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ON A CLASS OF OPERATORS RELATED TO PARANORMAL OPERATORS

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Abstract. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-paranormal* if

$$|||T|^p U|T|^p x||\ ||x|| \geq |||T|^p x||^2$$

for all $x \in \mathcal{H}$ and $p > 0$, where $T = U|T|$ is the polar decomposition of T. It is easy that every 1-paranormal operator is paranormal, and every p-paranormal operator is paranormal for $0 < p < 1$. In this note, we discuss some properties for p-paranormal operators.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert and $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$. Every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T|$ is the square root of T^*T. If U is determined uniquely by the kernel condition $N(U) = N(|T|)$, then this decomposition is called the *polar decomposition*, which is one of the most important results in operator theory ([10], [11], [12] and [13]). In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $N(U) = N(|T|)$.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is *positive*, $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$. Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators ([12], [14] and [15]). In particular, the Putnam inequality [14]

$$||T^*T - TT^*|| \leq \frac{1}{\pi} m_2(\sigma(T))$$

is fundamental for hyponormal operators T, where m_2 is the planer Lebesque measure and $\sigma(T)$ is the spectrum of T. Xia [19], Cho and Itoh [5] extended the Putnam inequality to p-hyponormal, i.e.,

$$(T^*T)^p \geq (TT^*)^p$$

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for $0 < p \leq \frac{1}{2}$:

$$||(T^*T)^p - (TT^*)^p|| \leq \frac{p}{\pi} \iint_{\sigma(T)} r^{2p-1} dr d\theta.$$

Recently, Tanahashi [17] introduced the log-hyponormality for operators. An invertible operator T is log-hyponormal if $\log T^*T \geq \log TT^*$. It is quite meaningful because every log-hyponormal operator T satisfies the following Putnam inequality [18]

$$|| \log T^*T - \log TT^*|| \leq \frac{1}{\pi} \iint_{\sigma(T)} r^{-1} dr d\theta.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *paranormal* if it satisfies the following norm inequality

$$||T^2x|| \, ||x|| \geq ||Tx||^2$$

for all $x \in \mathcal{H}$. Ando [4] proved that every log-hyponormal operators is paranormal. It was originally introduced as an intermediate class between hyponormal operators and normaloid. Furuta, Ito and Yamazaki [20] introduced new families of classes of operators, whose background is recent development of operator inequalities. As a matter of fact, these are defined by operator inequalities and norm inequalities, and named classes $A(k)$ and absolute k-paranormal operators. Among others, the class $A(1)$ is given by an operator inequality

$$|T^2| \geq |T|^2.$$

It occupies a desirable location between log-hyponormal and paranormal [20].

On the other hand, Fujii-Izumino-Nakamoto [7] introduced the p-paranormality for operators. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-paranormal* if

$$|||T|^pU|T|^px|| \, ||x|| \geq |||T|^px||^2$$

for all $x \in \mathcal{H}$ and $p > 0$, where U is the partial isometry appeared in the polar decomposition $T = U|T|$ of T. And they proved that every p-paranormal operator is paranormal for $0 < p < 1$. It is easy that every 1-paranormal operator is paranormal. In addition, the p-paranormality is based on the fact that $T = U|T|$ is p-hyponormal if and only if $S = U|T|^p$ is hyponormal [6]. Actually, $T = U|T|$ is p-paranormal if and only if $S = U|T|^p$ is paranormal. From this fact, a p-hyponormal operator is a p-paranormal operator for $p > 0$. Recently, Fujii-Jung-Nakamoto and authors [8] introduced a new class $A(p,p)$ of operators: For $p > 0$, an operator T belongs to $A(p,p)$ if it satisfies an operator inequality

$$(|T^*|^p|T|^{2p}|T^*|^p)^{\frac{1}{2}} \geq |T^*|^{2p}.$$

More generally, we can define the classes $A(p,q)$ for $p, q > 0$ by an operator inequality

$$(|T^*|^q|T|^{2p}|T^*|^q)^{\frac{q}{p+q}} \geq |T^*|^{2q}.$$

Note that $A(k,1)$ is the classes $A(k)$ due to Furuta-Ito-Yamazaki. Namely the family $\{A(p,q) : p, q > 0\}$ is a generalization of $\{A(k) : k > 0\}$ exactly. We

[8] discussed inclusion relations between $A(p,q)$ and p-paranormal operators. And we proved that every p-paranormality has monotone increasing property on $p > 0$ and every p-paranormal operator is normaloid.

