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ON A q-ANALOGUE OF THE p-ADIC GENERALIZED TWISTED L-FUNCTIONS AND p-ADIC q-INTEGRALS

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ABSTRACT. The purpose of this paper is to define generalized twisted q-Bernoulli numbers by using p-adic q-integrals. Furthermore, we construct a q-analogue of the p-adic generalized twisted L-functions which interpolate generalized twisted q-Bernoulli numbers. This is the generalization of Kim's h-extension of p-adic $q-L$-function which was constructed in [5] and is a partial answer for the open question which was remained in [3].

1. Introduction

Let us denote $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ sets of positive integer, integer, rational, real and complex numbers respectively. Let p be prime and $x \in \mathbb{Q}$. Then $x = p^{v(x)}\frac{m}{n}$, where m, n, $v = v(x) \in \mathbb{Z}$, m and n are not divisible by p. Let $|x|_p = p^{-v(x)}$ and $|0|_p = 0$. Then $|x|_p$ is valuation on $\mathbb{Q}$ satisfying

$$|x+y|_p \leq \max\{|x|_p, |y|_p\}.$$

Completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is denoted by $\mathbb{Q}_p$ and called the field of p-adic rational numbers. $\mathbb{C}_p$ is the completion of algebraic closure of $\mathbb{Q}_p$ and $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$ is called the ring of p-adic rational integers (see [1, 2, 10, 12, 16]).

Let l be a fixed integer and let p be a fixed prime number. We set

$$X = \varprojlim_N (\mathbb{Z}/lp^N\mathbb{Z}),$$

$$X^* = \bigcup_{\substack{0<a<lp \\ (a,p)=1}} (a + lp\mathbb{Z}_p),$$

$$a + lp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{lp^N}\},$$

where $N \in \mathbb{N}$ and $a \in \mathbb{Z}$ lies in $0 \leq a < lp^N$, cf. [3, 7, 8, 9].

When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$,

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one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for each $x \in X$. We use the notation as $[x] = [x;q] = \frac{1-q^x}{1-q}$ for each $x \in X$. Hence $\lim_{q \to 1}[x] = x$, cf. [4, 16, 18, 19, 20]. For any positive integer N, we set

$$\mu_q(a + lp^N \mathbb{Z}_p) = \frac{q^a}{[lp^N]}, \quad [5, 6, 7, 8, 9, 10, 11, 12, 13, 14],$$

and this can be extended to a distribution on X. This distribution yields an integral for each nonnegative integer n (see [7]) :

$$\int_{\mathbb{Z}_p} [x]^n \, d\mu_q(x) = \int_X [x]^n \, d\mu_q(x) = \beta_n(q),$$

where $\beta_n(q)$ are the n-th Carlitz's q-Bernoulli number, cf. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

In the paper [17], Koblitz constructed p-adic $q-L-$function which interpolates Carlitz's q-Bernoulli number at non-positive integers and suggested two questions. One of these two questions was solved by Kim (see [7]). In fact, Kim constructed p-adic q-integral and proved that Carlitz's q-Bernoulli number can be represented as an p-adic q-integral by the q-analogue of the ordinary p-adic invariant measure. And also Kim is constructed a h-extension of p-adic $q-L$-function which interpolates the h-extension of q-Bernoulli numbers at non-positive integers (see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]). In [5, 12, 13], Kim constructed p-adic $q-L$-functions and he studied their properties. In [5], Kim introduced h-extension of p-adic $q-L$-functions and investigated many interesting physical meaning. Also, in [15, 16], Koblitz defined p-adic twisted L-functions, and he constructed p-adic measures and integrations. And also Koblitz [3] constructed a q-analogue of the twisted Dirichlet's L-function which interpolated the twisted Carlitz's q-Bernoulli numbers, and they remained an open question in [3] as follows:

Find q-analogue of the p-adic twisted L-function which interpolates q-Bernoulli numbers $\beta^{(h)}_{m,w,\chi}(q)$, by means of a method provided by Kim, cf. [5].

