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NONEXISTENCE OF A CREPANT RESOLUTION OF SOME MODULI SPACES OF SHEAVES ON A K3 SURFACE

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ABSTRACT. Let $M_c = M(2,0,c)$ be the moduli space of $\mathcal{O}(1)$-semistable rank 2 torsion-free sheaves with Chern classes $c_1 = 0$ and $c_2 = c$ on a K3 surface X, where $\mathcal{O}(1)$ is a generic ample line bundle on X. When $c = 2n \geq 4$ is even, M_c is a singular projective variety equipped with a holomorphic symplectic structure on the smooth locus. In particular, M_c has trivial canonical divisor. In [22], O'Grady asks if there is any symplectic desingularization of M_{2n} for $n \geq 3$. In this paper, we show that there is no crepant resolution of M_{2n} for $n \geq 3$. This obviously implies that there is no symplectic desingularization.

1. Introduction

Let X be a complex projective K3 surface with polarization $H = \mathcal{O}_X(1)$ generic in the sense of [22] §0. Let $M(r, c_1, c_2)$ be the moduli space of rank r H-semistable torsion-free sheaves on X with Chern classes (c_1, c_2) in $H^*(X, \mathbb{Z})$. Let $M^s(r, c_1, c_2)$ be the open subscheme of H-stable sheaves in $M(r, c_1, c_2)$. In [19], Mukai shows that $M^s(r, c_1, c_2)$ is smooth and has a holomorphic symplectic structure. By [6], if either $(c_1.H)$ or c_2 is an odd number, then $M(2, c_1, c_2)$ is equal to $M^s(2, c_1, c_2)$ and thus $M(2, c_1, c_2)$ is a smooth projective irreducible symplectic variety. However if both $(c_1.H)$ and c_2 are even numbers then generally $M(2, c_1, c_2)$ admits singularities. We restrict our interest to the trivial determinant case i.e., $c_1 = 0$ and let $M_c = M(2, 0, c)$, where $c = 2n$ $(n \geq 2)$. It is well-known that M_{2n} is an irreducible, normal ([26] Theorem 3.18) and projective variety ([12] Theorem 4.3.4) of dimension $8n - 6$ ([19] Theorem 0.1) with only Gorenstein singularities ([12] Theorem 4.5.8, [5] Corollary 21.19). Since M_{2n} contains the smooth open subset M_{2n}^s, there arises a natural question: does there exist a resolution of M_{2n} such that the Mukai form on M_{2n}^s

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extends to the resolution without degeneration? When $c = 4$, O'Grady successfully extends the Mukai form on M_{2n}^s to some resolution without degeneration ([20, 22]). At the same time, he conjectures nonexistence of a symplectic desingularization of M_{2n} for $n \geq 3$ ([22], (0.1)). Our main result in this paper is the following.

Theorem 1.1. *If $n \geq 3$, there is no crepant resolution of M_{2n}.*

The highest exterior power of a symplectic form gives a non-vanishing section of the canonical sheaf on M_{2n}. Likewise any symplectic desingularization of M_{2n} has trivial canonical divisor and hence it must be a crepant resolution. Therefore, O'Grady's conjecture is a consequence of Theorem 1.1.

Corollary 1.2. *If $n \geq 3$, there is no symplectic desingularization of M_{2n}.*

The idea of the proof of Theorem 1.1 is to use a new invariant called the stringy E-function [1, 4]. Since M_{2n} is normal irreducible variety with log terminal singularities ([22], 6.1), the stringy E-function of M_{2n} is a well-defined rational function. If there is a crepant resolution $\widetilde{M}_{2n}$ of M_{2n}, then the stringy E-function of M_{2n} is equal to the Hodge-Deligne polynomial (E-polynomial) of $\widetilde{M}_{2n}$ (Theorem 2.1). In particular, we deduce that the stringy E-function $E_{st}(M_{2n}; u, v)$ must be a polynomial. Therefore, Theorem 1.1 is a consequence of the following.

Proposition 1.3. *The stringy E-function $E_{st}(M_{2n}; u, v)$ is not a polynomial for $n \geq 3$.*

To prove that $E_{st}(M_{2n}; u, v)$ is not a polynomial for $n \geq 3$, we show that $E_{st}(M_{2n}; z, z)$ is not a polynomial in z. Thanks to the detailed analysis of Kirwan's desingularization in [20] and [22] which is reviewed in section 4, we can find an expression for $E_{st}(M_{2n}; z, z)$ and then with some efforts on the combinatorics of rational functions we show that $E_{st}(M_{2n}; z, z)$ is not a polynomial in section 3. In section 2, we recall basic facts on stringy E-function and in section 5 we prove a lemma which computes the E-polynomial of a divisor.

In [22], O'Grady gets a symplectic desingularization $\widetilde{M}_{2n}$ of M_{2n} in the case when $n = 2$. This turns out to be a new irreducible symplectic variety, which means that it does not come from a generalized Kummer variety nor from a Hilbert scheme parameterizing 0-dimensional subschemes on a K3 surface [21, 2]. Corollary 1.2 shows that unfortunately we cannot find any more irreducible symplectic variety in this way.

After we finished the first draft of this paper, we learned that Kaledin and Lehn [13] proved Corollary 1.2 in a completely different way. We are grateful to D. Kaledin for informing us of their approach. The second named author thanks Professor Jun Li for useful discussions concerning the article [23]. Finally we would like to express our gratitude to the referee for careful reading and challenging us for many details which led us to improve the manuscript and correct an error in Proposition 3.2.

2. Preliminaries

In this section we collect some facts that we shall use later.

For a topological space V, the Poincaré polynomial of V is defined as

$$P(V;z)=\sum_i(-1)^i b_i(V)z^i, \tag{2.1}$$

where $b_i(V)$ is the i-th Betti number of V. It is well-known from [7] that the Betti numbers of the Hilbert scheme of points $X^{[n]}$ in X are given by the following:

$$\sum_{n\geq 0}P(X^{[n]};z)t^n=\prod_{k\geq 1}\prod_{i=0}^{4}(1-z^{2k-2+i}t^k)^{(-1)^{i+1}b_i(X)}. \tag{2.2}$$

Next we recall the definition and basic facts about stringy E-functions from [1, 4]. Let W be a normal irreducible variety with at worst log-terminal singularities, i.e.,

(1) W is $\mathbb{Q}$-Gorenstein;
(2) for a resolution of singularities $\rho: V\to W$ such that the exceptional locus of ρ is a divisor D whose irreducible components $D_1,\ldots,D_r$ are smooth divisors with only normal crossings, we have

$$K_V=\rho^*K_W+\sum_{i=1}^{r}a_iD_i$$

with $a_i>-1$ for all i, where D_i runs over all irreducible components of D. The divisor $\sum_{i=1}^r a_iD_i$ is called the *discrepancy divisor.*

For each subset $J\subset I=\{1,2,\ldots,r\}$, define $D_J=\cap_{j\in J}D_j$, $D_\emptyset=V$ and $D_J^0=D_J-\cup_{i\in I-J}D_i$. Then the stringy E-function of W is defined by

$$E_{st}(W;u,v)=\sum_{J\subset I}E(D_J^0;u,v)\prod_{j\in J}\frac{uv-1}{(uv)^{a_j+1}-1}, \tag{2.3}$$

where

$$E(Z;u,v)=\sum_{p,q}\sum_{k\geq 0}(-1)^k h^{p,q}(H_c^k(Z;\mathbb{C}))u^pv^q$$

is the Hodge-Deligne polynomial for a variety Z. Note that the Hodge-Deligne polynomials have

(1) the additive property: $E(Z;u,v)=E(U;u,v)+E(Z-U;u,v)$ if U is a smooth open subvariety of Z;
(2) the multiplicative property: $E(Z;u,v)=E(B;u,v)E(F;u,v)$ if Z is a Zariski locally trivial F-bundle over B.

By [1] Theorem 6.27, the function E_{st} is independent of the choice of a resolution (Theorem 3.4 in [1]) and the following holds.

Theorem 2.1 ([1] Theorem 3.12). *Suppose W is a $\mathbb{Q}$-Gorenstein algebraic variety with at worst log-terminal singularities. If $\rho : V \to W$ is a crepant desingularization (i.e., $\rho^* K_W = K_V$) then $E_{st}(W;u,v) = E(V;u,v)$. In particular, $E_{st}(W;u,v)$ is a polynomial.*

3. Proof of Proposition 1.3

In this section we first find an expression for the stringy E-function of the moduli space M_{2n} for $n \geq 3$ by using the detailed analysis of Kirwan's desingularization in [20, 22]. Then we show that it cannot be a polynomial, which proves Proposition 1.3.

We fix a generic polarization of X as in [22]. The moduli space M_{2n} has a stratification

$$M_{2n} = M_{2n}^s \sqcup (\Sigma - \Omega) \sqcup \Omega,$$

where M_{2n}^s is the locus of stable sheaves and $\Sigma \simeq (X^{[n]} \times X^{[n]})/\text{involution}$ is the locus of sheaves of the form $I_Z \oplus I_{Z'}$ ($[Z],[Z'] \in X^{[n]}$) while $\Omega \simeq X^{[n]}$ is the locus of sheaves $I_Z \oplus I_Z$. For $n \geq 3$, Kirwan's desingularization $\rho : \widehat{M_{2n}} \to M_{2n}$ is obtained by blowing up M_{2n} first along Ω, next along the proper transform of Σ and finally along the proper transform of a subvariety Δ in the exceptional divisor of the first blow-up. This is indeed a desingularization by [22] Proposition 1.8.3.