In this paper, we will obtain a characterization of p-paranormal operators using the polar decomposition $T = U|T|$ of T. i.e., $T = U|T|$ is p-paranormal for $p > 0$ if and only if

$$|T|^p U^* |T|^{2p} U |T|^p + 2\lambda |T|^{2p} + \lambda^2 \geq 0$$

for all real λ. Using this characterization, we will also obtain some properties for p-paranormal operators. Finally, we give some examples about p-paranormality for $p > 0$.

2. On p-paranormal operators

First, we need the followings in this section.

Theorem A. [10] *Let $T_1 = U_1 P_1$ and $T_2 = U_2 P_2$ be the polar decompositions of T_1 and T_2, respectively. Then the following are equivalent:*

(1) *T_1 doubly commutes with T_2.*

(2) *U_1^*, U_1 and P_1 commutes with U_2^*, U_2 and P_2.*

(3) $[P_1, P_2] = 0, [U_1, P_2] = 0, [P_1, U_2] = 0, [U_1, U_2] = 0$ *and* $[U_1^*, U_2] = 0$.

Theorem B. [10] *Let $T_1 = U_1 P_1$ and $T_2 = U_2 P_2$ be the polar decompositions of T_1 and T_2, respectively. If T_1 doubly commutes with T_2, then $T_1 T_2 = U_1 U_2 P_1 P_2$ is also the polar decomposition of $T_1 T_2$, that is, $U_1 U_2$ is partial isometry with $N(U_1 U_2) = N(P_1 P_2)$ and $P_1 P_2 = |T_1 T_2|$.*

The following lemma is an important characterization for p-para-normal operators.

Lemma 2.1. *Let an operator $T \in \mathcal{L}(\mathcal{H})$ have the polar decomposition $T = U|T|$. Then T is p-paranormal for $p > 0$ if and only if*

$$|T|^p U^* |T|^{2p} U |T|^p + 2\lambda |T|^{2p} + \lambda^2 \geq 0$$

for all real λ.

Proof. Suppose that

$$|T|^p U^* |T|^{2p} U |T|^p + 2\lambda |T|^{2p} + \lambda^2 \geq 0$$

for all real λ. This inequality is equivalent to

$$|||T|^p U |T|^p x||^2 + 2\lambda |||T|^p x||^2 + \lambda^2 ||x||^2 \geq 0$$

for all real λ and $x \in \mathcal{H}$. This is equivalent to

$$|||T|^p x||^4 \leq ||x||^2 |||T|^p U |T|^p x||^2$$

$x \in \mathcal{H}$, and hence T is p-paranormal. □

Shan and Sheth [16] proved that the inverse of an invertible paranormal operator is also paranormal. We have a generalization for p-paranormal operators. It is easy that if $T = U|T|$ is invertible, then U is unitary and $T^{-1} = U^*|T^*|^{-1}$ is the polar decomposition.

Theorem 2.2. *Let $T = U|T|$ be invertible p-paranormal for $p > 0$. Then T^{-1} is also p-paranormal.*

Proof. Suppose that $T = U|T|$ is invertible p-paranormal. Then

$$U|T|^{-r} = |T^*|^{-r}U \text{ and } |T^*|^{-r} = U|T|^{-r}U^*$$

for all $r > 0$. Since T is p-paranormal, we have

$$\begin{aligned} &I + 2\lambda|T^{-1}|^{2p} + \lambda^2|T^{-1}|^pU|T^{-1}|^{2p}U^*|T^{-1}|^p \\ &= I + 2\lambda U|T|^{-2p}U^* + \lambda^2 U|T|^{-p}U|T|^{-2p}U^*|T|^{-p}U^* \\ &= U|T|^{-p}U|T|^{-p}(|T|^pU^*|T|^{2p}U|T|^p + 2\lambda|T|^{2p} + \lambda^2)|T|^{-p}U^*|T|^{-p}U^* \end{aligned}$$

is positive for all real λ. By Lemma 2.1, T^{-1} is p-paranormal. □

Recall that an operator A is unitarily equivalent to B if there exists an unitary X such that $XA = BX$.