In this paper, we will apply the concept of "twisted" to p-adic generalized $q-L-$functions and generalized q-Bernoulli numbers to be a part of answer for the question which was remained by Kim et al. in [3] by means of the same method provided by Kim in [5: p.98]. In section 2, we construct generalized twisted q-Bernoulli polynomials by using p-adic q-integrals by the same method of Kim, cf. [3, 5, 12, 13, 14, 20, 21, 22, 23, 24, 25, 26]. We prove a formula between generalized twisted q-Bernoulli polynomials which is regarded as a generalization of Witt's formula for Carlitz's q-Bernoulli polynomials in [5, Eq (5.9)], [13] and [7, Theorem 2]. This means that the q-analogue of generalized twisted q-Bernoulli numbers occur in the coefficients of some stirling type series. We also give construction of the distribution of the p-adic generalized twisted q-Bernoulli distribution. In section 3, we define the p-adic generalized twisted L-function and construct a q-analogue of the p-adic generalized twisted L-function which interpolate generalized twisted q-Bernoulli numbers on X. This result is related as a generalization of a q-analogue of the p-adic L-function which interpolate Carlitz's q-bernoulli numbers in [5, 11, 12, 13], of p-adic generalized L-function which interpolates the h-extension of q-Bernoulli numbers at non-positive integers in [5, 6, 7].

2. Generalized twisted q-Bernoulli polynomials

In this section, we give generalized twisted q-Bernoulli polynomials by using p-adic q-integrals on X. Let $UD(X)$ be the set of uniformly differentiable functions on X. For any $f \in UD(X)$, T. Kim defined a q-analogue of an integral with respect to an p-adic invariant measure in [5] which is called p-adic q-integral. The p-adic q-integral was defined as follows:

$$\begin{aligned} I_q(f) &= \int_X f(x)\, d\mu_q(x) \\ &= \lim_{N\to\infty} \frac{1}{[lp^N]} \sum_{0\le x<lp^N} f(x)q^x, \end{aligned} \tag{1}$$

cf. [4, 5, 6, 7, 8]. Note that

$$\begin{aligned} I_1(f) &= \lim_{q\to 1} I_q(f) = \int_X f(x)\, d\mu_1(x) \\ &= \lim_{N\to\infty} \frac{1}{lp^N} \sum_{0\le x<lp^N} f(x), \end{aligned} \tag{2}$$

and that

$$I_1(f_1) = I_1(f) + f'(x), \tag{3}$$

where $f_1(x) = f(x+1)$.

Let $T_p = \cup_{n\ge1} C_{p^n} = \lim_{n\to\infty} \mathbb{Z}/p^n\mathbb{Z}$, where $C_{p^n} = \{\xi \in X \mid \xi^{p^n} = 1\}$ is the cyclic group of order p^n, see [9]. For $\xi \in T_p$, we denote by $\phi_\xi : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \mapsto \xi^x$. If we take $f(x) = \phi_\xi(x)e\xi^{tx}$, then we have that

$$\int_X e^{tx}\phi_\xi(x)d\mu_1(x) = \frac{t}{we^t - 1}, \tag{4}$$

cf. [5, 8]. It is obvious from (3) that

$$\int_X e^{tx}\chi(x)\phi_\xi(x)d\mu_1(x) = \frac{\sum_{a=1}^{l} \chi(a)\phi_\xi(a)e^{at}}{\xi^l e^{lt} - 1}. \tag{5}$$

Now we define the analogue of Bernoulli numbers as follows:

$$e^{xt}\frac{t}{\xi e^t - 1} = \sum_{n=0}^{\infty} B_{n,\xi}(x)\frac{t^n}{n!} \tag{6}$$

$$\frac{\sum_{a=1}^{l}\chi(a)\phi_\xi(a)e^{at}}{\xi^l e^{lt}-1}=\sum_{n=0}^{\infty}B_{n,\xi,\chi}\frac{t^n}{n!},$$