Let $D_1 = \hat{\Omega}$, $D_2 = \hat{\Sigma}$ and $D_3 = \hat{\Delta}$ be the (proper transforms of the) exceptional divisors of the three blow-ups. Then they are smooth divisors with only normal crossings as we will see in Proposition 3.2 and the discrepancy divisor of $\rho : \widehat{M_{2n}} \to M_{2n}$ is ([22], 6.1)

$$(6n-7)D_1 + (2n-4)D_2 + (4n-6)D_3.$$

Therefore the singularities are log-terminal for $n \geq 2$, and from (2.3) the stringy E-function of M_{2n} is given by

$$\begin{aligned}
&E(M_{2n}^s;u,v) + E(D_1^0;u,v)\tfrac{1-uv}{1-(uv)^{6n-6}} + E(D_2^0;u,v)\tfrac{1-uv}{1-(uv)^{2n-3}} \\
&+E(D_3^0;u,v)\tfrac{1-uv}{1-(uv)^{4n-5}} + E(D_{12}^0;u,v)\tfrac{1-uv}{1-(uv)^{6n-6}}\tfrac{1-uv}{1-(uv)^{2n-3}} \\
&+E(D_{23}^0;u,v)\tfrac{1-uv}{1-(uv)^{2n-3}}\tfrac{1-uv}{1-(uv)^{4n-5}} \\
&+E(D_{13}^0;u,v)\tfrac{1-uv}{1-(uv)^{4n-5}}\tfrac{1-uv}{1-(uv)^{6n-6}} \\
&+E(D_{123}^0;u,v)\tfrac{1-uv}{1-(uv)^{6n-6}}\tfrac{1-uv}{1-(uv)^{2n-3}}\tfrac{1-uv}{1-(uv)^{4n-5}}.
\end{aligned} \tag{3.1}$$

We need to compute the Hodge-Deligne polynomials of D_J^0 for $J \subset \{1,2,3\}$. Let $(\mathbb{C}^{2n},\omega)$ be a symplectic vector space. Let $\mathrm{Gr}^\omega(k,2n)$ be the Grassmannian of k-dimensional subspaces of $\mathbb{C}^{2n}$, isotropic with respect to the symplectic form ω (i.e., the restriction of ω to the subspace is zero).

Lemma 3.1. *For $k \le n$, the Hodge-Deligne polynomial of* $\mathrm{Gr}^\omega(k,2n)$ *is*

$$\prod_{1\le i\le k} \frac{1-(uv)^{2n-2k+2i}}{1-(uv)^i}.$$

Proof. Consider the incidence variety

$$Z = \{(a,b) \in \mathrm{Gr}^\omega(k-1,2n) \times \mathrm{Gr}^\omega(k,2n) | a \subset b\}.$$

This is a $\mathbb{P}^{2n-2k+1}$-bundle over $\mathrm{Gr}^\omega(k-1,2n)$ and a $\mathbb{P}^{k-1}$-bundle over $\mathrm{Gr}^\omega(k, 2n)$. We have the following equalities between Hodge-Deligne polynomials:

$$\begin{aligned} E(Z;u,v) &= \frac{1-(uv)^{2n-2k+2}}{1-uv} E(\mathrm{Gr}^\omega(k-1,2n);u,v) \\ &= \frac{1-(uv)^k}{1-uv} E(\mathrm{Gr}^\omega(k,2n);u,v). \end{aligned}$$

The desired formula follows recursively from $\mathrm{Gr}^\omega(1,2n) = \mathbb{P}^{2n-1}$. □

Let $\hat{\mathbb{P}}^5$ be the blow-up of $\mathbb{P}^5$ (projectivization of the space of 3×3 symmetric matrices) along $\mathbb{P}^2$ (the locus of rank 1 matrices). We have the following from [20] and [22]. The proof will be presented in §4.

Proposition 3.2. *Let $n \ge 3$.*

(1) *D_1 is a $\hat{\mathbb{P}}^5$-bundle over a $\mathrm{Gr}^\omega(3,2n)$-bundle over $X^{[n]}$.*

(2) *D_2^0 is a free $\mathbb{Z}_2$-quotient of a Zariski locally trivial I_{2n-3}-bundle over $X^{[n]} \times X^{[n]} - \boldsymbol{\Delta}$, where $\boldsymbol{\Delta}$ is the diagonal in $X^{[n]} \times X^{[n]}$ and I_{2n-3} is the incidence variety given by*

$$I_{2n-3} = \{(p,H) \in \mathbb{P}^{2n-3} \times \check{\mathbb{P}}^{2n-3} | p \in H\}.$$

(3) *D_3 is a $\mathbb{P}^{2n-4}$-bundle over a Zariski locally trivial $\mathbb{P}^2$-bundle over a Zariski locally trivial $\mathrm{Gr}^\omega(2,2n)$-bundle over $X^{[n]}$.*

(4) *D_{12} is a $\mathbb{P}^2$-bundle over a $\mathbb{P}^2$-bundle over a $\mathrm{Gr}^\omega(3,2n)$-bundle over $X^{[n]}$.*

(5) *D_{23} is a $\mathbb{P}^{2n-4}$-bundle over a $\mathbb{P}^1$-bundle over a $\mathrm{Gr}^\omega(2,2n)$-bundle over $X^{[n]}$.*

(6) *D_{13} is a $\mathbb{P}^2$-bundle over a $\mathbb{P}^2$-bundle over a $\mathrm{Gr}^\omega(3,2n)$-bundle over $X^{[n]}$.*

(7) *D_{123} is a $\mathbb{P}^1$-bundle over a $\mathbb{P}^2$-bundle over a $\mathrm{Gr}^\omega(3,2n)$-bundle over $X^{[n]}$.*

All the above bundles except in (2) and (3) are Zariski locally trivial. Moreover, D_i $(i=1,2,3)$ are smooth divisors such that $D_1 \cup D_2 \cup D_3$ is normal crossing.

From Lemma 3.1 and Proposition 3.2, we have the following corollary by the additive and multiplicative properties of the Hodge-Deligne polynomial.

Corollary 3.3.

$$\begin{aligned} E(D_1;u,v) = &\left(\tfrac{1-(uv)^6}{1-uv} - \tfrac{1-(uv)^3}{1-uv} + \left(\tfrac{1-(uv)^3}{1-uv}\right)^2\right) \\ &\times \prod_{1\le i\le 3} \left(\tfrac{1-(uv)^{2n-6+2i}}{1-(uv)^i}\right) \times E(X^{[n]};u,v), \end{aligned}$$

$$E(D_3;u,v)=\tfrac{1-(uv)^{2n-3}}{1-uv}\cdot\tfrac{1-(uv)^3}{1-uv}\times\prod_{1\le i\le 2}\left(\tfrac{1-(uv)^{2n-4+2i}}{1-(uv)^i}\right)\times E(X^{[n]};u,v),$$

$$E(D_{12};u,v)=\left(\tfrac{1-(uv)^3}{1-uv}\right)^2\times\prod_{1\le i\le 3}\left(\tfrac{1-(uv)^{2n-6+2i}}{1-(uv)^i}\right)\times E(X^{[n]};u,v),$$

$$E(D_{23};u,v)=\tfrac{1-(uv)^{2n-3}}{1-uv}\cdot\tfrac{1-(uv)^2}{1-uv}\times\prod_{1\le i\le 2}\left(\tfrac{1-(uv)^{2n-4+2i}}{1-(uv)^i}\right)\times E(X^{[n]};u,v),$$

$$E(D_{13};u,v)=\tfrac{1-(uv)^3}{1-uv}\cdot\tfrac{1-(uv)^{2n-4}}{1-uv}\times\prod_{1\le i\le 2}\left(\tfrac{1-(uv)^{2n-4+2i}}{1-(uv)^i}\right)\times E(X^{[n]};u,v),$$

$$E(D^0_{123};u,v)=\tfrac{1-(uv)^2}{1-uv}\cdot\tfrac{1-(uv)^{2n-4}}{1-uv}\times\prod_{1\le i\le 2}\left(\tfrac{1-(uv)^{2n-4+2i}}{1-(uv)^i}\right)\times E(X^{[n]};u,v).$$

Proof. Perhaps the only part that requires proof is the equation for $E(D_3;u,v)$. From Proposition 3.2 (3), D_3 is a projective variety which is a $\mathbb{P}^{2n-4}$-bundle over a smooth projective variety, say Y, whose E-polynomial is

$$E(\mathbb{P}^2;u,v)\times E(\mathrm{Gr}^\omega(2,2n);u,v)\times E(X^{[n]};u,v).$$

By the Leray-Hirsch theorem ([24] p.182), we have

$$\begin{aligned}H^*(D_3;\mathbb{C}) &\cong H^*(Y;\mathbb{C})\otimes H^*(\mathbb{P}^{2n-4};\mathbb{C})\cong H^*(Y;\mathbb{C})\otimes\mathbb{C}[\lambda]/(\lambda^{2n-3})\\ &\cong H^*(Y;\mathbb{C})\oplus H^*(Y;\mathbb{C})\lambda\oplus\cdots\oplus H^*(Y;\mathbb{C})\lambda^{2n-4},\end{aligned}$$

where λ is a class of type $(1,1)$ which comes from the Kähler class. The above determines the Hodge structure of D_3 because the Hodge structure is compatible with the cup product. Therefore we deduce that

$$E(D_3;u,v)=\tfrac{1-(uv)^{2n-3}}{1-uv}\times E(Y;u,v).$$

□

For the E-polynomial of D^0_2 we have the following lemma whose proof is presented in section 5. Recall that

$$I_{2n-3}=\{((x_i),(y_j))\in\mathbb{P}^{2n-3}\times\mathbb{P}^{2n-3}\mid\sum_{i=0}^{2n-3}x_iy_i=0\}$$

and there is an action of $\mathbb{Z}_2$ which interchanges (x_i) and (y_j). Let $H^r(I_{2n-3})^+$ denote the $\mathbb{Z}_2$-invariant subspace of $H^r(I_{2n-3})$.

