identity A.92

$$\sqrt{3} q^{\frac{1}{3}} \sum_{n=0}^{\infty} \frac{(q^3; q^3)_n q^{n(n+1)}}{(q; q)_{2n+1} (q; q)_n} = \prod_{m=1}^{\infty} \left( \sqrt{3} \frac{q^{\frac{9}{24}} (1 - q^{9m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \mathfrak{A}$$

Identity A.93

$$\begin{aligned}
& \left( 1 + \sum_{n=1}^{\infty} \frac{(q^3; q^3)_{n-1} q^{n^2}}{(q; q)_{2n-1} (q; q)_n} \right) \\
&= \prod_{m=1}^{\infty} \left( q^{\frac{-13}{12}} (1 - q^{27m-12})(1 - q^{27m-15}) \right) \left( \frac{q^{\frac{27}{24}} (1 - q^{27m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \overline{\mathbb{Q}}, \\
& \sum_{n=1}^{\infty} \frac{(q^3; q^3)_{n-1} q^{n^2}}{(q; q)_{2n-1} (q; q)_n} \in \overline{\mathbb{Q}}
\end{aligned}$$

Identity A.94

$$\begin{aligned} 2\sqrt{10}q^{\frac{9}{40}} \sum_{n=0} \frac{q^{n(n+1)}}{(q;q)_{2n+1}} &= \prod_{m=1}^{\infty} \left( \sqrt{10} \frac{q^{\frac{10}{24}}(1-q^{10m})}{q^{\frac{1}{24}}(1-q^m)} \right) \\ &\times \left( 2q^{\frac{-13}{60}}(1-q^{10m-3})(1-q^{10m-7}) \right) \\ &\times \left( q^{\frac{4}{60}}(1-q^{20m-4})(1-q^{20m-16}) \right) \in \mathfrak{A} \end{aligned}$$

Identity A.95,97

$$\begin{aligned} 2\sqrt{5}q^{\frac{7}{120}} \sum_{n=0} \frac{(-q;q^2)_n q^{n(3n-2)}}{(q^2;q^2)_{2n}} &= 2\sqrt{5}q^{\frac{7}{120}} \sum_{n=0} \frac{(-q;q^2)_{n+1} q^{n(3n+2)}}{(q^2;q^2)_{2n+1}} \\ &= \prod_{m=1}^{\infty} \left( \sqrt{5} \frac{q^{\frac{10}{24}}(1-q^{10m})}{q^{\frac{1}{24}}(1-q^{2m})} \right) \left( 2q^{\frac{-13}{60}}(1-q^{10m-3})(1-q^{10m-7}) \right) \\ &\times \left( q^{\frac{4}{60}}(1-q^{20m-4})(1-q^{20m-16}) \right) \left( q^{\frac{-1}{24}}(1+q^{2m-1}) \right) \in \mathfrak{A} \end{aligned}$$

Identity A.96

$$\begin{aligned} 2\sqrt{10}q^{\frac{31}{40}} \sum_{n=0} \frac{q^{n(n+2)}}{(q;q)_{2n+1}} &= \prod_{m=1}^{\infty} \left( \sqrt{10} \frac{q^{\frac{10}{24}}(1-q^{10m})}{q^{\frac{1}{24}}(1-q^m)} \right) \\ &\times \left( q^{\frac{-11}{30}}(1-q^{10m-4})(1-q^{10m-6}) \right) \\ &\times \left( 2q^{\frac{23}{30}}(1-q^{20m-2})(1-q^{20m-18}) \right) \in \mathfrak{A} \end{aligned}$$

Identity A.99

$$\begin{aligned} 2\sqrt{10}q^{\frac{1}{40}} \sum_{n=0} \frac{q^{n(n+1)}}{(q;q)_{2n}} &= \prod_{m=1}^{\infty} \left( \sqrt{10} \frac{q^{\frac{10}{24}}(1-q^{10m})}{q^{\frac{1}{24}}(1-q^m)} \right) \\ &\times \left( 2q^{\frac{23}{60}}(1-q^{10m-1})(1-q^{10m-9}) \right) \\ &\times \left( q^{\frac{-44}{60}}(1-q^{20m-8})(1-q^{20m-12}) \right) \in \mathfrak{A} \end{aligned}$$