cf. [5, 8]. By (4), (5) and (6), it is not difficult to see that

$$\int_X x^n\phi_\xi(x)\,d\mu_1(x)=B_{n,\xi} \tag{7}$$

and

$$\int_X \chi(x)x^n\phi_\xi(x)\,d\mu_1(x)=B_{n,\xi,\chi}. \tag{8}$$

From (7) and (8) we consider twisted q-Bernoulli numbers by using p-adic q-integral on $\mathbb{Z}_p$. For $\xi\in T_p$ and $h\in\mathbb{Z}$, we define twisted q-Bernoulli polynomials as

$$\beta_{m,\xi}^{(h)}(x,q)=\int_{\mathbb{Z}_p} q^{(h-1)y}\xi^y[x+y]^m d\mu_q(y). \tag{9}$$

Observe that

$$\lim_{q\to 1}\beta_{m,\xi}^{(h)}(x,q)=B_{m,\xi}(x).$$

When $x=0$, we write $\beta_{m,\xi}^{(h)}(0,q)=\beta_{m,\xi}^{(h)}(q)$, which are called twisted q-Bernoulli numbers. It follows from (9) that

$$\beta_{m,\xi}^{(h)}(x,q)=\frac{1}{(1-q)^{m-1}}\sum_{k=0}^{m}\binom{m}{k}q^{xk}(-1)^k\frac{k+h}{1-q^{h+k}\xi}. \tag{10}$$

The Eq.(10) is equivalent to

$$\beta_{m,\xi}^{(h)}(q)=-m\sum_{n=0}^{\infty}[n]^{m-1}q^{hn}\xi^n-(q-1)(m+h)\sum_{n=0}^{\infty}[n]^m q^{hn}\xi^n. \tag{11}$$

From (9), we obtain the below distribution relation for the twisted q-Bernoulli polynomials as follows. In fact, the proof of Lemma 1 is similar to the proof of Lemma 2 with $\chi=1$.

Lemma 1. *For $n\geq 1$, we have*

$$\beta_{n,\xi}^{(h)}(x,q)=d^{n-1}\sum_{a=0}^{d-1}\xi^a q^{ha}\beta_{n,\xi}^{(h)}(\frac{a+x}{d},q^d).$$

For $\xi\in T_p$ and $h\in X$, we define generalized twisted q-Bernoulli polynomials as

$$\beta_{n,\xi,\chi}^{(h)}(x,q)=\int_X \chi(y)q^{(h-1)y}\xi^y[x+y]^n d\mu_q(y). \tag{12}$$

Observe that when $\chi=1$,

$$\beta_{n,\xi,1}^{(h)}(x,q)=\int_X q^{(h-1)y}\xi^y[x+y]^n d\mu_q(y)=\beta_{n,\xi}^{(h)}(x,q) \tag{13}$$

and

$$\lim_{q\to 1}\beta^{(h)}_{n,\xi,\chi}(x,q)=\int_X \chi(y)\xi^y[x+y]^n d\mu_1(y)=B^{(h)}_{n,\xi,\chi}(x), \tag{14}$$

where $\beta^{(h)}_{n,\xi}(x,q)$ is a twisted q-Bernoulli polynomial and $B^{(h)}_{n,\xi,\chi}(x)$ is a generalized Bernoulli polynomial.

Lemma 2. *For any $n\geq 1$, we have*

$$\beta^{(h)}_{n,\xi,\chi}(x,q)=[l]^{n-1}\sum_{a=0}^{l-1}\chi(a)\xi^a q^{ha}\beta^{(h)}_{n,\xi^l,\chi^l}(\frac{a+x}{l},q^l). \tag{15}$$