Lemma 3.4.

$$\begin{aligned}(3.2)\qquad E(D^0_2;z,z) &= P(I_{2n-3};z)\Big(\frac{P(X^{[n]};z)^2-P(X^{[n]};z^2)}{2}\Big)\\ &\quad +P^+(I_{2n-3};z)\big(P(X^{[n]};z^2)-P(X^{[n]};z)\big),\end{aligned}$$

where $P^+(I_{2n-3};z)=\sum_{r\geq 0}(-1)^r z^r \dim H^r(I_{2n-3})^+$. *Moreover*

$$E(D_2^0;z,z)=\frac{1-(z^2)^{2n-3}}{1-z^2}Q(z^2) \tag{3.3}$$

for some polynomial Q.

Proof of Proposition 1.3. Let us prove that (3.1) cannot be a polynomial. Let

$$S(z)=E_{st}(M_{2n};z,z)-E(M_{2n}^s;z,z).$$

It suffices to show that $S(z)$ is not a polynomial for all $n\geq 3$ because $E(M_{2n}^s;z,z)$ is a polynomial.

Note that given any $n\geq 3$, we can explicitly compute $E(X^{[n]};z,z)$ and $E(D_2^0;z,z)$ by (2.2) and Lemma 3.4. If $n=3$, direct calculation shows that $S(z)$ is as follows:

$$\begin{aligned} & S(z) \\ = \; & 1+46z^2+852z^4+12308z^6+111641z^8+886629z^{10}+4233151z^{12} \\ & +4990239z^{14}+4999261z^{16}+4230852z^{18}+884441z^{20}+113877z^{22} \\ & +12928z^{24}+3749z^{26}+3200z^{28}+2877z^{30}+299z^{32}+\cdots . \end{aligned}$$

It is easy to see from (3.1) and Corollary 3.3 that if $S(z)$ were a polynomial, it should be of degree ≤ 30. Since the series $S(z)$ has a nonzero coefficient of z^{32}, $S(z)$ cannot be a polynomial. So we assume from now on that $n\geq 4$.

Express the rational function $S(z)$ as

$$\frac{N(z)}{(1-(z^2)^{2n-3})(1-(z^2)^{4n-5})(1-(z^2)^{6n-6})}.$$

All we need to show is that the numerator $N(z)$ is not divisible by the denominator $(1-(z^2)^{2n-3})(1-(z^2)^{4n-5})(1-(z^2)^{6n-6})$.

As $E(X^{[n]};z,z)$ and $E(D_2^0;z,z)$ do not have nonzero terms of odd degree by (2.2) and Lemma 3.4, all the nonzero terms in $S(z)$ are of even degree by (3.1) and Corollary 3.3. Hence, we can write $S(z)=s(z^2)=s(t)$ for some rational function $s(t)$ in $t=z^2$. The numerator $N(z)=n(z^2)=n(t)$ is not divisible by $1-(z^2)^{2n-3}$ if and only if $n(t)$ is not divisible by $1-t^{2n-3}$. By direct computation using (3.1), Corollary 3.3 and Lemma 3.4, $n(t)$ modulo $1-t^{2n-3}$ is congruent to

$$\begin{aligned} &(1-t)^2(1-t^{4n-5})\times\left(\tfrac{1-t^3}{1-t}\right)^2\times\prod_{1\leq i\leq 3}\left(\tfrac{1-t^{2n-6+2i}}{1-t^i}\right)\times p(X^{[n]};t) \\ &-(1-t)^2(1-t^{4n-5})\times\tfrac{1-t^2}{1-t}\cdot\tfrac{1-t^{2n-4}}{1-t}\times\prod_{1\leq i\leq 2}\left(\tfrac{1-t^{2n-4+2i}}{1-t^i}\right)\times p(X^{[n]};t) \\ &-(1-t)^2(1-t^{6n-6})\times\tfrac{1-t^2}{1-t}\cdot\tfrac{1-t^{2n-4}}{1-t}\times\prod_{1\leq i\leq 2}\left(\tfrac{1-t^{2n-4+2i}}{1-t^i}\right)\times p(X^{[n]};t) \end{aligned} \tag{3.4}$$

$$+(1-t)^3 \times \tfrac{1-t^2}{1-t} \cdot \tfrac{1-t^{2n-4}}{1-t} \times \prod_{1\le i\le 2} \left(\tfrac{1-t^{2n-4+2i}}{1-t^i}\right) \times p(X^{[n]};t),$$

where $p(X^{[n]};t) = P(X^{[n]};z)$ with $t = z^2$. We write (3.4) as a product $\bar{s}(t) \cdot p(X^{[n]};t)$ for some polynomial $\bar{s}(t)$. For the proof of our claim for $n \geq 4$, it suffices to prove the following:

(1) if n is not divisible by 3, then $1-t$ is the GCD of $1-t^{2n-3}$ and $\bar{s}(t)$, and $\frac{1-t^{2n-3}}{1-t}$ does not divide $p(X^{[n]};t)$;

(2) if n is divisible by 3, then $1-t^3$ is the GCD of $1-t^{2n-3}$ and $\bar{s}(t)$, and $\frac{1-t^{2n-3}}{1-t^3}$ does not divide $p(X^{[n]};t)$.

For (1), suppose n is not divisible by 3. From (3.4), $\bar{s}(t)$ is divisible by $1-t$. We claim that $\bar{s}(t)$ is not divisible by any irreducible factor of $\frac{1-t^{2n-3}}{1-t}$, i.e., for any root α of $1-t^{2n-3}$ which is not 1, $\bar{s}(\alpha) \neq 0$. Using the relation $\alpha^{2n-3} = 1$, we compute directly that

$$\bar{s}(\alpha) = -\frac{\alpha(1-\alpha^{-1})(1-\alpha^3)^2}{1+\alpha}, \tag{3.5}$$

which is not 0 because 3 does not divide $2n-3$.

Next we check that $\frac{1-t^{2n-3}}{1-t}$ does not divide $p(X^{[n]};t)$. We put

$$p(X^{[n]};t) = \sum_{0\le i\le 2n} c_i t^i$$

and write $p(X^{[n]};t)$ as follows:

$$\begin{aligned} &\sum_{0\le i\le 2n} c_i t^i \\ (3.6) \quad = \;& (c_0 + c_{2n-3}) + (c_1 + c_{2n-2})t + (c_2 + c_{2n-1})t^2 + (c_3 + c_{2n})t^3 \\ &+ \sum_{4\le i\le 2n-4} c_i t^i + c_{2n-3}(t^{2n-3} - 1) + c_{2n-2}t(t^{2n-3} - 1) \\ &+ c_{2n-1}t^2(t^{2n-3} - 1) + c_{2n}t^3(t^{2n-3} - 1). \end{aligned}$$

Therefore, the divisibility of $p(X^{[n]};t)$ by $\frac{1-t^{2n-3}}{1-t}$ is that of $(c_0 + c_{2n-3}) + (c_1 + c_{2n-2})t + (c_2 + c_{2n-1})t^2 + (c_3 + c_{2n})t^3 + \sum_{4\le i\le 2n-4} c_i t^i$ by $\frac{1-t^{2n-3}}{1-t}$. Since $\frac{1-t^{2n-3}}{1-t} = \sum_{0\le i\le 2n-4} t^i$, the polynomial $(c_0 + c_{2n-3}) + (c_1 + c_{2n-2})t + (c_2 + c_{2n-1})t^2 + (c_3 + c_{2n})t^3 + \sum_{4\le i\le 2n-4} c_i t^i$ is divisible by $\frac{1-t^{2n-3}}{1-t}$ if and only if it is a scalar multiple of $\sum_{0\le i\le 2n-4} t^i$, i.e., $c_0 + c_{2n-3} = c_1 + c_{2n-2} = c_2 + c_{2n-1} = c_3 + c_{2n} = c_4 = \cdots = c_{2n-4}$ $(n \geq 4)$.

Table 1 is the list of c_i $(1 \leq i \leq 4)$ for $n \geq 3$, which comes from direct computation using the generating functions (2.2) for the Betti numbers of $X^{[n]}$.

TABLE 1. list of c_i

	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n\geq 8$
c_1	23	23	23	23	23	23
c_2	299	300	300	300	300	300
c_3	2554	2852	2875	2876	2876	2876
c_4	299	19298	22127	22426	22449	22450

By Table 1, we can check that this is impossible. Indeed, for $n \geq 6$, $c_0 = 1$, $c_1 = 23$, $c_2 = 300$ and $c_3 = 2876$, which implies $c_{2n-3} = 2876$, $c_{2n-2} = 300$, $c_{2n-1} = 23$ and $c_{2n-2} = 1$ by Poincaré duality. Thus $c_0 + c_{2n-3} = 2877$ while $c_1 + c_{2n-2} = 323$. For $4 \leq n \leq 5$, the proof is also direct computation using Table 1.

For (2), suppose 3 divides n and $n \neq 3$. Then from (3.5), $(1-t^3)$ divides $\bar{s}(t)$. More precisely, for a third root of unity α, $\bar{s}(\alpha) = 0$. On the other hand, if α is a root of $1-t^{2n-3}$ but not a third root of unity then we can observe that $\bar{s}(\alpha) \neq 0$ by (3.5). Therefore, since every root of $1-t^{2n-3}$ is a simple root, any irreducible factor of $\frac{1-t^{2n-3}}{1-t^3}$ does not divide $\bar{s}(t)$.