Identity A.100

$$\begin{aligned} 2\sqrt{5}q^{\frac{-7}{120}} \sum_{n=0} \frac{(-q;q^2)_n q^{3n^2}}{(q^2;q^2)_{2n}} &= \prod_{m=1}^{\infty} \left( \sqrt{5} \frac{q^{\frac{10}{24}}(1-q^{10m})}{q^{\frac{1}{24}}(1-q^{2m})} \right) \left( q^{\frac{-1}{24}}(1+q^{2m-1}) \right) \\ &\times \left( 2q^{\frac{23}{60}}(1-q^{10m-1})(1-q^{10m-9}) \right) \\ &\times \left( q^{\frac{-44}{60}}(1-q^{20m-8})(1-q^{20m-12}) \right) \in \mathfrak{A} \end{aligned}$$

Identity A.101

$$\begin{aligned}
& 4\sqrt{2}q^{\frac{6}{24}} \frac{(q^2, q^6, q^8; q^8)_\infty}{(q; q)_\infty (q; q^2)_\infty} \\
&= \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{4}{24}} \frac{1}{(1 - q^{8m-4})} \right) \left( \frac{q^{\frac{2}{24}}(1 - q^{2m})}{q^{\frac{8}{24}}(1 - q^{8m})} \right) \\
&\quad \times \left( \sqrt{2} \frac{q^{\frac{1}{24}}}{(1 - q^{2m-1})} \right) \left( \sqrt{8} \frac{q^{\frac{8}{24}}(1 - q^{8m})}{q^{\frac{1}{24}}(1 - q^m)} \right) \in \mathfrak{A}, \\
& 8 \frac{(-q^{16}, -q^{16}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \\
&= \prod_{m=1}^{\infty} \left( q^{\frac{-16}{24}}(1 + q^{32m-16}) \right)^2 \left( \sqrt{2} \frac{q^{\frac{1}{24}}}{(1 - q^{2m-1})} \right) \\
&\quad \times \left( \sqrt{32} \frac{q^{\frac{32}{24}}(1 - q^{32m})}{q^{\frac{1}{24}}(1 - q^m)} \right) \in \mathfrak{A}, \\
& 16q \frac{(-q^8, -q^{24}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \\
&= \prod_{m=1}^{\infty} \left( \frac{\sqrt{2}q^{\frac{8}{24}}(1 + q^{8m})}{q^{\frac{32}{24}}(1 + q^{32m})} \right) \left( \frac{\sqrt{2}q^{\frac{16}{24}}}{(1 + q^{32m-16})} \right) \\
&\quad \times \left( \sqrt{2} \frac{q^{\frac{1}{24}}}{(1 - q^{2m-1})} \right) \left( \sqrt{32} \frac{q^{\frac{32}{24}}(1 - q^{32m})}{q^{\frac{1}{24}}(1 - q^m)} \right) \in \mathfrak{A}, \\
& \sum_{n=1}^{\infty} \frac{(-q; q)_n (-q^2; q^2)_{n-1} q^{n(n-1)/2}}{(q; q)_{2n}} = \frac{(q^2, q^6, q^8; q^8)_\infty}{(q; q)_\infty (q; q^2)_\infty} \\
&\quad + \frac{(-q^{16}, -q^{16}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} - q \frac{(-q^8, -q^{24}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} - 1 \in \mathfrak{T}
\end{aligned}$$

Identity A.102

$$\begin{aligned}
& 8\sqrt{2}\sqrt{2}q^{\frac{6}{24}} \frac{(-q^{12}, -q^{20}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \\
&= \left( \sqrt{2} \prod_{m=1}^{\infty} \frac{q^{\frac{1}{24}}}{(1 - q^{2m-1})} \right) \left( \sqrt{2\sqrt{2}q^{\frac{-13}{12}}} (1 + q^{32m-12})(1 + q^{32m-20}) \right) \\
&\quad \times \left( \sqrt{32} \frac{q^{\frac{32}{24}}(1 - q^{32m})}{q^{\frac{1}{24}}(1 - q^m)} \right) \in \mathfrak{A},
\end{aligned}$$

$$\begin{aligned}
& 8\sqrt{2}\sqrt{2}q^{\frac{54}{24}} \frac{(-q^4, -q^{28}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \\
&= \left( \sqrt{2} \prod_{m=1}^{\infty} \frac{q^{\frac{1}{24}}}{(1 - q^{2m-1})} \right) \left( \sqrt{2}\sqrt{2}q^{\frac{11}{12}}(1 + q^{32m-4})(1 + q^{32m-28}) \right) \\
&\quad \times \left( \sqrt{32} \frac{q^{\frac{32}{24}}(1 - q^{32m})}{q^{\frac{1}{24}}(1 - q^m)} \right) \in \mathfrak{A}, \\
& 16q \frac{(-q^8, -q^{24}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \in \mathfrak{A}, \\
& 6\sqrt{2}q^4 \frac{(-1, -q^{32}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \in \mathfrak{A}, \\
& \sum_{n=0}^{\infty} \frac{(-q; q)_{n+1}(-q^2; q^2)_n q^{n(n+1)/2 + \frac{6}{24}}}{(q; q)_{2n+2}} \\
&= q^{\frac{6}{24}} \frac{(-q^{12}, -q^{20}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \\
&\quad - q^{\frac{54}{24}} \frac{(-q^4, -q^{28}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} + q^{\frac{30}{24}} \frac{(-q^8, -q^{24}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \\
&\quad - q^{\frac{102}{24}} \frac{(-1, -q^{32}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \in \mathfrak{T}
\end{aligned}$$

