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Temperature effect on spherical Couette flow of Oldroyd-B fluid

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Abstract

The present paper is concerned with non-isothermal spherical Couette flow of Oldroyd-B fluid in the annular region between two concentric spheres. The inner sphere rotates with a uniform angular velocity while the outer sphere is kept at rest. Moreover, the two spherical boundaries are maintained at fixed temperature values. Hence, the fluid is effect by two heat sources; namely, the viscous heating and the temperature gradient between the two spheres. The viscoelasticity of the fluid is assumed to dominate the inertia such that the latter can be neglected. An approximate analytical solution of the energy and momentum equations is obtained through the expansion of the dynamical fields in power series of Nahme number.
The analysis show that, the temperature variation due to the external source appears in the zero order solution and its effect extends to the fluid velocity distribution up to present second order. Viscous heating contributes in the first and second order solutions. In contrast to isothermal case, a first order axial velocity and a second order stream function fields has been appeared. Moreover, at higher orders the temperature distribution depends on the gap width between the two spheres. Finally, there exist a thermal distribution of positive and negative values depend on their positions in the domain region between the two spheres.
keywords : spherical Couette, Oldtoyd-B, non-isothermal, viscous heating, Nahme number

1. Introduction

Rheological measurements of the parameters of the viscoelastic fluids are very important in the last few decades due to the large width using of this kind of fluids in industry. Many instruments that are used to detect these parameters stand on the postulate that independence of the fluid parameters on a temperature in spite of an appearance of viscous heating due to a conversion of mechanical energy into thermal energy inside the fluid. In fact the fluid parameters, such as viscosity and relaxation time are very sensitive to temperature changes. Moreover, the fluid suffering from a large temperature variation in industry due to forced mechanical operations. This application increases the gap between the measured parameters in laboratory and real parameters control the motion in industry. Hence, the temperature is considered as a source of error in rheological measurements. So the aim of the present work is to deduce the temperature effect on the fluid parameters not only due to viscous heating but also in application of external temperature source on the fluid boundaries. The later one; *i.e.* the external temperature source in fluid motion enables us to study the behavior of the fluid and parameter variation due to a wide range of temperature. This helps us to improve the control of the fluid motion in applications (Ferry, 1980).

It is interest to show that numerous works have been dealt with spherical Couette flow of Oldroyd-B fluid isothermally, both theoretically and experimentally. Yamaguchi et al. (1997a, 1997b, 1999) treated the present problem for Oldroyd-B fluid. Abu-El Hassan (2006, 2007a) investigated the same problem by using the successive approximate method for Oldroyd 8-constant constitutive model where Oldroyd-B fluid is taken as a special case. The solution shows an appearance of axial velocities in zero and second order approximation as well as a secondary flow in first order solution only.

Recently, Abu-El Hassan et. al (2007b) investigated the present boundary value problem (B.V.P.) for viscous heating only. The analysis show that two additional contributed terms had appeared due to viscous heating ; namely, a first order axial-velocity component and a second order stream function which are not exist in isothermal case. Up to the first order, the behavior of all viscoelastic fluids are identical. The difference appears in second order solution ; specially for temperature profile which depends on the gap width between the two spheres. Noteworthy, the temperature growth due to viscous heating is of order one degree Kelvin.

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In this paper we study spherical Couette flow of fluid particles with the influence of temperature profile due to viscous heating effect and the temperature-difference between the two boundaries. The main important of an external temperature source in fluid is giving us a chance to study the behavior of the fluid in a large width of a temperature change. Hence, the present B.V.P. is concerned with non-isothermal steady shear flow of Oldroyd-B fluid in the annular region between two concentric spheres of radii R_1 and R_2 ($R_2 \succ R_1$). The two inner and outer spheres are maintained at different temperatures T_1 and T_2 ($T_1 \succ T_2$). Moreover, the inner sphere rotates with a uniform angular velocity ω about the z-axis centered at the origin of the system and the outer sphere is at rest. The successive approximate method of solution is performed through the expansion of the dynamical variables in power series of Nahme number. Two viscoelastic fluids with different rheological properties has been investigated as test fluids.

2. Governing equations

In dimensional form, the equations governing the steady state flow of an incompressible viscoelastic fluid are stated as follows:

The continuity equation

$$\nabla \cdot \underline{\tilde{V}} = 0 \tag{2-1}$$

where $\underline{\tilde{V}}$ is the velocity field. In terms of spherical polar coordinate ($\tilde{r}$, ϑ, ϕ), the velocity field takes the form

$$\underline{\tilde{V}} = \underline{\tilde{V}}(\tilde{r},\vartheta)\underline{\hat{r}} + \underline{\tilde{V}}(\tilde{r},\vartheta)\underline{\hat{\vartheta}} + \underline{\tilde{V}}(\tilde{r},\vartheta)\underline{\hat{\varphi}} \tag{2-2}$$

where $\underline{\hat{r}}$, $\underline{\hat{\vartheta}}$ and $\underline{\hat{\varphi}}$ are the unit base vectors in r, ϑ and φ axis, respectively. The index "~" refers to the dimensional quantities.