Theorem 2.3. *An operator unitarily equivalent to p-paranormal is p-paranormal for $p > 0$.*

Proof. Let $T_1 = U|T_1|$ be p-paranormal, W be unitary and let $T_2 = W^*T_1W$. Note that

$$|T_2|^r = W^*|T_1|^rW$$

for every $r > 0$. By Theorem A and Theorem B, we have $T_2 = W^*U|T_1|W = W^*UWW^*|T_1|W$ and $N(W^*UW) = N(W^*|T_1|W)$ so that

$$T_2 = (W^*UW)(W^*|T_1|W)$$

is the polar decomposition of T_2. Thus we have

$$\begin{aligned} &|T_2|^p(W^*UW)^*|T_2|^{2p}(W^*WS)|T_2|^p + 2\lambda|T_2|^{2p} + \lambda^2 \\ &= W^*(|T_1|^pW^*|T_1|^{2p}W|T_1|^p + 2\lambda|T_1|^{2p} + \lambda^2)W \end{aligned}$$

is positive for all real λ. So, T_2 is p-paranormal. □

Ando [4] proved that if a paranormal operator T double commutes with a hyponormal operator S, then TS is paranormal. We prove the following:

Theorem 2.4. *If a p-paranormal operator T double commutes with a p-hyponormal operator S, then TS is p-paranormal for $p > 0$.*

Proof. Let $T = U|T|$ and $S = W|S|$ be the polar decompositions of T and S, respectively. Then $T_p = U|T|^p$ is paranormal and $S_p = W|S|^p$ is hyponormal. By Theorem A, T_p doubly commutes with S_p. By [5, Theorem 4], $T_pS_p = (TS)_p$ is paranormal. Thus TS is p-paranormal. □

Ando [4] proved that if T and T^* are paranormal with $N(T) = N(T^*)$, then T is normal. Let $T = U|T|$ be the polar decomposition of T and let $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. An operator T is said to be *w-hyponormal* if

$$|\tilde{T}| \leq |T| \leq |\tilde{T}^*|.$$

The classes of w-hyponormal operators contains both p-hyponormal and log-hyponormal operators [2]. Aluthge and Wang [3] proved that if T and T^* are w-hyponormal with $N(T) = N(T^*)$, then T is normal. The following theorems are generalizations of Ando's theorems [4].

Theorem 2.5. *If both T and T^* are p-paranormal with $N(T) = N(T^*)$, then T is normal.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Then $T_p = U|T|^p$ and $T_p^* = U^*|T^*|^p$ are paranormal with $N(T_p) = N(T_p^*)$. By [4, Theorem 5], T_p is normal, i.e., T is normal. □

It is known that if T is paranormal, $T \otimes I$ is also paranormal. However the tensor product of two doubly commuting paranormal operators is not necessarily paranormal [4].

Theorem 2.6. *If T is p-paranormal, $T \otimes I$ is also p-paranormal for $p > 0$.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Then

$$T \otimes I = (U \otimes I)(|T| \otimes I)$$

is the polar decomposition of $T \otimes I$. Since

$$\begin{aligned} &|T \otimes I|^p(U \otimes I)^*|T \otimes I|^{2p}(U \otimes I)|T \otimes I|^p + 2\lambda|T \otimes I|^{2p} + \lambda^2 \\ &= (|T|^pU^*|T|^{2p}U|T|^p + 2\lambda|T|^{2p} + \lambda^2) \otimes I \end{aligned}$$

is positive for all real λ, we have $T \otimes I$ is p-paranormal. □

3. Examples

In this section, we will give some examples which are related to p-paranormality. Now, we have next results:

Proposition 3.1. *Let $\mathcal{H} = \oplus_{-\infty}^{\infty}\mathbf{C}^2$ with $x = (x_n)_{n=-\infty}^{\infty}$ and $\sum ||x_n||^2 < \infty$. For a given two positive operators A, B on $\mathbf{C}^2$, define the operator $T =: T_{A,B}$ on $\mathcal{H}$ as follows:*

$$T = \begin{pmatrix} \ddots & & & & & & \\ & \ddots & 0 & 0 & 0 & & \\ & & A & 0 & 0 & & \\ & & 0 & A & \boxed{0} & & \\ & & 0 & 0 & B & 0 & \\ & & & & 0 & B & 0 \\ & & & & & \ddots & \ddots \end{pmatrix},$$

where $\boxed{\cdot}$ *is shows the place of the* (0,0) *matrix element. The following statements are hold:*

(1) T *is p-paranormal if and only if*

$$A^pB^{2p}A^p + 2\lambda A^{2p} + \lambda^2 \geq 0 \tag{3.1}$$

for all real λ.

(2) *If* AB *is positive, then* T^2 *is p-paranormal if and only if*

$$(AB)^pB^{4p}(AB)^p + 2\lambda(AB)^{2p} + \lambda^2 \geq 0 \tag{3.2}$$

for all real λ.