Identity A.103

$$\begin{aligned}
& 16q \frac{(-q^8, -q^{24}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \in \mathfrak{A}, \\
& 8q^4 \frac{(-1, -q^{32}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \\
&= \prod_{m=1}^{\infty} \left( \sqrt{32} \frac{q^{\frac{32}{24}}(1 - q^{32m})}{q^{\frac{1}{24}}(1 - q^m)} \right) \left( \sqrt{2} \frac{q^{\frac{1}{24}}}{(1 - q^{2m-1})} \right) \\
&\quad \times \left( \sqrt{2}q^{\frac{32}{24}}(1 + q^{32m}) \right)^2 \in \mathfrak{A}, \\
& 16 \sum_{n=0}^{\infty} \frac{(-q; q)_{n+1}(-q^2; q^2)_n q^{n(n+3)/2 + 1}}{(q; q)_{2n+2}} \\
&= 16q \frac{(-q^8, -q^{24}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \\
&\quad - 16q^4 \frac{(-1, -q^{32}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \in \mathfrak{A}
\end{aligned}$$

Identity A.104

$$\begin{aligned}
 & 8 \frac{(-q^{16}, -q^{16}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \in \mathfrak{A}, \\
 & 16q \frac{(-q^8, -q^{24}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \in \mathfrak{A}, \\
 & 16 \sum_{n=1} \frac{(-q; q)_n (-q^2; q^2)_{n-1} q^{n(n+1)/2}}{(q; q)_{2n}} = 16 \frac{(-q^{16}, -q^{16}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} \\
 & \quad - 16q \frac{(-q^8, -q^{24}, q^{32}; q^{32})_\infty}{(q; q)_\infty (q; q^2)_\infty} - 16 \in \mathfrak{A}
 \end{aligned}$$

Identity A.105-a

$$\begin{aligned}
 & 4\sqrt{2}q^{\frac{1}{4}} \sum_{n=0} \frac{(-q^2; q^2)_n q^{n(n+1)}}{(q; q^2)_{n+1} (q; q)_n} \\
 & = \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{-2}{24}} (1 - q^{8m-2})(1 - q^{8m-6}) \right) \left( \sqrt{2}q^{\frac{1}{24}} (1 + q^m) \right) \\
 & \quad \times \left( \sqrt{8} \frac{q^{\frac{8}{24}} (1 - q^{8m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \in \mathfrak{A}
 \end{aligned}$$

Identity A.107

$$\begin{aligned}
 & 2\sqrt{6}q^{1/8} \sum_{n=0} \frac{(q^3; q^6)_n (-q^2; q^2)_n q^{n(n+1)}}{(q^2; q^2)_{2n+1} (q; q^2)_n} = \prod_{m=1}^{\infty} \left( \sqrt{2} \frac{q^{\frac{3}{24}} (1 + q^{3m})}{q^{\frac{6}{24}} (1 + q^{6m})} \right) \\
 & \quad \times \left( \sqrt{6} \frac{q^{\frac{12}{24}} (1 - q^{12m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \left( \frac{q^{\frac{6}{24}} (1 - q^{6m})}{q^{\frac{12}{24}} (1 - q^{12m})} \right) \left( \sqrt{2}q^{\frac{2}{24}} (1 + q^{2m}) \right) \in \mathfrak{A}
 \end{aligned}$$

Identity A.108

$$\begin{aligned}
 & 2\sqrt{6}q \sum_{n=0} \frac{(q^6; q^6)_n (-q; q^2)_{n+1} (1 - q^{2n+2}) q^{n(n+2)}}{(q^2; q^2)_{2n+2} (q^2; q^2)_n} \\
 & = \prod_{m=1}^{\infty} \left( \sqrt{2}q^{\frac{-11}{24}} (1 + q^{12m-5})(1 + q^{12m-7}) \right) \left( \sqrt{6} \frac{q^{\frac{12}{24}} (1 - q^{12m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \\
 & \quad \times \left( \sqrt{2}q^{\frac{26}{24}} (1 - q^{24m-2})(1 - q^{24m-22}) \right) \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right) \in \mathfrak{A}
 \end{aligned}$$