The momentum equation is given by

$$-\nabla\tilde{P} + \nabla \cdot \underline{\underline{\tilde{\sigma}}} = 0 \tag{2-3}$$

where $\tilde{P}$ is the pressure and $\underline{\underline{\tilde{\sigma}}}$ is the stress tensor. The stress tensor can be written as

$$\underline{\underline{\tilde{\sigma}}} = \eta_s \underline{\underline{\dot{\tilde{\gamma}}}} + \underline{\underline{\tilde{\tau}}} \tag{2-4}$$

A non-isothermal version of Oldroyd-B model based on the pseudo-time hypothesis (Bird et al., 1987; Bird et al., 1995) gives the following equation for the extra stress $\underline{\underline{\tilde{\tau}}}$

$$\underline{\underline{\tilde{\tau}}} + \lambda\left(\overset{\nabla}{\underline{\underline{\tilde{\tau}}}} - \underline{\underline{\tilde{\tau}}}\frac{D}{Dt}\mathrm{In}(1+\Theta/\delta)\right) = \eta_p \underline{\underline{\dot{\tilde{\gamma}}}} \tag{2-5}$$

Where δ is the dimensionless thermal sensitivity defined by

$$\delta = \frac{\tilde{T}_0}{\eta_0}\left|\frac{\partial\eta}{\partial\tilde{T}}\right|_{\tilde{T}=\tilde{T}_0} \tag{2-6}$$

Θ is the dimensionless temperature related to the reference temperature $\tilde{T}_0$ by

$$\Theta = \delta\left(\frac{\tilde{T}}{\tilde{T}_0} - 1\right) \tag{2-7}$$

η_s and η_p are solvent and polymer viscosities; respectively defined by Nahme type law (Nahme, 1940; Kearsley, 1962; Turian, 1965; Becker and McKinley, 2000)

$$\eta_s = \eta_{s0}e^{-\Theta} \text{ and } \eta_p = \eta_{p0}e^{-\Theta} \tag{2-8}$$

and λ is the relaxation time given as

$$\lambda(T) = \frac{\lambda_0 e^{-\Theta}}{1+\Theta/\delta} \tag{2-9}$$

η_{s0}, η_{p0} and λ_0 are solvent and polymer viscosities and relaxation time, measured at $\tilde{T}_0$; respectively.

For simplicity, one may neglect the inertia (Bohme, 1987; Bird, 1987) such that the energy equation for the temperature T is given as

$$0 = k\nabla^2\tilde{T} + \underline{\underline{\tilde{\sigma}}}:\nabla\underline{\underline{\tilde{V}}} \tag{2-10}$$

where k is the thermal conductivity of the fluid.

It is more convenient to introduce the following dimensionless variables

$$P = \frac{\tilde{P}}{\eta_0\omega},\ \underline{\underline{\tau}} = \frac{\underline{\underline{\tilde{\tau}}}}{\eta_0\omega},\ \underline{\underline{\sigma}} = \frac{\underline{\underline{\tilde{\sigma}}}}{\eta_0\omega},\ \underline{\underline{d}} = \frac{\underline{\underline{\dot{\tilde{\gamma}}}}}{\omega},\ \underline{V} = \frac{\underline{\tilde{V}}}{\omega R_1},\ \underline{r} = \frac{\underline{\tilde{r}}}{R_1} \tag{2-11}$$

Hence, the governing equations in non-dimensional form can be written in the following manner. The constitutive equation

$$\underline{\underline{\sigma}} = e^{-\Theta}\underline{\underline{d}} + De\frac{e^{-\Theta}}{1+\Theta/\delta}(\overset{\nabla}{\underline{\underline{\tau}}} - \underline{\underline{\tau}}(\underline{V}.\nabla)\mathrm{In}(1+\Theta/\delta)) \tag{2-12}$$

where $De = \lambda_0\omega$ is the Debora number. The momentum equation

$$-\nabla P + \nabla.\underline{\underline{\sigma}} = 0 \tag{2-13}$$

and the energy equation

$$\nabla^2\Theta + Na\,\underline{\underline{\sigma}}:\nabla\underline{V} = 0 \tag{2-14}$$

where the Nahme number Na is given as

$$Na = \frac{\eta_0\delta R_1^2\omega^2}{kT_0} \tag{2-15}$$

3. Boundary Conditions

The boundary conditions are no slip at the surface of the two spheres which has different non-distributed temperatures, hence

$$V_\varphi = \begin{Bmatrix} \sin\vartheta \\ 0 \end{Bmatrix} \text{ and } \Theta = \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} \text{ at } r = \begin{Bmatrix} 1 \\ a \end{Bmatrix} \tag{3-1}$$

where $a = R_2/R_1$ is the geometrical parameter ratio and Θ_1

and Θ_2 are the dimensionless temperature of the spherical boundaries.