We next check that the polynomial $\frac{1-t^{2n-3}}{1-t^3}$ does not divide $p(X^{[n]};t)$. Write $p(X^{[n]};t) = \sum_{0\leq i\leq 2n} c_i t^i$ as follows:

$$\begin{aligned}
&\sum_{0\leq i\leq 2n} c_i t^i \\
(3.7) \quad = \; &(c_0 + c_{2n-3}) + (c_1 + c_{2n-2})t + (c_2 + c_{2n-1})t^2 + (c_3 + c_{2n})t^3 \\
&+ \sum_{4\leq i\leq 2n-6} c_i t^i - c_{2n-5}\Big(\sum_{i=0}^{\frac{2n-9}{3}} t^{3i+1}\Big) - c_{2n-4}\Big(\sum_{i=0}^{\frac{2n-9}{3}} t^{3i+2}\Big) \\
&+ c_{2n-5}t \cdot \tfrac{1-t^{2n-3}}{1-t^3} + c_{2n-4}t^2 \cdot \tfrac{1-t^{2n-3}}{1-t^3} + c_{2n-3}(t^{2n-3}-1) \\
&+ c_{2n-2}t(t^{2n-3}-1) + c_{2n-1}t^2(t^{2n-3}-1) + c_{2n}t^3(t^{2n-3}-1),
\end{aligned}$$

where the equality comes from

$$t^{2n-5} = -\sum_{i=0}^{\frac{2n-9}{3}} t^{3i+1} + t \cdot \tfrac{1-t^{2n-3}}{1-t^3}$$

and

$$t^{2n-4} = -\sum_{i=0}^{\frac{2n-9}{3}} t^{3i+2} + t^2 \cdot \tfrac{1-t^{2n-3}}{1-t^3}$$

since $\frac{1-t^{2n-3}}{1-t^3} = \sum_{i=0}^{\frac{2n-6}{3}} t^{3i}$. Therefore, $p(X^{[n]};t)$ modulo $\frac{1-t^{2n-3}}{1-t^3}$ is congruent to

$$\begin{aligned} R(t) &= (c_0+c_{2n-3})+(c_1+c_{2n-2})t+(c_2+c_{2n-1})t^2+(c_3+c_{2n})t^3 \\ &+ \sum_{4\le i\le 2n-6} c_i t^i - c_{2n-5}\Big(\sum_{i=0}^{\frac{2n-9}{3}} t^{3i+1}\Big) - c_{2n-4}\Big(\sum_{i=0}^{\frac{2n-9}{3}} t^{3i+2}\Big). \end{aligned}$$

Now $R(t)$ is divisible by $\frac{1-t^{2n-3}}{1-t^3} = \sum_{i=0}^{\frac{2n-6}{3}} t^{3i}$ if and only if $R(t)$ is a scalar multiple of $\sum_{i=0}^{\frac{2n-6}{3}} t^{3i}$ because $R(t)$ is of degree $\le 2n-6$. Thus the coefficient of $R(t)$ with respect to t^2 should be 0 i.e., $c_2+c_{2n-1}-c_{2n-4}=0$. However, $c_2+c_{2n-1}-c_{2n-4}=c_2+c_1-c_4$ is not zero by Table 1. This proves Proposition 1.3 for the case, where 3 divides n and $n\neq 3$. So the proof of Proposition 1.3 is completed for any $n\ge 3$. □

Remark 3.5. In case of smooth projective curves, we remark that the stringy E-function of the moduli space of rank 2 bundles is explicitly computed ([14] and [16]). We were not able to compute the stringy E-function of M_{2n} precisely, because we do not know how to compute the Hodge-Deligne polynomial $E(M_{2n}^s;u,v)$ of the locus M_{2n}^s of stable sheaves.

4. Analysis of Kirwan's desingularization

This section is devoted to the proof of Proposition 3.2. All can be extracted from [20] but we spell out the details for reader's convenience.

To begin with, note that for each $Z\in X^{[n]}$, the tangent space $T_{X^{[n]},Z}$ of the Hilbert scheme $X^{[n]}$ is canonically isomorphic to $\mathrm{Ext}^1(I_Z,I_Z)$, where I_Z is the ideal sheaf of the 0-dimensional closed subscheme Z. By the Yoneda pairing map and Serre duality, we have a skew-symmetric pairing $\omega : \mathrm{Ext}^1(I_Z,I_Z)\otimes \mathrm{Ext}^1(I_Z,I_Z)\to \mathrm{Ext}^2(I_Z,I_Z)\cong \mathbb{C}$, which gives us a symplectic form ω on the tangent bundle $T_{X^{[n]}}$ by [19] Theorem 0.1.

Note that the Killing form on $sl(2)$ gives an isomorphism $sl(2)^\vee \cong sl(2)$. Let $W=sl(2)^\vee\cong sl(2)\cong\mathbb{C}^3$. The adjoint action of $PGL(2)$ on W gives us an identification $SO(W)\cong PGL(2)$ ([20] §1.5). For a symplectic vector space (V,ω), let $\mathrm{Hom}^\omega(W,V)$ be the space of homomorphisms from W to V whose image is isotropic. Let $\mathrm{Hom}^\omega(W,T_{X^{[n]}})$ be the bundle over $X^{[n]}$ whose fiber over $Z\in X^{[n]}$ is $\mathrm{Hom}^\omega(W,T_{X^{[n]},Z})$. Clearly $\mathrm{Hom}^\omega(W,T_{X^{[n]}})$ is Zariski locally trivial over $X^{[n]}$. Let $\mathrm{Hom}^\omega_k(W,T_{X^{[n]}})$ be the subbundle of $\mathrm{Hom}^\omega(W,T_{X^{[n]}})$ of rank $\le k$ elements in $\mathrm{Hom}^\omega(W,T_{X^{[n]}})$. Also let $\mathrm{Gr}^\omega(3,T_{X^{[n]}})$ be the relative Grassmannian of isotropic 3-dimensional subspaces in $T_{X^{[n]}}$ and let $\mathcal{B}$ denote the tautological rank 3 bundle on $\mathrm{Gr}^\omega(3, T_{X^{[n]}})$. Obviously these bundles are all Zariski locally trivial as well.

Let $\mathbb{P}\mathrm{Hom}^\omega(W, T_{X^{[n]}})$ (resp. $\mathbb{P}\mathrm{Hom}^\omega_k(W, T_{X^{[n]}})$) be the projectivization of $\mathrm{Hom}^\omega(W, T_{X^{[n]}})$ (resp. $\mathrm{Hom}^\omega_k(W, T_{X^{[n]}})$). Likewise, let $\mathbb{P}\mathrm{Hom}(W, \mathcal{B})$ and $\mathbb{P}\mathrm{Hom}_k(W, \mathcal{B})$ denote the projectivizations of the bundles Hom $(W, \mathcal{B})$ and $\mathrm{Hom}_k(W, \mathcal{B})$. Note that there are obvious forgetful maps

$$f : \mathbb{P}\mathrm{Hom}(W, \mathcal{B}) \to \mathbb{P}\mathrm{Hom}^\omega(W, T_{X^{[n]}}) \text{ and}$$
$$f_k : \mathbb{P}\mathrm{Hom}_k(W, \mathcal{B}) \to \mathbb{P}\mathrm{Hom}^\omega_k(W, T_{X^{[n]}})$$

Since the pull-back of the defining ideal of $\mathbb{P}\mathrm{Hom}^\omega_1(W, T_{X^{[n]}})$ is the ideal of $\mathbb{P}\mathrm{Hom}_1(W, \mathcal{B})$ (both are actually given by the determinants of 2×2 minor matrices), f gives rise to a map between blow-ups

$$\overline{f} : Bl_{\mathbb{P}\mathrm{Hom}_1(W,\mathcal{B})}\mathbb{P}\mathrm{Hom}(W, \mathcal{B}) \to Bl_{\mathbb{P}\mathrm{Hom}^\omega_1(W,T_{X^{[n]}})}\mathbb{P}\mathrm{Hom}^\omega(W, T_{X^{[n]}}).$$

Let us denote $Bl_{\mathbb{P}\mathrm{Hom}_1(W,\mathcal{B})}\mathbb{P}\mathrm{Hom}(W, \mathcal{B})$ by $Bl^{\mathcal{B}}$ and $Bl_{\mathbb{P}\mathrm{Hom}^\omega_1(W,T_{X^{[n]}})}\mathbb{P}\mathrm{Hom}^\omega$ $(W, T_{X^{[n]}})$ by Bl^T. We denote the proper transform of $\mathbb{P}\mathrm{Hom}_2(W, \mathcal{B})$ in $Bl^{\mathcal{B}}$ by $Bl^{\mathcal{B}}_2$ and the proper transform of $\mathbb{P}\mathrm{Hom}^\omega_2(W, T_{X^{[n]}})$ by Bl^T_2. Since $Bl^{\mathcal{B}}_2$ is a Cartier divisor which is mapped onto Bl^T_2 and the pull-back of the defining ideal of Bl^T_2 is the ideal sheaf of $Bl^{\mathcal{B}}_2$, $\overline{f}$ lifts to

$$\hat{f} : Bl^{\mathcal{B}} \to Bl_{Bl^T_2} Bl^T. \tag{4.1}$$

By [20] §3.1 IV, $\hat{f}$ is an isomorphism on each fiber over $X^{[n]}$, so in particular $\hat{f}$ is bijective. Therefore, $\hat{f}$ is an isomorphism by Zariski's main theorem.

Note that $\mathbb{P}\mathrm{Hom}(W, \mathcal{B})/\!\!/SO(W)$ (resp. $\mathbb{P}\mathrm{Hom}_k(W, \mathcal{B})/\!\!/SO(W)$) is isomorphic to the space of conics $\mathbb{P}(S^2\mathcal{B})$ (resp. rank $\le k$ conics $\mathbb{P}(S^2_k\mathcal{B})$), where the $SO(W)$-quotient map is given by $[\alpha] \mapsto [\alpha \circ \alpha^t]$, where α^t denotes the transpose of $\alpha \in \mathrm{Hom}(W, \mathcal{B})$ ([20] §3.1). Let $\hat{\mathbb{P}}(S^2\mathcal{B}) = Bl_{\mathbb{P}(S^2_1\mathcal{B})}\mathbb{P}(S^2\mathcal{B})$ denote the blow-up along the locus of rank 1 conics. Then $Bl^{\mathcal{B}}/\!\!/SO(W)$ is canonically isomorphic to $\hat{\mathbb{P}}(S^2\mathcal{B})$ by [17] Lemma 3.11. Since $\mathcal{B}$ is Zariski locally trivial, so is $\hat{\mathbb{P}}(S^2\mathcal{B})$ over $\mathrm{Gr}^\omega(3, T_{X^{[n]}})$.