Proof. For each $n\in\mathbb{N}$, we have

$$\begin{aligned}
\beta^{(h)}_{n,\xi,\chi}(x,q)&=\int_X \chi(y)q^{(h-1)y}\xi^y[x+y]^n d\mu_q(y)\\
&=\lim_{N\to\infty}\sum_{x_1=0}^{lp^N-1}\chi(x_1)\xi^{x_1}[x+x_1]^n\mu_q(x_1+lp^N\mathbb{Z}_p)\\
&=\lim_{N\to\infty}\frac{1}{[lp^N]}\sum_{x_1=0}^{lp^N-1}\chi(x_1)\xi^{x_1}[x+x_1]^n q^{x_1}\\
&=[l]^{n-1}\sum_{a=0}^{l-1}\chi(a)q^{ha}\xi^a\\
&\quad\times\lim_{N\to\infty}\frac{1}{[p^N:q^l]}\sum_{m=0}^{p^N-1}(q^l)^{(h-1)m}(\xi^l)^m[\frac{x+a}{l}+m:q^l]^n(q^l)^m\\
&=[l]^{n-1}\sum_{a=0}^{l-1}\chi(a)\xi^a q^{ha}\beta^{(h)}_{n,\xi^l,\chi^l}(\frac{a+x}{l},q^l).
\end{aligned}$$

□

We note that when $x=0$, we have the distribution relation for the generalized twisted q-Bernoulli numbers as follows: for $n\geq 1$,

$$\beta^{(h)}_{n,\xi,\chi}(q)=\beta^{(h)}_{n,\xi,\chi}(0,q)=[l]^{n-1}\sum_{a=0}^{l-1}\chi(a)\xi^a q^{ha}\beta^{(h)}_{n,\xi^l}(\frac{a}{l},q^l) \tag{16}$$

and that when $x=0$ and $q=1$, we have the distribution relation for the generalized twisted Bernoulli numbers as follows: for $n\geq 1$,

$$\beta^{(h)}_{n,\xi,\chi}=\beta^{(h)}_{n,\xi,\chi}(1)=l^{n-1}\sum_{a=0}^{l-1}\chi(a)\xi^a\beta^{(h)}_{n,\xi^l}(\frac{a}{l}) \tag{17}$$

and that when $x = 0$ and $\chi = 1$, we have the distribution relation for the twisted q-Bernoulli polynomials as follows : for $n \geq 1$,

$$\beta_{n,\xi}^{(h)}(0,q) = \beta_{n,\xi,1}^{(h)}(q) = [l]^{n-1} \sum_{a=0}^{l-1} \xi^a q^{ha} \beta_{n,\xi^l}^{(h)}(\frac{a}{l}, q^l). \tag{18}$$

Lemma 1 and Lemma 2 are important for the construction of the p-adic generalized twisted q-Bernoulli distribution as follows.

Theorem 3. *Let $q \in \mathbb{C}_p$. For any positive integers N, n and l, let $\mu_{n,\xi}^{(h)}$ be defined by*

$$\mu_{n,\xi}^{(h)}(a + lp^N \mathbb{Z}_p) = [lp^N]^{n-1} q^{ha} \xi^a \beta_{n,\xi^{lp^N}}(\frac{a}{lp^N}, q^{lp^N}).$$

Then $\mu_{n,\xi}^{(h)}$ extends uniquely to a distribution on X.

Proof. It is suffices to show

$$\sum_{i=1}^{p-1} \mu_{n,\xi}^{(h)}(a + ip^N + p^{N+1}\mathbb{Z}_p) = \mu_{n,\xi}^{(h)}(a + p^N \mathbb{Z}_p).$$

Indeed, Lemma 1 and the definition of $\mu_{n,\xi}^{(h)}$ imply that

$$\begin{aligned}
&\sum_{i=1}^{p-1} \mu_{n,\xi}^{(h)}(a + ip^N + p^{N+1}\mathbb{Z}_p) \\
&= \sum_{x=0}^{p-1} [p^{N+1}]^{n-1} q^{h(a+xp^N)} \xi^{a+xp^N} \beta_{n,\xi^{p^{N+1}}}^{(h)}(\frac{a + xp^N}{p^{N+1}}, q^{p^{N+1}}) \\
&= [p]^{n-1} q^{ha} \xi^a [p^N : q^p]^{n-1} \sum_{x=0}^{p-1} (q^{p^N})^{xh} (\xi^{p^N})^x \beta_{n,(\xi^{p^N})^p}^{(h)}(\frac{\frac{a}{p^N} + x}{p}, (q^{p^N})^p) \\
&= [p]^{n-1} q^{ha} \xi^a \beta_{n,\xi^{p^N}}^{(h)}(\frac{a}{p^N}, q^{p^N}) \\
&= \mu_{n,\xi}^{(h)}(a + p^N \mathbb{Z}_p).
\end{aligned}$$