Identity A.109

$$\begin{aligned}
& 2\sqrt{3}q^{-\frac{1}{8}} \frac{(-q^2, -q^{10}, q^{12}; q^{12})_\infty (q^8, q^{16}; q^{24})_\infty}{(q^4; q^4)_\infty (q; q^2)_\infty} \\
&= \prod_{m=1}^{\infty} \left( \frac{\sqrt{2}q^{\frac{1}{24}}}{(1 - q^{2m-1})} \right) \left( \sqrt{2}q^{\frac{1}{8}}(1 + q^{12m-2})(1 + q^{12m-10}) \right) \\
&\quad \times \left( \sqrt{2}q^{-\frac{8}{12}}(1 - q^{24m-8})(1 - q^{24m-16}) \right) \\
&\quad \times \left( \sqrt{3} \frac{q^{\frac{12}{24}}(1 - q^{12m})}{q^{\frac{4}{24}}(1 - q^{4m})} \right) \in \mathfrak{A}, \\
& \sqrt{3}q^{9/24} \frac{(q^{12}; q^{12})_\infty}{(q^4; q^4)_\infty (q; q^2)_\infty} \\
&= \prod_{m=1}^{\infty} \left( \sqrt{3} \frac{q^{\frac{12}{24}}(1 - q^{12m})}{q^{\frac{4}{24}}(1 - q^{4m})} \right) \left( \frac{q^{1/24}}{1 - q^{2m-1}} \right) \in \mathfrak{A}, \\
& \sum_{n=0}^{\infty} \frac{(q^3; q^6)_n (-q; q^2)_{n+1} q^{n^2 - \frac{1}{8}}}{(q^2; q^2)_{2n+1} (q; q^2)_n} \\
&= q^{-\frac{1}{8}} \left( \frac{(-q^2, -q^{10}, q^{12}; q^{12})_\infty (q^8, q^{16}; q^{24})_\infty}{(q^4; q^4)_\infty (q; q^2)_\infty} \right. \\
&\quad \left. + q \frac{(q^{12}; q^{12})_\infty}{(q^4; q^4)_\infty (q; q^2)_\infty} \right) \in \mathfrak{T}
\end{aligned}$$

Identity A.109-a

$$\begin{aligned}
& 2\sqrt{6}q^{\frac{1}{24}} \sum_{n=0}^{\infty} \frac{(q^3; q^6)_n q^{n^2}}{(q^4; q^4)_n (q; q^2)_n^2} \\
&= \prod_{m=1}^{\infty} \left( 2q^{\frac{4}{24}}(1 + q^{12m-2})(1 + q^{12m-10}) \right) \\
&\quad \times \left( \sqrt{6} \frac{q^{\frac{12}{24}}(1 - q^{12m})}{q^{\frac{2}{24}}(1 - q^{2m})} \right) \left( q^{\frac{-12}{24}}(1 - q^{24m-8})(1 - q^{24m-16}) \right) \\
&\quad \times \left( q^{\frac{-1}{24}}(1 + q^{2m-1}) \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.110

$$\begin{aligned}
& \sqrt{6}q^{\frac{3}{8}} \sum_{n=0}^{\infty} \frac{(q^3; q^6)_n (-q; q^2)_{n+1} q^{n(n+2)}}{(q^2; q^2)_{2n+1} (q; q^2)_n} \\
&= \prod_{m=1}^{\infty} \left( q^{\frac{-1}{24}}(1 + q^{2m-1}) \right) \left( \sqrt{6} \frac{q^{\frac{12}{24}}(1 - q^{12m})}{q^{\frac{2}{24}}(1 - q^{2m})} \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.111