Now consider Simpson's construction of the moduli space M_{2n} ([20] §1.1). Let Q be the closure of the set of semistable points Q^{ss} in the Quot-scheme whose quotient by the natural $PGL(N)$ action is M_{2n} for some even integer N. Then Q^{ss} parameterizes semistable torsion-free sheaves F together with surjective homomorphisms $h : \mathcal{O}^{\oplus N} \to F(k)$ which induces an isomorphism $\mathbb{C}^N \cong H^0(F(k))$ and $H^1(F(k)) = 0$. Let Ω_Q denote the subset of Q^{ss} which parameterizes sheaves of the form $I_Z \oplus I_Z$ for some $Z \in X^{[n]}$. This is precisely the locus of closed orbits with maximal dimensional stabilizers, isomorphic to $PGL(2)$ and the quotient of Ω_Q by $PGL(N)$ is $X^{[n]}$.

We can give a more precise description of Ω_Q as follows. Let $\mathcal{L} \to X^{[n]} \times X$ be the universal rank 1 sheaf such that $\mathcal{L}|_{Z \times X}$ is isomorphic to the ideal sheaf I_Z. By [12] Theorem 10.2.1, the tangent bundle $T_{X^{[n]}}$ is in fact isomorphic to $\mathcal{E}xt^1_{X^{[n]}}(\mathcal{L},\mathcal{L})$. Let $p : X^{[n]} \times X \to X^{[n]}$ be the projection onto the first component. For $k \gg 0$, $p_*\mathcal{L}(k)$ is a vector bundle of rank $N/2$. Let

$$q : \mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k)) \to X^{[n]} \tag{4.2}$$

be the $PGL(N)$-bundle over $X^{[n]}$ whose fiber over Z is $\mathbb{P}\mathrm{Isom}(\mathbb{C}^N, H^0\ (I_Z(k) \oplus I_Z(k)))$. Note that the standard action of $GL(N)$ on $\mathbb{C}^N$ and the obvious action of $GL(2)$ on $p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k)$ induce a $PGL(N) \times PGL(2)$-action on $\mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k)) \to X^{[n]}$.

Lemma 4.1. (1) $\Omega_Q \cong \mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k))/\!\!/SO(W)$.

(2) *Via the above isomorphism, the normal cone of* Ω_Q *in* Q^{ss} *is*

$$q^*\mathrm{Hom}^\omega(W, T_{X^{[n]}})/\!\!/SO(W) \to \mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k))/\!\!/SO(W)$$

whose fiber over a point lying over $Z \in X^{[n]}$ *is* $\mathrm{Hom}^\omega(W, T_{X^{[n]},Z})$.

Proof. (1) Let $\hat{p} : \mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k)) \times X \to \mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k))$ be the obvious projection so that we have $q \circ \hat{p} = p \circ (q \times 1_X)$. Let H be the dual of the tautological line bundle over $\mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k))$. There is a canonical isomorphism $\mathcal{O}^{\oplus N} \cong q^*(p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k)) \otimes H$. This induces a surjective homomorphism

$$\begin{aligned} \mathcal{O}^{\oplus N} &\to \hat{p}^*q^*(p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k)) \otimes H \\ &= (q \times 1)^*(p^*p_*\mathcal{L}(k) \oplus p^*p_*\mathcal{L}(k)) \otimes H \\ &\to (q \times 1)^*(\mathcal{L}(k) \oplus \mathcal{L}(k)) \otimes H \end{aligned}$$

over $\mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k)) \times X$. By the universal property of the Quot-scheme, we get a morphism $\mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k)) \to Q^{ss}$ whose image is clearly contained in Ω_Q. This map is $PGL(2)$-invariant and hence we get a morphism

$$\phi_\Omega : \mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k))/\!\!/SO(W) \to \Omega_Q. \tag{4.3}$$

It is easy to check that ϕ_Ω is bijective. Since Ω_Q is smooth ([20] (1.5.1)), ϕ_Ω is an isomorphism by Zariski's main theorem.

(2) Let $\mathcal{O}^{\oplus N} \to \mathcal{E}(k)$ denote the universal quotient sheaf on $Q^{ss} \times X$ restricted to Ω_Q and let $\mathcal{F}$ be the kernel of the twisted homomorphism $\mathcal{O}^{\oplus N}(-k) \to \mathcal{E}$ so that we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}^{\oplus N}(-k) \to \mathcal{E} \to 0$$

over $\Omega_Q \times X$. The induced long exact sequence gives us

$$\begin{aligned} &\mathcal{H}om_{\Omega_Q}(\mathcal{O}^{\oplus N}(-k), \mathcal{E}) \to \mathcal{H}om_{\Omega_Q}(\mathcal{F}, \mathcal{E}) \\ &\quad \to \mathcal{E}xt^1_{\Omega_Q}(\mathcal{E}, \mathcal{E}) \to \mathcal{E}xt^1_{\Omega_Q}(\mathcal{O}^{\oplus N}(-k), \mathcal{E}). \end{aligned} \tag{4.4}$$

Let $\pi : \Omega_Q \times X \to \Omega_Q$ be the obvious projection. Note that $\mathcal{E}xt^1_{\Omega_Q}(\mathcal{O}^{\oplus N}(-k), \mathcal{E}) = R^1\pi_*(\mathcal{E}(k))^{\oplus N} = 0$ and that $\mathcal{H}om_{\Omega_Q}(\mathcal{O}^{\oplus N}(-k), \mathcal{E}) \cong \mathcal{H}om_{\Omega_Q}(\mathcal{O}^{\oplus N}, \mathcal{E}(k))$ is a vector bundle over Ω_Q whose fiber is $gl(N)$ because $\mathcal{O}_X^{\oplus N} \cong H^0(E(k))$ for any $[\mathcal{O}_X^{\oplus N} \to E(k)] \in Q^{ss}$. Let $T^*_{Q^{ss}}, T^*_{\Omega_Q}$ be cotangent sheaves over Q^{ss} and Ω_Q respectively. By a famous result of Grothendieck ([10] §5) we know

$$(T^*_{Q^{ss}}|_{\Omega_Q})^\vee \cong \mathcal{H}om_{\Omega_Q}(\mathcal{F}, \mathcal{E})$$

which contains the tangent bundle of Ω_Q as a subbundle. So the first homomorphism in (4.4) is the tangent map of the group action of $PGL(N)$[1] on Ω_Q and the second homomorphism is the Kodaira-Spencer map.

Via the isomorphism ϕ_Ω (4.3), we have a map

$$\begin{aligned}\delta: \quad & \mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k)) \\ & \to \mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}(k) \oplus p_*\mathcal{L}(k))/\!\!/SO(W) \cong \Omega_Q.\end{aligned}$$

From the proof of (1) above, the pull-back of $\mathcal{E}$ by $\delta \times 1$ is isomorphic to $(q\times 1)^*(\mathcal{L}(k)\oplus\mathcal{L}(k))\otimes H$ and thus the vector bundle $\delta^*\mathcal{E}xt^1_{\Omega_Q}(\mathcal{E},\mathcal{E})$ is isomorphic to

$$q^*\mathcal{E}xt^1_{X^{[n]}}(\mathcal{L},\mathcal{L}) \otimes gl(2) \cong q^*T_{X^{[n]}} \otimes gl(2).$$

The pull-back of the tangent sheaf of $X^{[n]}$ sits in $q^*T_{X^{[n]}} \otimes gl(2)$ as $q^*T_{X^{[n]}} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence the pull-back by δ of the normal bundle of Ω_Q (in the sense of [20] §1.3) is isomorphic to

$$q^*T_{X^{[n]}} \otimes sl(2) \cong q^*\mathrm{Hom}(W, T_{X^{[n]}}).$$

By [20] (1.5.10), the normal cone is fiberwisely the same as the Hessian cone. (See [20] §1.3 for more details on the Hessian cone.) Since the normal cone is contained in the Hessian cone, the normal cone is equal to the Hessian cone which is the inverse image of zero by the Yoneda square map $\Upsilon : \mathcal{E}xt^1_{\Omega_Q}(\mathcal{E},\mathcal{E}) \to \mathcal{E}xt^2_{\Omega_Q}(\mathcal{E},\mathcal{E})$. It is elementary to see that $\delta^*\Upsilon^{-1}(0)$ is precisely $q^*\mathrm{Hom}^\omega(W, T_{X^{[n]}})$. Since $SO(W)$ acts freely we obtain (2). See [20] (1.5.1) for a description of the normal cone at each point. □

Let Σ_Q denote the subset of Q^{ss} whose sheaves are of the form $I_Z \oplus I_W$ for some $Z, W \in X^{[n]}$. Then $\Sigma_Q - \Omega_Q$ is precisely the locus of points in Q^{ss} whose stabilizer is isomorphic to $\mathbb{C}^*$. Let $\pi_R : R \to Q^{ss}$ be the blow-up of Q^{ss} along Ω_Q and let Ω_R denote the exceptional divisor. By the above lemma, we have

$$\Omega_R \cong q^*\mathbb{P}\mathrm{Hom}^\omega(W, T_{X^{[n]}})/\!\!/SO(W). \tag{4.5}$$

The following lemma is an easy exercise.

Lemma 4.2. (1) *The locus of points in* $\mathbb{P}\mathrm{Hom}^\omega(W, T_{X^{[n]},Z})^{ss}$ *whose stabilizer is 1-dimensional by the action of* $SO(W)$ *is precisely* $\mathbb{P}\mathrm{Hom}^\omega_1(W, T_{X^{[n]},Z})^{ss}$.

(2) *The locus of nontrivial stabilizers is* $\mathbb{P}\mathrm{Hom}^\omega_2(W, T_{X^{[n]},Z})^{ss}$.