4. Method of solution

Using Eq. (2-12) into Eq. (2-13), the momentum equation can be written in the form

$$-\nabla P+e^{-\Theta}(\nabla\cdot \underline{\underline{d}}-\nabla\Theta\cdot \underline{\underline{d}})+Na\ e^{-\Theta}(\nabla\Theta\cdot\Gamma-\nabla\cdot\Gamma)=0 \quad (4\text{-}1)$$

where

$$\Gamma=\frac{x}{1+\Theta/\delta}(\overset{\nabla}{\underline{\underline{\tau}}}-\underline{\underline{\tau}}(\underline{V}\nabla)\text{In}(1+\Theta/\delta)) \quad (4\text{-}2)$$

$$\text{and } x=\frac{De}{Na}=\frac{kT_0\lambda_0}{\eta_0R_1^2\omega}. \quad (4\text{-}3)$$

The momentum equation, Eq. (4-1), may be decomposed into a scalar equation governing the $\varphi-$ component and a vector equation including the $r-$ and $\vartheta-$ components and then simplified to take the form

$$\hat{D}_1V_\varphi-\Sigma_\varphi+Na\ \Lambda_\varphi=0 \quad (4\text{-}4)$$

$$-\nabla P-e^{-\Theta}\nabla^2\underline{U}-e^{-\Theta}(\Sigma_r\hat{\underline{r}}+\Sigma_\vartheta\hat{\underline{\vartheta}})+Na\ e^{-\Theta}(\Lambda_r\hat{\underline{r}}+\Lambda_\vartheta\hat{\underline{\vartheta}})=0 \quad (4\text{-}5)$$

where

$$\hat{D}_1=\frac{1}{r^2}\left[\partial_r(r^2\partial_r)+\partial_\vartheta\left(\frac{1}{\sin\vartheta}\partial_\vartheta(\sin\vartheta)\right)\right] \quad (4\text{-}6)$$

$\underline{\Sigma}=\nabla\Theta\cdot\underline{\underline{d}}=\Sigma_r\hat{\underline{r}}+\Sigma_\vartheta\hat{\underline{\vartheta}}+\Sigma_\varphi\hat{\underline{\varphi}}$ with its components (4-7 a)

$\Sigma_r=\Theta_{,r}d_{rr}+\frac{\Theta_{,\vartheta}}{r}d_{r\vartheta}$, $\Sigma_\vartheta=\Theta_{,r}d_{r\vartheta}+\frac{\Theta_{,\vartheta}}{r}d_{\vartheta\vartheta}$ and

$$\Sigma_\varphi=\Theta_{,r}d_{r\varphi}+\frac{\Theta_{,\vartheta}}{r}d_{\vartheta\varphi} \quad (4\text{-}7\text{ b})$$

and

$$\underline{\Lambda}=\nabla\Theta\cdot\underline{\underline{\Gamma}}-\nabla\cdot\underline{\underline{\Gamma}}=\Lambda_r\hat{\underline{r}}+\Lambda_\vartheta\hat{\underline{\vartheta}}+\Lambda_\varphi\hat{\underline{\varphi}} \quad (4\text{-}8\text{ a})$$

with its components

$$\Lambda_r=\Theta_{,r}\Gamma_{rr}+\frac{\Theta_{,\vartheta}}{r}\Gamma_{r\vartheta}-\left(\frac{1}{r^2}\partial_r(r^2\Gamma_{rr})+\frac{1}{r\sin\vartheta}\partial_\vartheta(\Gamma_{r\vartheta}\sin\vartheta)-\frac{\Gamma_{\vartheta\vartheta}+\Gamma_{\varphi\varphi}}{r}\right)$$

$$\Lambda_\vartheta=\Theta_{,r}\Gamma_{r\vartheta}+\frac{\Theta_{,\vartheta}}{r}\Gamma_{\vartheta\vartheta}-\left(\frac{1}{r^3}\partial_r(r^3\Gamma_{r\vartheta})+\frac{1}{r\sin\vartheta}\partial_\vartheta(\Gamma_{\vartheta\vartheta}\sin\vartheta)-\frac{\Gamma_{\varphi\varphi}\cot\vartheta}{r}\right)$$

$$\Lambda_\varphi=\Theta_{,r}\Gamma_{r\varphi}+\frac{\Theta_{,\vartheta}}{r}\Gamma_{\vartheta\varphi}-\left(\frac{1}{r^3}\partial_r(r^3\Gamma_{r\varphi})+\frac{1}{r\sin\vartheta}\partial_\vartheta(\Gamma_{\vartheta\varphi}\sin\vartheta)-\frac{\Gamma_{\vartheta\varphi}\cot\vartheta}{r}\right) \quad (4\text{-}8\text{ b})$$

Using the stream function Ψ which can be defined as

$$V_r\hat{\underline{r}}+V_\vartheta\hat{\underline{\vartheta}}=-\nabla\wedge\left(\frac{\Psi}{r\sin\vartheta}\hat{\underline{\varphi}}\right) \quad (4\text{-}9)$$

and taking the curl of Eq.(4-5) one gets

$$\hat{D}_2^2\Psi-\nabla\Theta\nabla(\hat{D}_2\Psi)-\sin\vartheta(-r\Theta_{,r}\Sigma_\vartheta+\Theta_{,\vartheta}\Sigma_r+\partial_r(r\Sigma_\vartheta)-\partial_\vartheta\Sigma_r)$$
$$+Na\ \sin\vartheta(-r\Theta_{,r}\Lambda_\vartheta+\Theta_{,\vartheta}\Lambda_r+\partial_r(r\Lambda_\vartheta)-\partial_\vartheta\Lambda_r)=0 \quad (4\text{-}10)$$

where

$$\hat{D}_2=\partial_r^2+\frac{\sin\vartheta}{r^2}\partial_\vartheta\left(\frac{1}{\sin\vartheta}\partial_\vartheta\right) \quad (4\text{-}11)$$