$$\begin{aligned}
6 \frac{(q^{15}, q^{21}, q^{36}; q^{36})_\infty}{(q^4; q^4)_\infty (q; q^2)_\infty} &= \prod_{m=1}^{\infty} \left( \sqrt{2} \frac{q^{\frac{1}{24}}}{(1 - q^{2m-1})} \right) \\
&\times \left( 3 \frac{q^{\frac{36}{24}} (1 - q^{36m})}{q^{\frac{4}{24}} (1 - q^{4m})} \right) \left( \sqrt{2} q^{\frac{-33}{24}} (1 - q^{36m-15}) (1 - q^{36m-21}) \right) \in \mathfrak{A}, \\
3\sqrt{2}q \frac{(q^9, q^{27}, q^{36}; q^{36})_\infty}{(q^4; q^4)_\infty (q; q^2)_\infty} &= \prod_{m=1}^{\infty} \left( \sqrt{2} \frac{q^{\frac{1}{24}}}{(1 - q^{2m-1})} \right) \\
&\times \left( 3 \frac{q^{\frac{36}{24}} (1 - q^{36m})}{q^{\frac{4}{24}} (1 - q^{4m})} \right) \left( \frac{q^{\frac{9}{24}} (1 - q^{9m})}{q^{\frac{18}{24}} (1 - q^{18m})} \right) \in \mathfrak{A}, \\
6 \sum_{n=1} \frac{(q^6; q^6)_{n-1} (-q; q^2)_n q^{n(n+2)}}{(q^2; q^2)_{2n} (q^2; q^2)_{n-1}} &= 6 \frac{(q^{15}, q^{21}, q^{36}; q^{36})_\infty}{(q^4; q^4)_\infty (q; q^2)_\infty} \\
&- 6q \frac{(q^9, q^{27}, q^{36}; q^{36})_\infty}{(q^4; q^4)_\infty (q; q^2)_\infty} - 6 \in \mathfrak{A}
\end{aligned}$$

Identity A.112

$$\begin{aligned}
\frac{(q^9, q^{27}, q^{36}; q^{36})_\infty}{(q^4; q^4)_\infty (q; q^2)_\infty} &\in \mathfrak{T}, \\
6q^3 \frac{(q^3, q^{33}, q^{36}; q^{36})_\infty}{(q^4; q^4)_\infty (q; q^2)_\infty} &= \prod_{m=1}^{\infty} \left( \sqrt{2} \frac{q^{\frac{1}{24}}}{(1 - q^{2m-1})} \right) \\
&\times \left( 3 \frac{q^{\frac{36}{24}} (1 - q^{36m})}{q^{\frac{4}{24}} (1 - q^{4m})} \right) \left( \sqrt{2} q^{\frac{39}{24}} (1 - q^{36m-3}) (1 - q^{36m-33}) \right) \in \mathfrak{A}, \\
\sum_{n=0} \frac{(q^6; q^6)_n (-q; q^2)_{n+2} q^{n(n+2)}}{(q^2; q^2)_{2n+2} (q^2; q^2)_n} &= \frac{(q^9, q^{27}, q^{36}; q^{36})_\infty}{(q^4; q^4)_\infty (q; q^2)_\infty} \\
&+ q^3 \frac{(q^3, q^{33}, q^{36}; q^{36})_\infty}{(q^4; q^4)_\infty (q; q^2)_\infty} \in \mathfrak{T}
\end{aligned}$$

Identity A.113

$$\begin{aligned}
6 \sum_{n=1} \frac{(q^6; q^6)_{n-1} (-q; q^2)_n q^{n(n+2)}}{(q^2; q^2)_{2n-1} (q^2; q^2)_{n-1}} &= 6 \frac{(q^{15}, q^{21}, q^{36}; q^{36})_\infty}{(q^4; q^4)_\infty (q; q^2)_\infty} \\
&- 6q^3 \frac{(q^3, q^{33}, q^{36}; q^{36})_\infty}{(q^4; q^4)_\infty (q; q^2)_\infty} - 6 \in \mathfrak{A}
\end{aligned}$$

Identity A.114

$$3 \cdot 2^{\frac{3}{4}} \left( 1 + \sum_{n=1}^{\infty} \frac{(q^6; q^6)_{n-1}(-q; q^2)_n q^{n^2}}{(q^2; q^2)_{2n-1} (q^2; q^2)_n} \right) = \prod_{m=1}^{\infty} \left( \sqrt{18} \frac{q^{\frac{36}{24}} (1 - q^{36m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right)$$

$$\times \left( \sqrt{\sqrt{2} q^{\frac{-33}{24}}} (1 - q^{36m-15})(1 - q^{36m-21}) \right) \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right) \in \mathfrak{A},$$

$$6 \sum_{n=1}^{\infty} \frac{(q^6; q^6)_{n-1}(-q; q^2)_n q^{n^2}}{(q^2; q^2)_{2n-1} (q^2; q^2)_n} \in \mathfrak{A}$$

Identity A.115

$$3 \cdot 2^{\frac{3}{4}} q \sum_{n=0}^{\infty} \frac{(q^6; q^6)_n (-q; q^2)_{n+1} q^{n(n+2)}}{(q^2; q^2)_{2n+2} (q^2; q^2)_n} = \prod_{m=1}^{\infty} \left( \sqrt{18} \frac{q^{\frac{36}{24}} (1 - q^{36m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right)$$

$$\times \left( \sqrt{\sqrt{2}} \frac{q^{\frac{18}{24}}}{(1 - q^{36m-18})} \right) \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right) \left( \frac{q^{\frac{9}{24}} (1 - q^{9m})}{q^{\frac{36}{24}} (1 - q^{36m})} \right) \in \mathfrak{A}$$