[1] In fact the term prior to the first term of (4.4) is $\mathcal{H}om_{\Omega_Q}(\mathcal{E},\mathcal{E})$ which contains $\mathcal{O}$ obviously and the quotient of $\mathcal{H}om_{\Omega_Q}(\mathcal{O}^{\oplus N}(-k), \mathcal{E})$ by $\mathcal{O}$ is a vector bundle whose fiber is the Lie algebra of $PGL(N)$.

Let

$$\Delta_R = q^*\mathbb{P}\mathrm{Hom}_2^\omega(W, T_{X^{[n]}})/\!\!/SO(W). \tag{4.6}$$

Let Σ_R be the proper transform of Σ_Q. Then Σ_R^{ss} is precisely the locus of points in R^{ss} with 1-dimensional stabilizers by [17]. Therefore we have the following from Lemma 4.2.

Corollary 4.3. $\Sigma_R^{ss} \cap \Omega_R = q^*\mathbb{P}\mathrm{Hom}_1^\omega(W, T_{X^{[n]}})^{ss}/\!\!/SO(W)$.

We have an explicit description of Σ_R^{ss} from [20] §1.7 III as follows. Let

$$\beta : \mathcal{X}^{[n]} \to X^{[n]} \times X^{[n]}$$

be the blow-up along the diagonal and let $\mathcal{X}_0^{[n]} = X^{[n]} \times X^{[n]} - \mathbf{\Delta}$, where $\mathbf{\Delta}$ is the diagonal. Let $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) be the pull-back to $\mathcal{X}^{[n]} \times X$ of the universal sheaf $\mathcal{L} \to X^{[n]} \times X$ by $p_{13} \circ (\beta \times 1)$ (resp. $p_{23} \circ (\beta \times 1)$), where p_{ij} is the projection onto the first (resp. second) and third components. Let $p : \mathcal{X}^{[n]} \times X \to \mathcal{X}^{[n]}$ be the projection onto the first component. Then for $k \gg 0$, $p_*\mathcal{L}_1(k) \oplus p_*\mathcal{L}_2(k)$ is a vector bundle of rank N. Let

$$q : \mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}_1(k) \oplus p_*\mathcal{L}_2(k)) \to \mathcal{X}^{[n]}$$

be the $PGL(N)$-bundle. There is an action of $O(2)$ on $\mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}_1\ (k) \oplus p_*\mathcal{L}_2(k))$. We quote [20] (1.7.10) and (1.7.1).

Lemma 4.4. (1) $\Sigma_R^{ss} \cong \mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}_1(k) \oplus p_*\mathcal{L}_2(k))/\!\!/O(2)$.

(2) *The normal cone of* Σ_R^{ss} *in* R^{ss} *is a locally trivial bundle over* Σ_R^{ss} *with fiber the cone over a smooth quadric in* $\mathbb{P}^{4n-5}$.

In fact we can give a more explicit description of the normal cone when restricted to $\Sigma_R^0 := \Sigma_R^{ss} - \Omega_R$. Similarly as in the proof of Lemma 4.1, the normal vector bundle to Σ_R^0 is isomorphic to the vector bundle (of rank $4n-4$)

$$q^*[\mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_1, \mathcal{L}_2) \oplus \mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_2, \mathcal{L}_1)]/\!\!/O(2) \tag{4.7}$$

over $\mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}_1(k) \oplus p_*\mathcal{L}_2(k))/\!\!/O(2)$, where $O(2)$ acts as follows: if we realize $O(2)$ as the subgroup of $PGL(2)$ generated by

$$SO(2) = \{\theta_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\}/\{\pm Id\}, \qquad \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

θ_α multiplies α (resp. α^{-1}) to $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) and τ interchanges $\mathcal{L}_1$ and $\mathcal{L}_2$ by the induced action on $\mathcal{X}^{[n]}$ of interchanging the first and second factors of $X^{[n]} \times X^{[n]}$. The normal cone is the inverse image $q^*\Upsilon^{-1}(0)$ of zero in terms of the Yoneda pairing

$$\Upsilon : \mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_1, \mathcal{L}_2) \oplus \mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_2, \mathcal{L}_1) \to \mathcal{E}xt^2_{\mathcal{X}_0^{[n]}}(\mathcal{L}_1, \mathcal{L}_1). \tag{4.8}$$

Let $\pi_S : S \to R^{ss}$ denote the blow-up of R^{ss} along Σ_R^{ss} and let Σ_S be the exceptional divisor of π_S while Ω_S (resp. Δ_S) denotes the proper transform of

Ω_R (resp. Δ_R). By (4.8), we have

$$\Sigma_S|_{\pi_S^{-1}(\Sigma_R^0)} \cong q^*\mathbb{P}\Upsilon^{-1}(0)/\!\!/O(2)$$
$$\subset q^*\mathbb{P}[\mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_1,\mathcal{L}_2)\oplus \mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_2,\mathcal{L}_1)]/\!\!/O(2). \tag{4.9}$$

By [20] (1.8.10), $S^s = S^{ss}$ and S^s is smooth. The quotient $S/\!\!/ PGL(N)$ has only $\mathbb{Z}_2$-quotient singularities along $\Delta_S/\!\!/PGL(N)$. Let $\pi_T : T \to S^s$ be the blow-up of S^s along Δ_S^s. Then $T/\!\!/PGL(N)$ is nonsingular and this is Kirwan's desingularization $\rho : \widehat{M}_{2n} \to M_{2n}$.

Let Ω_T and Σ_T denote the proper transforms of Ω_S and Σ_S respectively. Let Δ_T be the exceptional divisor of π_T. Their quotients $\Omega_T/\!\!/PGL(N)$, $\Sigma_T/\!\!/PGL(N)$ and $\Delta_T/\!\!/PGL(N)$ are denoted by $D_1 = \hat{\Omega}$, $D_2 = \hat{\Sigma}$ and $D_3 = \hat{\Delta}$ respectively.

With this preparation, we now embark on the proof of Proposition 3.2.

Proof of (1). This is just [20] (3.0.1). More precisely, by (4.5) and Corollary 4.3, Ω_S is the blow-up of

$$q^*\mathbb{P}\mathrm{Hom}^\omega(W, T_{X^{[n]}})/\!\!/SO(W) \text{ along } q^*\mathbb{P}\mathrm{Hom}_1^\omega(W, T_{X^{[n]}})/\!\!/SO(W).$$

By (4.6), Ω_T is the blow-up of Ω_S along the proper transform of

$$q^*\mathbb{P}\mathrm{Hom}_2^\omega(W, T_{X^{[n]}})/\!\!/SO(W)$$

and $D_1 = \hat{\Omega}$ is the quotient of Ω_T by the action of $PGL(N)$. Since the action of $PGL(N)$ commutes with the action of $SO(W)$, D_1 is in fact the quotient by $SO(W) \times PGL(N)$ of the variety obtained from $q^*\mathbb{P}\mathrm{Hom}^\omega(W, T_{X^{[n]}})$ by two blow-ups. So D_1 is also the consequence of taking the quotient by $PGL(N)$ first and then the quotient by $SO(W)$ second. Since q in (4.2) is a principal $PGL(N)$ bundle, the result of the first quotient is just $Bl_{Bl_2^T}Bl^T$ in (4.1) which is isomorphic to $Bl^{\mathcal{B}}$. If we take further the quotient by $SO(W)$, then as discussed above the result is $D_1 = \hat{\mathbb{P}}(S^2\mathcal{B})$. □

Proof of (2). We use Lemma 4.4, (4.7), and (4.9). Note that Σ_R^0 does not intersect with Ω_R and Δ_R. Hence D_2^0 is the quotient of $q^*\mathbb{P}\Upsilon^{-1}(0)/\!\!/O(2)$ which is a subset of $q^*\mathbb{P}[\mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_1,\mathcal{L}_2)\oplus\mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_2,\mathcal{L}_1)]/\!\!/O(2)$, by the action of $PGL(N)$. The above are bundles over the restriction of

$$\mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}_1(k)\oplus p_*\mathcal{L}_2(k))/\!\!/O(2)$$

to the complement $\mathcal{X}_0^{[n]}$ of the diagonal $\mathbf{\Delta}$ in $X^{[n]}\times X^{[n]}$. As in the proof of (1), observe that D_2^0 is in fact the quotient of $q^*\mathbb{P}\Upsilon^{-1}(0)$ by the action of $PGL(N)\times O(2)$ since the actions commute. So we can first take the quotient by the action of $PGL(N)$, then by the action of $SO(2)$, and finally by the action of $\mathbb{Z}_2 = O(2)/SO(2)$. Since $\mathbb{P}\mathrm{Isom}(\mathbb{C}^N, p_*\mathcal{L}_1(k)\oplus p_*\mathcal{L}_2(k))$ is a principal $PGL(N)$-bundle, the quotient by $PGL(N)$ gives us

$$\mathbb{P}\Upsilon^{-1}(0)\subset \mathbb{P}[\mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_1,\mathcal{L}_2)\oplus\mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_2,\mathcal{L}_1)]$$

over $\mathcal{X}_0^{[n]}$. The algebraic vector bundles $\mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_1,\mathcal{L}_2)$ and $\mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_2,\mathcal{L}_1)$ are certainly Zariski locally trivial and in fact these bundles are dual to each other by the Yoneda pairing Υ which is non-degenerate (possibly after tensoring with a line bundle). In particular, $\Upsilon^{-1}(0)$ is Zariski locally trivial.