For $\Theta/\delta\prec\prec1$ we can make use of the expansions

$$\frac{1}{1+\Theta/\delta}\cong1-\Theta/\delta \quad (4\text{-}12\text{ a})$$

$$(\underline{V}\nabla)\text{In}(1+\Theta/\delta)\cong(\underline{V}\nabla)\Theta/\delta \quad (4\text{-}12\text{ b})$$

and

$$e^{-\Theta}=e^{-\Theta_s}e^{-(\Theta-\Theta_s)}=e^{-\Theta_s}(1+\Theta_s-\Theta),\ \Theta_s=(\Theta_1+\Theta_2)/2 \quad (4\text{-}12\text{ c})$$

such that

$$|\Theta_s-\Theta|\prec1 \text{ or } \left|\frac{2T-(T_1+T_2)}{2T_0}\right|\prec\frac{1}{\delta} \quad (4\text{-}12\text{ d})$$

Hence, the energy and constitutive equations for extra stress are ;respectively,

$$\nabla^2\Theta+e^{-\Theta_s}(1+\Theta_s-\Theta)[\underline{\underline{d}}-De(1-\Theta/\delta)(\overset{\nabla}{\underline{\underline{\tau}}}-\underline{\underline{\tau}}(\underline{V}\nabla)(\Theta/\delta))]:\nabla\underline{V}=0 \quad (4\text{-}13)$$

$$\underline{\underline{\tau}}=e^{-\Theta_s}(1+\Theta_s-\Theta)[B\underline{\underline{d}}-Nax(1-\Theta/\delta)(\overset{\nabla}{\underline{\underline{\tau}}}-\underline{\underline{\tau}}(\underline{V}\nabla)(\Theta/\delta))] \quad (4\text{-}14)$$

where $B=\frac{\eta_{p0}}{\eta_0}$. The problem now is to solve Eqs. (4-4), (4-10) and (4-13). Employing the constitutive equation , Eq. (4-14), in order to find the axial velocity, the stream function and the temperature profile. The solution is performed by using the method of successive approximation where the variable functions $\underline{\underline{d}}$, $\underline{\underline{\tau}}$, V_φ, Ψ and Θ are expanded in power series with respect to the Nahme number in the form

$$A=A^{(0)}+Na\ A^{(1)}+Na^2A^{(2)}+... \quad (4\text{-}15)$$

with A is any one of the above variable functions.

4.1. solution of zero order approximation

The governing set of equations in this case is reduced to

$$\hat{D}_2^2\Psi^{(0)}-\left(\Theta_{,r}^{(0)}\partial_r+\frac{\Theta_{,\vartheta}^{(0)}}{r^2}\partial_\vartheta\right)(\hat{D}_2\Psi^{(0)})$$
$$+\sin\vartheta[(\Theta_{,r}^{(0)}-\partial_r)r\Sigma_\vartheta^{(0)}-(\Theta_{,\vartheta}^{(0)}-\partial_\vartheta)\Sigma_r^{(0)}]=0 \quad (4\text{-}16)$$

$$\nabla^2\Theta^{(0)}=0 \quad (4\text{-}17)$$

$$\hat{D}_1V_\varphi^{(0)}-\Sigma_\varphi^{(0)}=0 \quad (4\text{-}18)$$

with the boundary conditions

$$\Psi^{(0)}=\Psi^{(0)}_{,r}=0 \quad \text{at } r=1\ ,\ a \tag{4-19 a}$$

$$\Theta^{(0)}=\Theta_1,\ \Theta_2 \quad \text{at } r=1\ ,\ a \tag{4-19 b}$$

$$V^{(0)}_{\varphi}=1,0 \quad \text{at } r=1\ ,\ a \tag{4-19 c}$$

Equation (4-16) shows that $\Sigma^{(0)}_r$ and $\Sigma^{(0)}_\vartheta$ are just operators acting on $\Psi^{(0)}$ so the solution subjected to the boundary conditions (4-19 a) gives

$$\Psi^{(0)}=0 \tag{4-20}$$

The solution of the temperature profile $\Theta^{(0)}$, Eq. (4-17), with the boundary conditions, Eq. (4-19 b), is

$$\Theta^{(0)}=c_0+\frac{c_1}{r}, \tag{4-21}$$

where the c's are determined from the boundary conditions.

For $V^{(0)}_\varphi$ using Eqs. (4-7),we can see that $\Sigma^{(0)}_\varphi=0$ in case of relatively small temperature difference, Eqs.(4-12d), between the two spheres so that Eq. (4-18) reduces to

$$\frac{1}{r^2}\left[\partial_r(r^2\partial_r)+\partial_\vartheta\left(\frac{1}{\sin\vartheta}\partial_\vartheta \sin\vartheta\right)\right]V^{(0)}_\varphi=0 \tag{4-22}$$

The solution of this equation which satisfy boundary conditions (4-19 c) is given as

$$V^{(0)}_\varphi=\frac{a^3r^{-2}-r}{a^3-1}\sin\vartheta. \tag{4-23}$$

This is the classical Newtonian velocity field.