Identity A.116

$$6q^3 \sum_{n=0}^{\infty} \frac{(q^6; q^6)_n (-q; q^2)_{n+1} q^{n(n+4)}}{(q^2; q^2)_{2n+2} (q^2; q^2)_n} = \prod_{m=1}^{\infty} \left( \sqrt{18} \frac{q^{\frac{36}{24}} (1 - q^{36m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right)$$

$$\times \left( \sqrt{2} q^{\frac{39}{24}} (1 - q^{36m-3})(1 - q^{36m-33}) \right) \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right) \in \mathfrak{A}$$

Identity A.117

$$q^{\frac{-13}{168}} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(q^2; q^2)_{2n}} = \prod_{m=1}^{\infty} \left( \frac{q^{\frac{14}{24}} (1 - q^{14m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right)$$

$$\times \left( q^{\frac{-1}{84}} (1 - q^{14m-3})(1 - q^{14m-11}) \right)$$

$$\times \left( q^{\frac{-11}{21}} (1 - q^{28m-8})(1 - q^{28m-20}) \right) \in \overline{\mathbb{Q}}$$

Identity A.118

$$q^{\frac{11}{168}} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n(n+2)}}{(q^2; q^2)_{2n}} = \prod_{m=1}^{\infty} \left( \frac{q^{\frac{14}{24}} (1 - q^{14m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right)$$

$$\times \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right) \left( q^{\frac{59}{84}} (1 - q^{14m-1})(1 - q^{14m-13}) \right)$$

$$\times \left( q^{\frac{-23}{21}} (1 - q^{28m-12})(1 - q^{28m-16}) \right) \in \overline{\mathbb{Q}}$$

Identity A.119

$$\begin{aligned} q^{\frac{107}{168}} \sum_{n=0}^{\infty} \frac{(-q; q^2)_{n+1} q^{n(n+2)}}{(q^2; q^2)_{2n+1}} &= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{14}{24}} (1 - q^{14m})}{q^{\frac{2}{24}} (1 - q^{2m})} \right) \\ &\times \left( q^{\frac{-1}{24}} (1 + q^{2m-1}) \right) \left( q^{\frac{-37}{84}} (1 - q^{14m-5})(1 - q^{14m-9}) \right) \\ &\times \left( q^{\frac{13}{21}} (1 - q^{28m-4})(1 - q^{28m-24}) \right) \in \overline{\mathbb{Q}} \end{aligned}$$

Identity A.120

$$\begin{aligned} \frac{(-q^{22}, -q^{26}, q^{48}; q^{48})_{\infty}}{(q; q)_{\infty}} &= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{48}{24}} (1 - q^{48m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \\ &\times \left( q^{\frac{-47}{24}} (1 + q^{48m-22})(1 + q^{48m-26}) \right) \in \overline{\mathbb{Q}}, \\ q \frac{(-q^{14}, -q^{34}, q^{48}; q^{48})_{\infty}}{(q; q)_{\infty}} &= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{48}{24}} (1 - q^{48m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \\ &\times \left( q^{\frac{-23}{24}} (1 + q^{48m-14})(1 + q^{48m-34}) \right) \in \overline{\mathbb{Q}}, \\ 1 + \sum_{n=1}^{\infty} \frac{(-q^2; q^2)_{n-1} q^{n(n+1)}}{(q; q)_{2n}} &= \frac{(-q^{22}, -q^{26}, q^{48}; q^{48})_{\infty}}{(q; q)_{\infty}} \\ - \frac{q(-q^{14}, -q^{34}, q^{48}; q^{48})_{\infty}}{(q; q)_{\infty}} &\in \overline{\mathbb{Q}} \end{aligned}$$

Identity A.121

$$\begin{aligned} 8\sqrt{2} \left( 1 + \sum_{n=1}^{\infty} \frac{(-q^2; q^2)_{n-1} q^{n^2}}{(q; q)_{2n}} \right) &= \prod_{m=1}^{\infty} \left( 4 \frac{q^{\frac{16}{24}} (1 - q^{16m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \\ &\times \left( \sqrt{2\sqrt{2}} q^{\frac{11}{24}} (1 - q^{16m-2})(1 - q^{16m-14}) \right) \\ &\times \left( \sqrt{2\sqrt{2}} q^{\frac{-13}{12}} (1 - q^{32m-12})(1 - q^{32m-20}) \right) \in \mathfrak{A}, \\ 8\sqrt{2} \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_{n-1} q^{n^2}}{(q; q)_{2n}} &\in \mathfrak{A} \end{aligned}$$