Proof. (1) By a simple calculation, we have

$$|T| = \begin{pmatrix} \ddots & & & & & & \\ & \ddots & A & 0 & 0 & & \\ & & 0 & A & 0 & & \\ & & 0 & 0 & \boxed{B} & & \\ & & 0 & 0 & 0 & B & \\ & & & & 0 & 0 & B \\ & & & & & & \ddots & \ddots \end{pmatrix}$$

so that

$$T = \begin{pmatrix} \ddots & & & & & & \\ & \ddots & 0 & 0 & 0 & & \\ & & I & 0 & 0 & & \\ & & 0 & I & \boxed{0} & & \\ & & 0 & 0 & I & 0 & \\ & & & & 0 & I & 0 \\ & & & & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \ddots & & & & & & \\ & \ddots & A & 0 & 0 & & \\ & & 0 & A & 0 & & \\ & & 0 & 0 & \boxed{B} & & \\ & & 0 & 0 & 0 & B & \\ & & & & 0 & 0 & B \\ & & & & & & \ddots & \ddots \end{pmatrix}$$

is the polar decomposition of T. According to Lemma 2.1, T is p-paranormal if and only if

$$A^pB^{2p}A^p + 2\lambda A^{2p} + \lambda^2 \geq 0$$

by comparing the matrix elements of

$$|T|^pU^*|T|^{2p}U|T|^p + 2\lambda|T|^{2p} + \lambda^2 \geq 0.$$

(2) If AB is positive, we have

$$T^2 = \begin{pmatrix} \ddots & & & & & \\ & \ddots & 0 & 0 & 0 & \\ & & 0 & 0 & 0 & \\ & & I & 0 & \boxed{0} & \\ & & 0 & I & 0 & 0 \\ & & & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \ddots & & & & & \\ & \ddots & A^2 & 0 & 0 & \\ & & 0 & AB & 0 & \\ & & 0 & 0 & \boxed{B^2} & \\ & & 0 & 0 & 0 & B^2 \\ & & & & & \ddots & \ddots \end{pmatrix}$$

is the polar decomposition of T^2. Similarly to (1), T^2 is p-paranormal if and only if

$$(AB)^p B^{4p} (AB)^p + 2\lambda (AB)^{2p} + \lambda^2 \geq 0$$

for all real λ. □

Fujii-Jung-Nakamoto and authors [8] proved that every p-paranormal operator is a q-paranormal operator for $0 < p < q$. The following example shows that the converse relation is not true.

Example 3.2. Let

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

for $\alpha, \beta \geq 0$ and let

$$f(p) = (\frac{9^p + 1}{2})^{\frac{1}{2p}}.$$

Then

$$B^{2p} = \frac{1}{2} \begin{pmatrix} 9^p + 1 & 9^p - 1 \\ 9^p - 1 & 9^p + 1 \end{pmatrix}$$

and $f(p)$ is strictly increasing for $p > 0$. By (3.1), $T =: T_{A,B}$ is p-paranormal if and only if

$$\begin{aligned} 0 &\leq A^p B^{2p} A^p + 2\lambda A^{2p} + \lambda^2 \\ &= \begin{pmatrix} \frac{1}{2}\alpha^{2p}(9^p + 1) + 2\lambda\alpha^{2p} + \lambda^2 & \frac{1}{2}\alpha^p(9^p - 1)\beta^p \\ \frac{1}{2}\alpha^p(9^p - 1)\beta^p & \frac{1}{2}\beta^{2p}(9^p + 1) + 2\lambda\beta^{2p} + \lambda^2 \end{pmatrix} \\ &=: X_{\alpha,\beta,p} \end{aligned} \tag{3.3}$$

for all real λ. This is equivalent to

$$\frac{1}{2}\alpha^{2p}(9^p + 1) + 2\lambda\alpha^{2p} + \lambda^2 \geq 0 \text{ for all real } \lambda, \tag{3.4}$$

$$\frac{1}{2}\beta^{2p}(9^p + 1) + 2\lambda\beta^{2p} + \lambda^2 \geq 0 \text{ for all real } \lambda \tag{3.5}$$

and

$$\det(X_{\alpha,\beta,p}) \geq 0. \tag{3.6}$$

Thus (3.4), (3.5) and (3.6) are equivalent to

$$\alpha \le f(p) \tag{3.7}$$

and

$$\beta \le f(p), \tag{3.8}$$

respectively. Finally, we can take $0 < p_1 < p_2$ and $0 < \alpha < f(p_1) < \beta < f(p_2)$. Then T is p_2-paranormal, but T is not p_1-paranormal.