4.2. solution of first order approximation

The governing set of equations in this case is

$$\hat{D}^2_2\Psi^{(1)}-\sin\vartheta(\partial_r(r\Sigma^{(1)}_\vartheta)-\partial_\vartheta\Sigma^{(1)}_r)$$
$$+\sin\vartheta(\partial_r(r\Lambda^{(0)}_\vartheta)-\partial_\vartheta\Lambda^{(0)}_r-r\Theta^{(0)}_{,r}\Lambda^{(0)}_\vartheta+\Theta^{(0)}_{,\vartheta}\Lambda^{(0)}_r)=0 \tag{4-24}$$

$$\nabla^2\Theta^{(1)}+e^{-\Theta_s}\left[(1-\Theta_s-\Theta^{(0)})\left(\underline{\underline{d}}^{(0)}-De\left(1-\frac{\Theta^{(0)}}{\delta}\right)\right)\right]:\nabla\underline{V}^{(0)}=0 \tag{4-25}$$

$$\hat{D}_1V^{(1)}_\varphi-\Sigma^{(1)}_\varphi+\Lambda^{(1)}_\varphi=0 \tag{4-26}$$

and the stress tensor in first order approximation is

$$\underline{\underline{\tau}}^{(1)}=e^{-\Theta_s}(1-\Theta_s-\Theta^{(0)})\left(\frac{\eta_{P_0}}{\eta_0}\underline{\underline{d}}^{(1)}-xe^{-\Theta_s}\left(1+\frac{\Theta^{(0)}}{\delta}\right)\overset{\nabla}{\underline{\underline{\tau}}}{}^{(0)}\right)$$
$$-e^{-\Theta_s}\frac{\eta_{P_0}}{\eta_0}\Theta^{(1)}\underline{\underline{d}}^{(0)} \tag{4-27}$$

with the boundary conditions

$$\Psi^{(1)}=\Psi^{(1)}_{,r}=0 \quad \text{at } r=0,\ 1 \tag{4-28 a}$$

$$\Theta^{(1)}=0 \quad \text{at } r=0,\ 1 \tag{4-28 b}$$

$$V^{(1)}_\varphi=0 \quad \text{at } r=0,\ 1 \tag{4-28c}$$

Equations (4-24) and (4-25) can be simplified to take the form

$$\hat{D}^2_2\Psi^{(1)}+(\partial_r(rF_1(r))-2F_1(r)-r\Theta^{(0)}_{,r}F_1(r))\sin^2\vartheta\cos\vartheta=0 \tag{4-29}$$

$$\nabla^2\Theta^{(1)}+\frac{3}{2}F_2(r)P_0-\frac{3}{2}F_2(r)P_2=0 \tag{4-30}$$

where

$$F_1(r)=-18xBe^{-\Theta_s}(a^3-1)^{-2}a^6\left(1-\frac{\Theta^{(0)}}{\delta}\right)(1+\Theta_s-\Theta^{(0)})r^{-7} \tag{4-31 a}$$

$$F_2(r)=9e^{-\Theta_s}(a^3-1)^{-2}a^6(1+\Theta_s-\Theta^{(0)})r^{-6} \tag{4-31 b}$$

with P_0 and P_2 are the associated Lagender polynomials. For $\Psi^{(1)}$, the general solution of Eq.(4-29) is written as

$$\Psi^{(1)}=(c_2r^{-2}+c_3+c_4r^3+c_5r^5+c_6r^{-6}+c_7r^{-5}+c_8r^7+c_9r^{-3})$$
$$\sin^2\vartheta\cos\vartheta \tag{4-32}$$

The parameters c_2 up to c_5 can be determined from the boundary conditions Eq. (4-28 a), see on App. A.

Also the general solution of Eq. (4-30) for $\Theta^{(1)}$ is

$$\Theta^{(1)}=(c_{10}+c_{11}r^{-1}+c_{14}r^{-5}+c_{15}r^{-4})P_0$$
$$+\left(c_{12}r^2+c_{13}r^{-3}-\frac{20}{14}c_{14}r^{-5}-\frac{12}{6}c_{15}r^{-4}\right)P_2 \tag{4-33}$$

the c's parameters are also determined from the boundary conditions, Eq. (4-28 b), see App. A. Using the solution for $\Theta^{(1)}$, the differential equation that governing the first order axial velocity $V^{(1)}_\varphi$ reduced to

$$\hat{D}_1V^{(1)}_\varphi+3a^3(a^3-1)^{-1}\left[\left(2c_{12}-c_{11}r^{-3}-3c_{13}r^{-4}+2c_{15}r^{-5}-\frac{15}{2}c_{14}r^{-6}\right)\right.$$
$$\left.\sin\vartheta-\frac{3}{2}\left(c_{12}r+c_{13}r^{-4}-\frac{12}{6}c_{15}r^{-5}-\frac{10}{7}c_{14}r^{-6}\right)\sin^3\vartheta\right]=0 \tag{4-34}$$

Hence, the general solution for $V^{(1)}_\varphi$ is given as

$$V^{(0)}_\varphi=3(a^3-1)^{-1}a^3\left[\left(c_{16}r+c_{17}r^{-2}-\frac{8}{10}(c_{18}r^3+c_{19}r^{-4})+c_{24}r^{-7}\right.\right.$$
$$+c_{25}r^{-6}+c_{25}r^{-5}+c_{27}r^{-3})\sin\vartheta+(c_{18}r^3+c_{19}r^{-4}+c_{20}r^{-7}+c_{21}r^{-6})$$
$$+c_{22}r^{-5}+c_{23})\sin^3\vartheta] \tag{4-35}$$

The c's parameters can be determined from the boundary conditions Eq. (4-28 c), see App. A.