Identity A.122

$$\begin{aligned} q^2 \frac{(-q^{10}, -q^{38}, q^{48}; q^{48})_{\infty}}{(q; q)_{\infty}} &= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{48}{24}} (1 - q^{48m})}{q^{\frac{1}{24}} (1 - q^m)} \right) \\ &\times \left( q^{\frac{1}{24}} (1 + q^{48m-10})(1 + q^{48m-38}) \right) \in \overline{\mathbb{Q}}, \end{aligned}$$

$$\begin{aligned}
q^5 \frac{(-q^2, -q^{46}, q^{48}; q^{48})_\infty}{(q; q)_\infty} &= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{48}{24}}(1-q^{48m})}{q^{\frac{1}{24}}(1-q^m)} \right) \\
&\times \left( q^{\frac{73}{24}}(1+q^{48m-2})(1+q^{48m-46}) \right) \in \overline{\mathbb{Q}}, \\
q^2 \sum_{n=0} \frac{(-q^2; q^2)_n q^{n(n+3)}}{(q; q)_{2n+2}} &= \frac{q^2(-q^{10}, -q^{38}, q^{48}; q^{48})_\infty}{(q; q)_\infty} \\
&- \frac{q^5(-q^2, -q^{46}, q^{48}; q^{48})_\infty}{(q; q)_\infty} \in \overline{\mathbb{Q}}
\end{aligned}$$

Identity A.123

$$\begin{aligned}
8\sqrt{2}q \sum_{n=0} \frac{(-q^2; q^2)_n q^{n(n+2)}}{(q; q)_{2n+2}} &= \prod_{m=1}^{\infty} \left( 4 \frac{q^{\frac{16}{24}}(1-q^{16m})}{q^{\frac{1}{24}}(1-q^m)} \right) \\
&\times \left( \sqrt{2\sqrt{2}}q^{\frac{-13}{24}}(1-q^{16m-6})(1-q^{16m-10}) \right) \\
&\times \left( \sqrt{2\sqrt{2}}q^{\frac{11}{12}}(1-q^{32m-4})(1-q^{32m-28}) \right) \in \mathfrak{A}
\end{aligned}$$

Identity A.124

$$\begin{aligned}
q^{\frac{1}{4}} \sum_{n=0} \frac{(q^3; q^6)_n q^{2n(n+1)}}{(q^2; q^2)_{2n+1}(q; q^2)_n} &= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{18}{24}}(1-q^{18m})}{q^{\frac{2}{24}}(1-q^{2m})} \right) \\
&\times \left( q^{\frac{-11}{36}}(1-q^{18m-5})(1-q^{18m-13}) \right) \\
&\times \left( q^{\frac{-1}{9}}(1-q^{36m-8})(1-q^{36m-28}) \right) \in \overline{\mathbb{Q}}
\end{aligned}$$

Identity A.125

$$\begin{aligned}
q^{\frac{5}{4}} \sum_{n=0} \frac{(q^3; q^6)_n q^{2n(n+2)}}{(q^2; q^2)_{2n+1}(q; q^2)_n} &= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{18}{24}}(1-q^{18m})}{q^{\frac{2}{24}}(1-q^{2m})} \right) \\
&\times \left( q^{\frac{-23}{36}}(1-q^{18m-7})(1-q^{18m-11}) \right) \\
&\times \left( q^{\frac{11}{9}}(1-q^{36m-4})(1-q^{36m-32}) \right) \in \overline{\mathbb{Q}}
\end{aligned}$$

Identity A.126

$$\begin{aligned}
\frac{(-q^{28}, -q^{36}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty(q^4; q^4)_\infty} &= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{64}{24}}(1-q^{64m})}{q^{\frac{4}{24}}(1-q^{4m})} \right) \\
&\times \left( q^{\frac{-61}{24}}(1+q^{64m-28})(1+q^{64m-36}) \right) \left( \frac{q^{\frac{1}{24}}}{(1-q^{2m-1})} \right) \in \overline{\mathbb{Q}},
\end{aligned}$$

$$\begin{aligned}
q^3 \frac{(-q^{12}, -q^{52}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} &= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{64}{24}}(1-q^{64m})}{q^{\frac{4}{24}}(1-q^{4m})} \right) \\
&\times \left( q^{\frac{11}{24}}(1+q^{64m-12})(1+q^{64m-52}) \right) \left( \frac{q^{\frac{1}{24}}}{(1-q^{2m-1})} \right) \in \overline{\mathbb{Q}}, \\
1 + \sum_{n=1} \frac{(-q^4; q^4)_{n-1} (-q; q^2)_n q^{n^2}}{(q^2; q^2)_{2n}} &= \frac{(-q^{28}, -q^{36}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} \\
-\frac{q^3(-q^{12}, -q^{52}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} &\in \overline{\mathbb{Q}}
\end{aligned}$$