Next we take the quotient by the action of $SO(2)\cong\mathbb{C}^*$. This action is trivial on the base $\mathcal{X}_0^{[n]}$ and $SO(2)$ acts on the fibers. Hence $\mathbb{P}\Upsilon^{-1}(0)/SO(2)$ is a Zariski locally trivial subbundle of

$$\begin{aligned}&\mathbb{P}[\mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_1,\mathcal{L}_2)\oplus\mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_2,\mathcal{L}_1)]/\!/\mathbb{C}^*\\ &\cong\mathbb{P}\mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_1,\mathcal{L}_2)\times_{\mathcal{X}_0^{[n]}}\mathbb{P}\mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_2,\mathcal{L}_1)\end{aligned}$$

over $\mathcal{X}_0^{[n]}$ given by the incidence relations in terms of the identification

$$\mathbb{P}\mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_1,\mathcal{L}_2)\cong\mathbb{P}\mathcal{E}xt^1_{\mathcal{X}_0^{[n]}}(\mathcal{L}_2,\mathcal{L}_1)^\vee.$$

Finally, D_2^0 is the $\mathbb{Z}_2$-quotient of $\mathbb{P}\Upsilon^{-1}(0)/SO(2)$. □

Proof of (3). By [20] (1.7.10), the intersection of Σ_R^{ss} and Ω_R is smooth. By Corollary 4.3, Δ_S is the blow-up of $q^*\mathbb{P}\mathrm{Hom}_2^\omega(W,T_{X^{[n]}})/\!/\ SO(W)$ along $q^*\mathbb{P}$ $\mathrm{Hom}_1^\omega(W,T_{X^{[n]}})/\!/SO(W)$. Hence $\Delta_S/\!/PGL(N)$ is the quotient of

$$Bl_{q^*\mathbb{P}\mathrm{Hom}_1^\omega(W,T_{X^{[n]}})}q^*\mathbb{P}\mathrm{Hom}_2^\omega(W,T_{X^{[n]}})$$

by the action of $SO(W)\times PGL(N)$. By taking the quotient by the action of $PGL(N)$ we get

$$Bl_{\mathbb{P}\mathrm{Hom}_1^\omega(W,T_{X^{[n]}})}\mathbb{P}\mathrm{Hom}_2^\omega(W,T_{X^{[n]}})$$

since q is a principal $PGL(N)$-bundle. Next we take the quotient by the action of $SO(W)$. Let $\mathrm{Gr}^\omega(2,T_{X^{[n]}})$ be the relative Grassmannian of isotropic 2-dimensional subspaces in $T_{X^{[n]}}$ and let $\mathcal{A}$ be the tautological rank 2 bundle on $\mathrm{Gr}^\omega(2,T_{X^{[n]}})$. We claim

$$Bl_{\mathbb{P}\mathrm{Hom}_1^\omega(W,T_{X^{[n]}})}\mathbb{P}\mathrm{Hom}_2^\omega(W,T_{X^{[n]}})/\!/SO(W)\simeq\mathbb{P}(S^2\mathcal{A}) \tag{4.10}$$

which is a $\mathbb{P}^2$-bundle over a $\mathrm{Gr}^\omega(2,2n)$-bundle over $X^{[n]}$. It is obvious that the bundles are Zariski locally trivial.

There are forgetful maps

$$\begin{aligned}f&:\mathbb{P}\mathrm{Hom}(W,\mathcal{A})\to\mathbb{P}\mathrm{Hom}_2^\omega(W,T_{X^{[n]}})\\ f_1&:\mathbb{P}\mathrm{Hom}_1(W,\mathcal{A})\to\mathbb{P}\mathrm{Hom}_1^\omega(W,T_{X^{[n]}}),\end{aligned}$$

where the subscript 1 denotes the locus of rank ≤ 1 homomorphisms. Because the ideal of $\mathbb{P}\mathrm{Hom}_1^\omega(W,T_{[n]})$ pulls back to the ideal of $\mathbb{P}\mathrm{Hom}_1(W,\mathcal{A})$, f lifts to

$$\hat{f}:Bl_{\mathbb{P}\mathrm{Hom}_1(W,\mathcal{A})}\mathbb{P}\mathrm{Hom}(W,\mathcal{A})\to Bl_{\mathbb{P}\mathrm{Hom}_1^\omega(W,T_{X^{[n]}})}\mathbb{P}\mathrm{Hom}_2^\omega(W,T_{X^{[n]}}).$$

This map is bijective ([20] (3.5.1)) and hence $\hat{f}$ is an isomorphism by Zariski's main theorem because the varieties are smooth. Now observe that the quotient $\mathbb{P}\mathrm{Hom}(W,\mathcal{A})/\!/SO(W)$ is $\mathbb{P}(S^2\mathcal{A})$, where the quotient map is given by $[\alpha]\mapsto$

$[\alpha\circ\alpha^t]$. Hence $\Delta_S /\!\!/ PGL(N)$ is the blow-up of $\mathbb{P}\mathrm{Hom}(W,\mathcal{A})/\!\!/SO(W) \cong \mathbb{P}(S^2\mathcal{A})$ along the locus of rank 1 quadratic forms $\mathbb{P}(S^2_1\mathcal{A})$ ([17] Lemma 3.11) which is a Cartier divisor. So we proved that

$$\Delta_S /\!\!/ PGL(N) \cong \mathbb{P}(S^2\mathcal{A}).$$

Finally $S/\!\!/PGL(N)$ is singular only along $\Delta_S/\!\!/PGL(N)$ and the singularities are $\mathbb{C}^{2n-3}/\{\pm 1\}$ by Luna's slice theorem [20] (1.2.1). Since D_3 is the exceptional divisor of the blow-up of $S/\!\!/PGL(N)$ along $\Delta_S/\!\!/ PGL(N)$, we conclude that D_3 is a $\mathbb{P}^{2n-4}$-bundle over $\mathbb{P}(S^2\mathcal{A})$. □

Proof of (4). By Corollary 4.3, $\Sigma^s_S \cap \Omega_S$ is the exceptional divisor of the blow-up $Bl_{q^*\mathbb{P}\mathrm{Hom}^\omega_1(W,T_{X^{[n]}})} q^*\mathbb{P}\mathrm{Hom}^\omega(W,T_{X^{[n]}})/\!\!/SO(W)$ and $\Sigma^s_T \cap \Omega_T$ is now the blow-up of the exceptional divisor along the proper transform of $q^*\mathbb{P}\mathrm{Hom}^\omega_2(W,T_{X^{[n]}})$ $/\!\!/SO(W)$. Using the isomorphism (4.1), this is the exceptional divisor of

$$q^* Bl_{\mathbb{P}(S^2_1\mathcal{B})}\mathbb{P}(S^2\mathcal{B}) \to q^*\mathbb{P}(S^2\mathcal{B})$$

over $\mathrm{Gr}^\omega(3,T_{X^{[n]}})$. Since q is a principal $PGL(N)$-bundle, $D_1\cap D_2 = \Sigma^s_T \cap \Omega_T/\!\!/PGL(N)$ is the exceptional divisor of the blow-up $Bl_{\mathbb{P}(S^2_1\mathcal{B})}\ \mathbb{P}(S^2\mathcal{B})$. Because the exceptional divisor is a Zariski locally trivial $\mathbb{P}^2$-bundle over $\mathbb{P}(S^2_1\mathcal{B})$ and $\mathbb{P}(S^2_1\mathcal{B})$ itself is a Zariski locally trivial $\mathbb{P}^2$-bundle over $\mathrm{Gr}^\omega(3,T_{X^{[n]}})$, we proved (4). □

Proof of (5). From the above proof of (3) it follows immediately that $\Sigma^s_S \cap \Delta_S/\!\!/PGL(N)$ is $\mathbb{P}(S^2_1\mathcal{A})$ and $D_2\cap D_3$ is a $\mathbb{P}^{2n-4}$ bundle over $\mathbb{P}(S^2_1\mathcal{A})$ which is Zariski locally trivial. □

Proof of (6). As in the above proof of (4), we start with (4.6) and use the isomorphism (4.1) to see that $D_1\cap D_3$ is the proper transform of $\mathbb{P}(S^2_2\mathcal{B})$ in the blow-up $Bl_{\mathbb{P}(S^2_1\mathcal{B})}\mathbb{P}(S^2\mathcal{B})$. This is a Zariski locally trivial $\mathbb{P}^2$-bundle over a Zariski locally trivially $\mathbb{P}^2$-bundle over $\mathrm{Gr}^\omega(3,T_{X^{[n]}})$. □

Proof of (7). This follows immediately from the proof of (4) and (6). □

From the above descriptions, it is clear that D_i $(i = 1,2,3)$ are normal crossing smooth divisors.

5. Hodge-Deligne polynomial of D_2^0

In this section we prove Lemma 3.4. Recall

$$I_{2n-3} = \{((x_i),(y_j)) \in \mathbb{P}^{2n-3}\times\mathbb{P}^{2n-3} \mid \sum_{i=0}^{2n-3} x_i y_i = 0\}.$$

It is elementary ([8] p.606) to see that

$$\begin{aligned} &H^*(I_{2n-3};\mathbb{Q})\\ \cong\ &\mathbb{Q}[a,b]/\langle a^{2n-2}, b^{2n-2}, a^{2n-3}+a^{2n-4}b+a^{2n-5}b^2+\cdots+b^{2n-3}\rangle, \end{aligned}$$

where a (resp. b) is the pull-back of the first Chern class of the tautological line bundle of the first (resp. second) $\mathbb{P}^{2n-3}$. The $\mathbb{Z}_2$-action interchanges a and b and the invariant subspace of $H^*(I_{2n-3};\mathbb{Q})$ is generated by classes of the form $a^ib^j + a^jb^i$. As a vector space $H^*(I_{2n-3};\mathbb{Q})$ is

$$\mathbb{Q}\text{-span}\{a^ib^j \mid 0 \le i \le 2n-3, 0 \le j \le 2n-4\} \tag{5.1}$$

while the invariant subspace is

$$\mathbb{Q}\text{-span}\{a^ib^j + a^jb^i \mid 0 \le i \le j \le 2n-4\}.$$

The index set $\{(i,j) \mid 0 \le i \le j \le 2n-4\}$ is mapped to its complement in $\{(i,j) \mid 0 \le i \le 2n-3, 0 \le j \le 2n-4\}$ by the map $(i,j) \mapsto (j+1,i)$. This immediately implies that

$$P(I_{2n-3};z) = (1+z^2)P^+(I_{2n-3};z) \tag{5.2}$$

By (5.1) or the observation that I_{2n-3} is the Zariski locally trivial $\mathbb{P}^{2n-4}$-bundle over $\mathbb{P}^{2n-3}$, we have

$$P(I_{2n-3};z) = \frac{1-(z^2)^{2n-2}}{1-z^2} \cdot \frac{1-(z^2)^{2n-3}}{1-z^2}. \tag{5.3}$$

Because $1+z^2$ divides $\frac{1-(z^2)^{2n-2}}{1-z^2}$, $\frac{1-(z^2)^{2n-3}}{1-z^2}$ also divides $P^+(I_{2n-3};z)$. Therefore, (3.3) is a direct consequence of (3.2) since $P(X^{[n]};z)$ has no odd degree terms by (2.2).