4.3. solution of second order approximation

The governing set of equations in this case is outlined as follows

$$\hat{D}^2_2\Psi^{(2)}-\left[\Theta^{(1)}_{,r}\partial_r(\hat{D}_2\Psi^{(1)})+\frac{\Theta^{(1)}_{,\vartheta}}{r^2}\partial_\vartheta(\hat{D}_2\Psi^{(1)})\right]$$

$$-\frac{1}{r^2}[\Theta^{(0)}_{,\vartheta}\partial_\vartheta(\hat{D}_2\Psi^{(2)})++\Theta^{(2)}_{,\vartheta}\partial_\vartheta(\hat{D}_2\Psi^{(0)})]+\sin\vartheta[\Theta^{(2)}_{,r}r\Sigma^{(0)}_\vartheta$$

$$+\Theta^{(1)}_{,r}r(\Sigma^{(1)}_\vartheta-\Lambda^{(0)}_\vartheta)+\Theta^{(1)}_{,r}r(\Sigma^{(2)}_\vartheta-\Lambda^{(1)}_\vartheta)-\Theta^{(2)}_{,\vartheta}\Sigma^{(0)}_r-\Theta^{(1)}_{,\vartheta}(\Sigma^{(1)}_r-x\Lambda^{(0)}_r)$$

$$-\Theta^{(0)}_{,\vartheta}(\Sigma^{(2)}_{r}-\Lambda^{(1)}_{r})-\partial_r(r\Sigma^{(2)}_{\vartheta}-r\Lambda^{(1)}_{\vartheta})+\partial_{\vartheta}(\Sigma^{(2)}_{r}-\Lambda^{(1)}_{r})]=0 \quad (4\text{-}36)$$

$$\nabla^2\Theta^{(2)}+e^{-\Theta_r}[(\underline{\underline{d}}^{(1)}-De\overset{\nabla}{\underline{\underline{\tau}}}^{(1)}):(\nabla V)^{(0)}+$$
$$+(\underline{\underline{d}}^{(0)}-De\overset{\nabla}{\underline{\underline{\tau}}}^{(0)}):(\nabla V)^{(1)}]+\Theta^{(1)}\underline{\underline{d}}^{(0)}:(\nabla V)^{(0)}=0 \quad (4\text{-}37)$$

with the boundary conditions

$$\Psi^{(2)}=\Psi^{(2)}_{,r}=0 \text{ at } r=0,\ 1 \quad (4\text{-}38\text{ a})$$

$$\Theta^{(2)}=0 \text{ at } r=0,\ 1 \quad (4\text{-}38\text{ b})$$

The solution of Eq.(4-36) for the second order stream function $\Psi^{(2)}$, subjected to the boundary conditions Eq. (4-38 a), can be treated as in the first order case. Hence the general solution of $\Psi^{(2)}$ is

$$\Psi^{(2)}=\left(a_{40}r^{-2}+a_{41}+a_{42}r^{3}+a_{43}r^{5}-\frac{288}{504}(a_{36}r^{-4}+a_{39}r^{7})+H_2\right)$$

$$sin^2\vartheta\cos\vartheta+(a_{36}r^{-4}+a_{37}r^{-2}+a_{38}r^{5}+a_{39}r^{7}+H_1)sin^4\vartheta\cos\vartheta \quad (4\text{-}39)$$

where

$$H_1=a_1r^{-12}+a_2r^{-11}+a_3r^{-10}+a_4r^{-9}+a_5r^{-8}+a_6r^{-7}+a_7r^{-6}$$
$$+a_9log(r)r^{-4}+a_{18}r^{-3}+a_{11}log(r)r^{-2}+a_{12}r^{-1}+a_{13}+a_{14}r+a_{15}r^2 \quad (4\text{-}40\text{ a})$$

$$H_2=a_{16}r^{-12}+a_{17}r^{-11}+a_{18}r^{-10}+a_{19}r^{-9}+a_{20}r^{-8}+a_{21}r^{-7}+a_{22}r^{-6}$$
$$+a_{23}r^{-5}+a_{24}log(r)r^{-4}+a_{25}r^{-4}+a_{26}r^{-3}+a_{27}log(r)r^{-2}+a_{28}r^{-1}$$
$$+a_{29}log(r)+a_{30}r+a_{31}r^2+a_{32}log(r)r^3+a_{33}r^4+a_{34}r^6+a_{35}r^7 \quad (4\text{-}40\text{ b})$$