Identity A.127

$$\begin{aligned}
\frac{(-q^{28}, -q^{36}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} &\in \overline{\mathbb{Q}}, \\
q \frac{(-q^{20}, -q^{44}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} &= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{64}{24}}(1-q^{64m})}{q^{\frac{4}{24}}(1-q^{4m})} \right) \\
&\times \left( q^{\frac{-37}{24}}(1+q^{64m-20})(1+q^{64m-44}) \right) \left( \frac{q^{\frac{1}{24}}}{(1-q^{2m-1})} \right) \in \overline{\mathbb{Q}}, \\
1 + \sum_{n=1} \frac{(-q^4; q^4)_{n-1} (-q; q^2)_n q^{n(n+2)}}{(q^2; q^2)_{2n}} &= \frac{(-q^{28}, -q^{36}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} \\
-\frac{q(-q^{20}, -q^{44}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} &\in \overline{\mathbb{Q}}
\end{aligned}$$

Identity A.128

$$\begin{aligned}
q \frac{(-q^{20}, -q^{44}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} &\in \overline{\mathbb{Q}}, \\
q^6 \frac{(-q^4, -q^{60}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} &= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{64}{24}}(1-q^{64m})}{q^{\frac{4}{24}}(1-q^{4m})} \right) \\
&\times \left( q^{\frac{83}{24}}(1+q^{64m-4})(1+q^{64m-60}) \right) \left( \frac{q^{\frac{1}{24}}}{(1-q^{2m-1})} \right) \in \overline{\mathbb{Q}}, \\
q \sum_{n=0} \frac{(-q^4; q^4)_n (-q; q^2)_{n+1} q^{n(n+2)}}{(q^2; q^2)_{2n+2}} &= \frac{q(-q^{20}, -q^{44}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} \\
-\frac{q^6(-q^4, -q^{60}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} &\in \overline{\mathbb{Q}}
\end{aligned}$$

Identity A.129

$$\begin{aligned} q^6 \frac{(-q^4, -q^{60}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} &\in \overline{\mathbb{Q}}, \\ q^3 \frac{(-q^{12}, -q^{52}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} &\in \overline{\mathbb{Q}}, \\ q^3 \sum_{n=0} \frac{(-q^4; q^4)_n (-q; q^2)_{n+1} q^{n(n+4)}}{(q^2; q^2)_{2n+2}} &= \frac{q^3 (-q^{12}, -q^{52}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} \\ - \frac{q^6 (-q^4, -q^{60}, q^{64}; q^{64})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} &\in \overline{\mathbb{Q}} \end{aligned}$$

Identity A.130

$$\begin{aligned} &2\sqrt{2}q^{-1/8} \frac{(q^8, q^8, q^{16}; q^{16})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} \\ &= \prod_{m=1}^{\infty} \left( q^{\frac{-16}{24}} (1 - q^{16m-8})^2 \right) \\ &\quad \times \left( \sqrt{2} \frac{q^{\frac{1}{24}}}{(1 - q^{2m-1})} \right) \left( 2 \frac{q^{\frac{16}{24}} (1 - q^{16m})}{q^{\frac{4}{24}} (1 - q^{4m})} \right) \in \mathfrak{A}, \\ &2\sqrt{2}q^{3/8} \frac{(q^4, q^{12}, q^{16}; q^{16})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} \\ &= \prod_{m=1}^{\infty} \left( \frac{q^{\frac{4}{24}} (1 - q^{4m})}{q^{\frac{8}{24}} (1 - q^{8m})} \right) \left( \sqrt{2} \frac{q^{\frac{1}{24}}}{(1 - q^{2m-1})} \right) \left( 2 \frac{q^{\frac{16}{24}} (1 - q^{16m})}{q^{\frac{4}{24}} (1 - q^{4m})} \right) \in \mathfrak{A}, \\ &\sum_{n=0} \frac{(-q^2; q^4)_n (-q; q^2)_{n+1} q^{n^2}}{(q^2; q^2)_{2n+1}} \\ &= \frac{(q^8, q^8, q^{16}; q^{16})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} - \frac{q(q^4, q^{12}, q^{16}; q^{16})_\infty}{(q; q^2)_\infty (q^4; q^4)_\infty} \in \mathfrak{T}. \end{aligned}$$

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