□

3. A q-analogue of the p-adic twisted L-functions

Let $\alpha \in X^*, \alpha \neq 1, n \geq 1$. By the definition of $\mu_{n,\xi,\chi}^{(h)}$, we easily see :

$$\int_X \chi(x) d\mu_{n,\xi}^{(h)}(x) = \beta_{n,\xi,\chi}^{(h)}(q)$$

$$\int_{pX} \chi(x) d\mu_{n,\xi}^{(h)}(x) = [p]^{n-1} \chi(p) \beta_{n,\xi^p,\chi}^{(h)}(q^p)$$

$$\begin{aligned}(19)\qquad &\int_X \chi(x)d\mu^{(h)}_{n;q^{\frac{1}{\alpha}},\xi^{\frac{1}{\alpha}}}(\alpha x)=\chi(\frac{1}{\alpha})\beta^{(h)}_{n,\xi^{\frac{1}{\alpha}},\chi}(q^{\frac{1}{\alpha}})\\ &\int_{pX} \chi(x)d\mu^{(h)}_{n;q^{\frac{1}{\alpha}},\xi^{\frac{1}{\alpha}}}(\alpha x)=[p;q^{\frac{1}{\alpha}}]^{n-1}\chi(\frac{p}{\alpha})\beta^{(h)}_{n,\xi^{\frac{p}{\alpha}},\chi}(q^{\frac{p}{\alpha}}).\end{aligned}$$

For compact open set $U\subset X$, we define

$$(20)\qquad \mu^{(h)}_{n;q,\alpha,\xi}(U)=\mu^{(h)}_{n;q,\xi}(U)-\alpha^{-1}[\alpha^{-1};q]^{n-1}\mu^{(h)}_{n;q^{\frac{1}{\alpha}},\xi^{\frac{1}{\alpha}}}(U).$$

By the definition of $\mu^{(h)}_{n;q,\xi}$ and (19), we note that

$$\begin{aligned}(21)\qquad \int_{X^*}\chi(x)d\mu^{(h)}_{n;q,\alpha,\xi}(x)&=\beta^{(h)}_{n,\xi,\chi}(q)-[p]^{n-1}\chi(p)\beta_{n,\xi^p,\chi}(q^p)\\ &\quad-\frac{1}{\alpha}[\frac{1}{\alpha}]^{n-1}\chi(\frac{1}{\alpha})\beta^{(h)}_{n,\xi^{\frac{1}{\alpha}},\chi}(q^{\frac{1}{\alpha}})\\ &\quad+\frac{1}{\alpha}[\frac{p}{\alpha}]^{n-1}\chi(\frac{p}{\alpha})\beta^{(h)}_{n,\xi^{\frac{p}{\alpha}},\chi}(q^{\frac{p}{\alpha}})\\ &=(1-\chi^p)(1-\frac{1}{\alpha}\chi^{\frac{1}{\alpha}})\beta^{(h)}_{n,\xi,\chi},\end{aligned}$$

where the operator $\chi^y=\chi^{y,n;q,\xi}$ on $f(q,\xi)$ defined by

$$\chi^y f(q,\xi)=[y]^{n-1}\chi(y)f(q^y,\xi^y),\quad \chi^x\chi^y=\chi^{x,n;q^y,\xi^y}\circ\chi^{y,n;q,\xi}.$$

Let $x\in X$. We recall that $\{x\}_N$ denote the least nonnegative residue (mod lp^N) and that if $[x]_N=x-\{x\}_N$, then $[x]_N\in lp^N\mathbb{Z}_p$. Now we can define in [5] as follows:

$$\mu^{(h)}_{Mazur,1,\alpha}(a+lp^N\mathbb{Z}_p)=(\frac{\frac{1}{\alpha}-1}{h+1}+\frac{h}{\alpha}\cdot\frac{[a\alpha]_N}{lp^N}).$$