The second order temperature profile $\Theta^{(2)}$, Eq. (4-37), subjected to the boundary conditions Eq. (4-38 b) can be solved as follows

$$\Theta^{(2)}=(b_{69}+b_{70}r^{-1}+H_3)P_0+(a_{71}r^2+a_{72}r^{-3}+H_4)P_2$$
$$+(a_{73}r^4+a_{74}r^{-5}+H_5)P_4 \quad (4\text{-}41)$$

where

$$H_3=b_1r^{-16}+b_2r^{-15}+b_3r^{-14}+b_4r^{-13}+b_5r^{-12}+b_6r^{-11}+b_7r^{-10}$$
$$+b_8r^{-9}+b_9r^{-8}+b_{10}r^{-7}+b_{11}r^{-6}+b_{12}r^{-5}+b_{13}r^{-4}+b_{14}r^{-3}+b_{15}r^{-2}$$
$$+b_{16}log(r)r^{-1}+b_{17}log(r)+b_{18}r+b_{19}r^2+b_{20}r^3+b_{21}r^4+b_{22}r^5$$
$$+b_{23}r^6+b_{24}r^7 \quad (4\text{-}42\text{ a})$$

$$H_4=b_{25}r^{-16}+b_{26}r^{-15}+b_{27}r^{-14}+b_{28}r^{-13}+b_{29}r^{-12}+b_{30}r^{-11}+b_{31}r^{-10}$$
$$+b_{32}r^{-9}+b_{33}r^{-8}+b_{34}r^{-7}+b_{35}r^{-6}+b_{36}r^{-5}+b_{37}r^{-4}+b_{38}log(r)r^{-3}$$
$$+b_{39}r^{-2}+b_{40}r^{-1}+b_{41}+b_{42}r+b_{43}log(r)r^2+b_{44}r^3+b_{45}r^4+b_{46}r^5$$
$$+b_{47}r^7 \quad (4\text{-}42\text{b})$$

$$H_5=b_{48}r^{-16}+b_{49}r^{-15}+b_{50}r^{-14}+b_{51}r^{-13}+b_{52}r^{-12}+b_{53}r^{-11}+b_{54}r^{-10}$$
$$+b_{55}r^{-9}+b_{56}r^{-8}+b_{57}r^{-7}+b_{58}r^{-6}+b_{59}log(r)r^{-5}+b_{60}r^{-4}+b_{61}r^{-3}$$
$$+b_{62}r^{-2}+b_{63}r^{-1}+b_{64}+b_{65}r+b_{66}r^2+b_{67}r^5+b_{68}r^7 \quad (4\text{-}42\text{ c})$$

The parameters a's, and b's are determined by using the boundary conditions (4-38); App. A.

5. Discussion

In the present work the non-isothermal steady state shear flow of an incompressible Oldroyd-B fluid in the annular region between two concentric spheres R_1 and R_2 ($R_2 \succ R_1$) is investigated theoretically. The inner sphere rotates with an angular velocity ω about z-axis which passes through the center of the spheres and the outer sphere is kept at rest. The two spherical shells R_1 and R_2 are maintained at fixed temperatures T_1 and T_2 ($T_1 > T_2$); respectively. The viscoelasticity of the fluid is assumed to dominate the inertia such that the latter can be neglected. Using the constitutive equation of the non-isothermal Oldroyd-B fluid, an approximate analytical solution of the energy and momentum equations is obtained through expanding the dynamical variables in power series of Naham number.

In order to investigate the effect of viscosity and elasticity on the fluid rheology, the parameters of two test fluids are used. Boger fluid is considered as one of them. This fluid first described in details by Boger and co-workers (2000). It consists of 0.05% solution of monodisperse polystyrene (PS) with a polydispersities of 1.05 and mass average molecular weights of 6.5×106 g/mol. The parameters related to this fluid are η_p=12 Pa·s, η_s=34 Pa·s, λ_0= 17.7 s, k=0.11 W/m·k, T_0=298 K and δ=68, (Boger, 2000; Pothstein and McKinley, 1977). The resulting solution falls into a class of fluids that are highly elastic. The large relaxation time and large viscosity of the fluid eliminates its inertial effects and also permits the study of viscoelastic flow at high Deborah numbers. Moreover, we notice from the experimental point of view, that this fluid has a constant viscosity and first normal stress and zero second normal stress (obeys Oldroyd-B fluid). This fluid will be assigned as (Fluid I) in the present analysis. We will choose the second fluid with a relatively small relaxation time to show the effect of elasticity on its motion. This fluid is 0.2 wt% aqueous solution of PAA-water solution (E-10 Allied Colloids (UK) Ltd), (Nakamura et al., 1995). The parameters related to this fluid are η_p=1.3125 Pa·s,

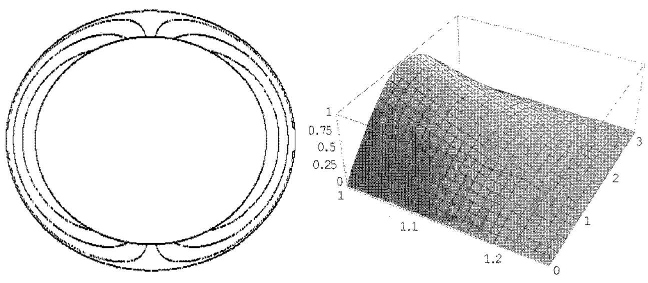

Fig. 1. Fluid I The velocity $V^{(0)}_{\varphi}$, a=1.25 in ρz- plane and in 3-dim. configuration.