Now let us prove (3.2). Let

$$\psi : \widetilde{D}_2^0 := \mathbb{P}\Upsilon^{-1}(0)/SO(2) \to \mathfrak{X}_0^{[n]} = X^{[n]} \times X^{[n]} - \Delta$$

be the Zariski locally trivial I_{2n-3}-bundle in the proof of Proposition 3.2 (2) in §4. Recall that $D_2^0 = \widetilde{D}_2^0/\mathbb{Z}_2$. We have seen in the proof of Proposition 3.2 (2) in §4 that there is a $\mathbb{Z}_2$-equivariant embedding

$$\imath : \widetilde{D}_2^0 \hookrightarrow \mathbb{P}\mathcal{E}xt^1_{\mathfrak{X}_0^{[n]}}(\mathcal{L}_1,\mathcal{L}_2) \times_{\mathfrak{X}_0^{[n]}} \mathbb{P}\mathcal{E}xt^1_{\mathfrak{X}_0^{[n]}}(\mathcal{L}_2,\mathcal{L}_1),$$

where the $\mathbb{Z}_2$-action interchanges $\mathcal{L}_1$ and $\mathcal{L}_2$.

Let λ (resp. η) be the pull-back to $\widetilde{D}_2^0$ of the first Chern class of the tautological line bundle over $\mathbb{P}\mathcal{E}xt^1_{\mathfrak{X}_0^{[n]}}(\mathcal{L}_1,\mathcal{L}_2)$ (resp. $\mathbb{P}\mathcal{E}xt^1_{\mathfrak{X}_0^{[n]}}(\mathcal{L}_2,\mathcal{L}_1)$). By definition, λ and η restrict to a and b respectively. The $\mathbb{Z}_2$-action interchanges λ and η. By the Leray-Hirsch theorem[2] we have an isomorphism

$$H_c^*(\widetilde{D}_2^0) \;\cong\; H_c^*(\mathfrak{X}_0^{[n]}) \otimes H^*(I_{2n-3}). \tag{5.4}$$

As the pull-back and the cup product preserve mixed Hodge structure, (5.4) determines the mixed Hodge structure of $H_c^*(\widetilde{D}_2^0)$. The $\mathbb{Z}_2$-invariant part is

$$H_c^*(\widetilde{D}_2^0)^+ \cong \left(H_c^*(\mathfrak{X}_0^{[n]})^+ \otimes H^*(I_{2n-3})^+\right) \oplus \left(H_c^*(\mathfrak{X}_0^{[n]})^- \otimes H^*(I_{2n-3})^-\right), \tag{5.5}$$

[2]The Leray-Hirsch theorem in [24] p.182 is stated for ordinary cohomology but the statement holds also for compact support cohomology. See the proof in [24] p.195

where the superscript $\pm$ denotes the ± 1-eigenspace of the $\mathbb{Z}_2$-action. Because $H_c^*(D_2^0) \cong H_c^*(\widetilde{D}_2^0/\mathbb{Z}_2) \cong H_c^*(\widetilde{D}_2^0)^+$ ([9] Theorem 5.3.1 and Proposition 5.2.3), $E(D_2^0; u, v)$ is equal to

$$\begin{aligned}(5.6)\qquad & E^+(\widetilde{D}_2^0; u, v) \\ = \ & E^+(\mathfrak{X}_0^{[n]}; u, v)E^+(I_{2n-3}; u, v) + E^-(\mathfrak{X}_0^{[n]}; u, v)E^-(I_{2n-3}; u, v),\end{aligned}$$

where $E^\pm(Y; u, v) = \sum_{p,q}\sum_{k\geq 0}(-1)^k h^{p,q}(H_c^k(Y)^\pm)u^p v^q$.

It is easy to see

$$P^+(X^{[n]} \times X^{[n]}; z) = \frac{P(X^{[n]}; z)^2 + P(X^{[n]}; z^2)}{2},$$

$$P^-(X^{[n]} \times X^{[n]}; z) = \frac{P(X^{[n]}; z)^2 - P(X^{[n]}; z^2)}{2}.$$

(Macdonald's formula). Since $X^{[n]} \times X^{[n]}$ is smooth projective, we have

$$E^+(X^{[n]} \times X^{[n]}; z, z) = \frac{P(X^{[n]}; z)^2 + P(X^{[n]}; z^2)}{2}$$

$$E^-(X^{[n]} \times X^{[n]}; z, z) = \frac{P(X^{[n]}; z)^2 - P(X^{[n]}; z^2)}{2}.$$

Now as $\mathfrak{X}_0^{[n]} = X^{[n]} \times X^{[n]} - \boldsymbol{\Delta}$ and $\boldsymbol{\Delta} \cong X^{[n]}$ is $\mathbb{Z}_2$-invariant, by the additive property of the E-polynomial we have

$$\begin{aligned}E^+(\mathfrak{X}_0^{[n]}; z, z) &= E^+(X^{[n]} \times X^{[n]}; z, z) - E(X^{[n]}; z, z) \\ &= \frac{P(X^{[n]}; z)^2 + P(X^{[n]}; z^2)}{2} - P(X^{[n]}; z),\end{aligned}$$

$$\begin{aligned}E^-(\mathfrak{X}_0^{[n]}; z, z) &= E^-(X^{[n]} \times X^{[n]}; z, z) \\ &= \frac{P(X^{[n]}; z)^2 - P(X^{[n]}; z^2)}{2}.\end{aligned}$$

The equation (3.2) is an immediate consequence of the above equations and (5.6).

References

[1] V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, Integrable systems and algebraic geometry (Kobe/Kyoto,1997) (1998), 1–32.

[2] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), no. 4, 755–782.

[3] V. Danilov and G. Khovanskiĭ, *Newton polyhedra and an algorithm for computing Hodge-Deligne numbers*, Math. USSR-Izv. **29** (1987), no. 2, 279–298.

[4] J. Denef and F. Loeser, *Germs of arcs on singular varieties and motivic integration*, Invent. Math. **135** (1999), no. 1, 201–232.

[5] D. Eisenbud, *Commutative algebra*, With a view toward algebraic geometry, Graduate Texts in Mathematics 150, Springer-Verlag, 1995.

[6] D. Gieseker, *On the moduli of vector bundles on an algebraic surface*, Ann. of Math.(2) **106** (1977), no. 1, 45–60.

[7] L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. **286** (1990), no. 1-3, 193–207.

[8] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience Publication, John Wiley & Sons, 1978.

[9] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J.(2) **9** (1957), 119–221.

[10] ———, *Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert*, Seminaire Bourbaki **6** Exp. No. 221 (1995), 249–276.

[11] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52. Sprin-ger-Verlag, 1977.

[12] D. Huybrechts and M. Lehn, *The Geometry of moduli spaces of sheaves*, A Publication of the Max-Planck-Institut für Mathematik, Bonn, 1997.

[13] D. Kaledin and M. Lehn, *Local structure of hyperkähler singularities in O'Grady's examples*, math.AG/0405575.

[14] Y.-H. Kiem, *The stringy E-function of the moduli space of rank 2 bundles over a Riemann surface of genus 3*, Trans. Amer. Math. Soc. **355** (2003), no. 5, 1843–1856.

[15] ———, *On the existence of a symplectic desingularization of some moduli spaces of sheaves on a K3 surface*, Compos. Math. **141** (2005), no. 4, 902–906.

[16] Y.-H. Kiem and J. Li, *Desingularizations of the moduli space of rank 2 bundles over a curve*, Math. Ann. **330** (2004), no. 3, 491–518.

[17] F. Kirwan, *Partial desingularisations of quotients of nonsingular varieties and their Betti numbers*, Ann. of Math.(2) **122** (1985), no. 1, 41–85.

[18] I. G. Macdonald, *The Poincaré polynomial of a symmetric product.* Proc. Cambridge Philos. Soc. **58** (1962), 563–568.

[19] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. Math. **77** (1984), no. 1, 101–116.

[20] K. G. O'Grady, *Desingularized moduli spaces of sheaves on a K3. I*, math. AG/9708009.

[21] ———, *Desingularized moduli spaces of sheaves on a K3. II*, math.AG/9805099.

[22] ———, *Desingularized moduli spaces of sheaves on a K3*, J. Reine Angew. Math. **512** (1999), 49–117.

[23] C. Vafa and E. Witten, *A strong coupling test of S-duality.* Nuclear Phys. B **431** (1994), no. 1–2, 3–77.

[24] C. Voisin, *Hodge theory and complex algebraic geometry. I*, Cambridge studies in Advanced Mathematics **76**, Cambridge University Press, 2002.

[25] ———, *Hodge theory and complex algebraic geometry. II*, Cambridge studies in Advanced Mathematics **77**, Cambridge University Press, 2002.

[26] K. Yoshioka, *Twisted stability and Fourier-Mukai transform. I*, Compositio Math. **138** (2003), no. 3, 261–288.

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