By the same method of Kim in [5], we easily see:

$$\begin{aligned}(22)\qquad &\lim_{N\to\infty}\mu^{(h)}_{n;q,\alpha,\xi}(a+lp^N\mathbb{Z}_p)\\ &=\lim_{N\to\infty}[l]^{n-1}((h+n)q^{(h+1)a}-hq^a)\xi^a(\frac{\frac{1}{\alpha}-1}{h+1}+\frac{h}{\alpha}\cdot\frac{[a\alpha]_N}{lp^N}).\end{aligned}$$

Thus we have

$$(23)\qquad \mu^{(h)}_{n;q,\alpha,\xi}(x)=[x]^{n-1}((h+n)q^{(h+1)x}-hq^{xh})\xi^x\mu^{(h)}_{Mazur,1,\alpha}(x).$$

Theorem 4. *$\mu^{(h)}_{n;q,\alpha,\xi}$ are bounded $\mathbb{C}_p$-valued measure on X for all $n\geq 1$ and $\alpha\in X^*,\alpha\neq 1$.*

Now we define $<x>=<x;q>=[x;q]/w(x)$, where $w(x)$ is the Teichmüller character. For $|q-1|_p<p^{-\frac{1}{p-1}}$, we note that $<x>^{p^N}\equiv 1\ (\mathrm{mod}\ p^N)$. By (21) and (23), we have the following:

$$\begin{aligned}
&\int_{X^*} \chi_n(x) d\mu^{(h)}_{n;q,\alpha\xi}(x) \\
(24) \qquad &= \int_{X^*} \chi_n(x)[x]^{n-1}((h+n)q^{(h+1)x} - hq^{xh})\xi^x \mu^{(h)}_{Mazur,1,\alpha}(x) \\
&= \int_{X^*} ((h+n)q^{(h+1)x} - hq^{xh}) < x >^{n-1} \xi^x \chi_1(x)\mu^{(h)}_{Mazur,1,\alpha}(x)
\end{aligned}$$

where $\chi_n(x) = \chi w^{-n}(x)$. By using (24), we can construct a q-analogue of p-adic generalized twisted L-function.

Definition 5. For fixed $\alpha \in X^*, \alpha \neq 1$, we define a h-extension of p-adic generalized twisted L-function as follows;

(25)

$$\begin{aligned}
&L^{(h)}_{p,q,\xi}(s,\chi) \\
&= \frac{1}{1-s}\int_{X^*} ((h+1-s)q^{(h+1)x} - hq^{hx})\xi^x < x >^{-s} \chi_1(x) d\mu^{(h)}_{Mazur,1,\alpha}(x)
\end{aligned}$$

for $s \in X$.

Theorem 6. *For each $s \in \mathbb{Z}_p$ and $\alpha \in X^*, \alpha \neq 1$, we have*

(26)

$$\begin{aligned}
L^{(h)}_{p,q,\xi}(s,\chi) = &\frac{1-s+h}{1-s}(q-1)\sum_{n=1}^{\infty}{}^{*} \frac{q^{nh}\xi^n w^{s-1}(n)}{[n]^{s-1}}\chi(n)(\frac{\frac{1}{\alpha}-1}{h+1} + \frac{h}{\alpha}\cdot\frac{[n\alpha]_N}{lp^N}) \\
&+ \sum_{n=1}^{\infty}{}^{*} q^{hn}\xi^n[n]^{-s}w^{s-1}(n)\chi(n)(\frac{\frac{1}{\alpha}-1}{h+1} + \frac{h}{\alpha}\cdot\frac{[n\alpha]_N}{lp^N}),
\end{aligned}$$

where $\sum_{n=1}^{\infty}{}^{*}$ means to sum over the rational integers prime to p in the given range.