Furuta [9] proved that if T is paranormal, then T^n is also paranormal for all $n \in \mathbf{N}$. But the p-paranormality of T does not imply the p-paranormality of T^n except $p = 1$.

Example 3.3. Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Since $f(\frac{1}{2}) = 2$, by Example 3.2, $T =: T_{A,B}$ is $\frac{1}{2}$-paranormal. Now,

$$\begin{aligned}(AB)^{\frac{1}{2}}B^{4\frac{1}{2}}(AB)^{\frac{1}{2}} + 2\lambda(AB)^{2\frac{1}{2}} + \lambda^2 &= 2B^3 + 4\lambda B + \lambda^2 \\ &= \begin{pmatrix} 28+8\lambda+\lambda^2 & 26+4\lambda \\ 26+4\lambda & 28+8\lambda+\lambda^2 \end{pmatrix}\end{aligned}$$

is not positive for $\lambda = -2$. So, T is $\frac{1}{2}$-paranormal, but T^2 is not $\frac{1}{2}$-paranormal.

Remark 3.4. Let T be an unilateral weighted shift with weight sequence $\{\alpha_k\}_{k=0}^{\infty}$ and let $n \in \mathbf{N}$. Let $\alpha_k = |\alpha_k|e^{i\theta_k}$ for all $k = 0, 1, 2, \ldots$. Then the polar decomposition $T^n = U|T^n|$ of T is as follows: U is the unilateral weighted shift of multiplicity n with weight $\{e^{i(\theta_k+\cdots+\theta_{k+n-1})}\}_{k=0}^{\infty}$ and $|T^n| = \mathrm{Diag}\{|\alpha_k \cdots \alpha_{k+n-1}|\}_{k=0}^{\infty}$. By Lemma 2.1, T^n is p-paranormal for $n \in \mathbf{N}$ if and only if

$$\begin{aligned}&|\alpha_k\alpha_{k+1}\cdots\alpha_{k+n-1}|^{2p}|\alpha_{k+1}\alpha_{k+2}\cdots\alpha_{k+n}|^{2p} \\ &+ 2\lambda|\alpha_k\alpha_{k+1}\cdots\alpha_{k+n-1}|^{2p} + \lambda^2 \ge 0\end{aligned}$$

for all real λ and $k = 0, 1, 2, \ldots$. Thus which is equivalent to

$$|\alpha_k\alpha_{k+1}\cdots\alpha_{k+n-1}|^{4p} - |\alpha_k\alpha_{k+1}\cdots\alpha_{k+n-1}|^{2p}|\alpha_{k+1}\alpha_{k+2}\cdots\alpha_{k+n}|^{2p} \le 0$$

for all $k = 0, 1, 2, \ldots$ and hence the sequence $\{|\alpha_k\alpha_{k+1}\cdots\alpha_{k+n-1}|\}_{k=0}^{\infty}$ is increasing. Thus, if T is an unilateral weighted shift p-paranormal then T^n is also p-paranormal for all $n \in \mathbf{N}$.

By Theorem 2.6, we proved that if T is p-paranormal, the tensor product $T \otimes I$ is p-paranormal. However the tensor product of two doubly commuting p-paranormal operators is not necessarily p-paranormal for $p > 0$.

Example 3.5. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{\frac{1}{2p}}, \ B = (A^{-\frac{1}{2}} \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} A^{-\frac{1}{2}})^{\frac{1}{2p}}.$$

Then

$$A^pB^{2p}A^p + 2\lambda A^{2p} + \lambda^2 = \begin{pmatrix} (1+\lambda)^2 & 2(1+\lambda) \\ 2(1+\lambda) & (2+\lambda)^2+4 \end{pmatrix}$$

is positive for all real λ and hence T is p-paranormal. And

$$\begin{aligned} &(T\otimes T)^{*2}(T\otimes T)^2 - 2(T\otimes T)^*(T\otimes T) + (-1)\otimes(-1) \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} - 2\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + 1\otimes 1 \\ &= \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 5 & 2 & 12 \\ 0 & 2 & 5 & 12 \\ 2 & 12 & 12 & 57 \end{pmatrix} \end{aligned}$$

is not positive, hence $T\otimes T$ is not p-paranormal.

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