Proof. For each $s \in \mathbb{Z}_p$ and $x \in X^*$, we have

$$\begin{aligned}
&(h+1-s)q^{(h+1)x} - hq^{hx} \\
&= hq^{hx}(q^x - 1) + (1-s)q^x q^{hx} \\
&= (q-1)q^{hx}[x](h+1-s) + q^{hx}(1-s).
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{1-s}\int_X^* ((h+1-s)q^{(h+1)x} - hq^{hx})\xi^x < x >^{-s} \chi_1(x) d\mu^{(h)}_{Mazur,1,\alpha}(x) \\
&= \frac{1}{1-s}\int_X^* [(q-1)q^{hx}[x](h+1-s) + q^{hx}(1-s)]\xi^x < x >^{-s} \\
&\qquad \times \chi_1(x) d\mu^{(h)}_{Mazur,1,\alpha}(x)
\end{aligned}$$

$$= \frac{1-s+h}{1-s}(q-1)\sum_{n=1}^{\infty}{}^{*} \frac{q^{nh}\xi^n w^{s-1}(n)}{[n]^{s-1}}\chi(n)(\frac{\frac{1}{\alpha}-1}{h+1}+\frac{h}{\alpha}\cdot\frac{[n\alpha]_N}{lp^N})$$
$$+\sum_{n=1}^{\infty}{}^{*} q^{hn}\xi^n[n]^{-s}w^{s-1}(n)\chi(n)(\frac{\frac{1}{\alpha}-1}{h+1}+\frac{h}{\alpha}\cdot\frac{[n\alpha]_N}{lp^N}).$$

□

The equation (26) with $h = s-1$ implies that

$$L^{(s-1)}_{p,q,\xi,\alpha}(s,\chi) = \sum_{n=1}^{\infty}{}^{*} q^{(s-1)n}\xi^n[n]^{-s}w^{s-1}(n)\chi(n)(\frac{\frac{1}{\alpha}-1}{s}+\frac{s-1}{\alpha}\cdot\frac{[n\alpha]_N}{lp^N}). \tag{27}$$

Finally for each positive integer m, we can construct a q-analogue of the p-adic twisted L-function which interpolate a generalized q-Bernoulli number.

Theorem 7. *For each $m \in \mathbb{N}$ and $\alpha \in X^*, \alpha \neq 1$, we have*

$$L^{(h)}_{p,q,\xi}(1-m,\chi) = -\frac{1}{m}(1-\chi^p_m)(1-\frac{1}{\alpha}\chi^{\frac{1}{\alpha}}_m)w^{-m}\beta^{(h)}_{m,\xi,\chi}(q). \tag{28}$$

Proof. For each $s \in \mathbb{Z}_p$, by using (21), we have

$$L^{(h)}_{p,q,\xi,\alpha}(s,\chi)$$
$$= \frac{1}{1-s}\int_{X^*}((h+1-s)q^{(h+1)x}-hq^{hx})\xi^x < x >^{-s} \chi_1(x)d\mu^{(h)}_{Mazur,1,\alpha}(x)$$
$$= \frac{1}{1-s}\int_{X^*}\chi_{1-s}(x)d\mu_{1-s;q,\alpha,\xi}(x)$$
$$= \frac{1}{1-s}(1-\chi^p_{1-s})(1-\frac{1}{\alpha}\chi^{\frac{1}{\alpha}}_{1-s})\beta^{(h)}_{n,\xi,\chi}(q).$$

Thus

$$L^{(h)}_{p,q,\xi}(1-m,\chi)$$
$$= \frac{1}{m}(1-\chi^p_m)(1-\frac{1}{\alpha}\chi^{\frac{1}{\alpha}}_m)\beta^{(h)}_{n,\xi,\chi}(q).$$

□

Remark. In [5], Kim constructed the h-extension of p-adic $q-L$-functions. And the question to inquire the existence of the twisted p-adic $q-L$-functions was remained in [3]. This is still open. By means of the method provided by Kim [5], we constructed the twisted p-adic $q-L$-function to be a part of an answer for the question which was remained in [3].

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