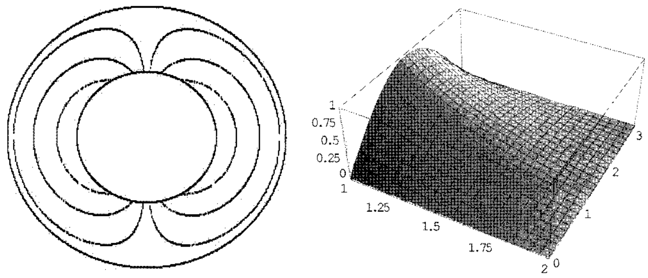

Fig. 2. Fluid I The velocity $V_\varphi^{(0)}$, a=2 in ρz- plane and in 3-dim. configuration.

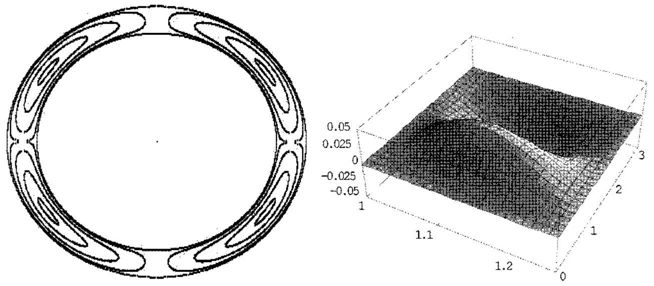

Fig. 3. Fluid I, The stream function $\Psi^{(1)}$, a=1.25 in ρz- plane and in 3- dim. configuration.

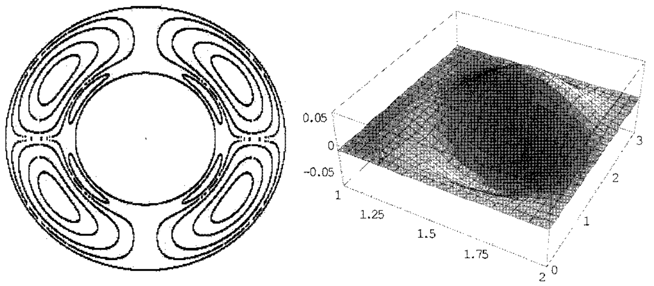

Fig. 4. Fluid I, The stream function $\Psi^{(1)}$, a=2 in ρz- plane and in 3- dim. configuration.

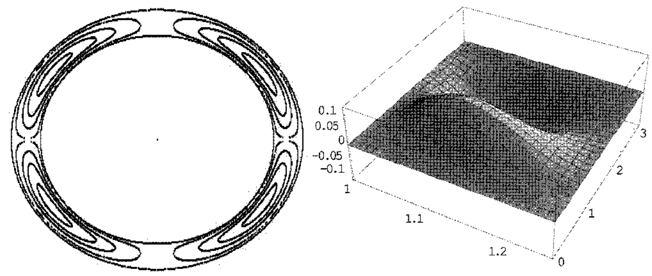

Fig. 5. Fluid II, The stream function $\Psi^{(1)}$, a=1.25 in ρz- plane and in 3- dim. configuration.

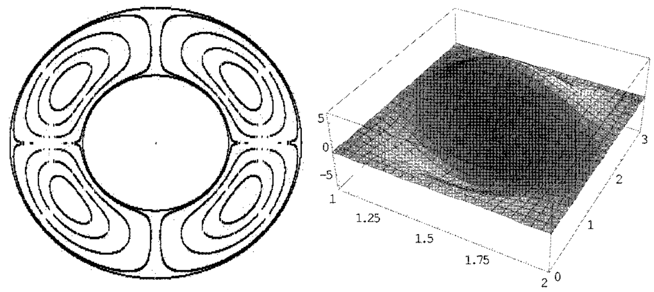

Fig. 6. Fluid II, The stream function $\Psi^{(1)}$, a=2 in ρz- plane and in 3- dim. configuration.

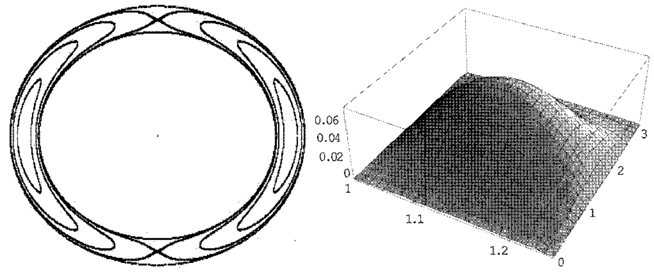

Fig. 7. Fluid I. The temperature field $\Theta^{(1)}$, a = 1.25 in ρz- plane and in 3- dim. configuration.

η_s=0.1875 Pa·s, λ_0= 2 s, k=0.2 W/m·k, T_0=298 K, and δ has chosen as δ=30. This fluid will be assigned as (Fluid II).

In zero order solution, the Newtonian field $V_\varphi^{(0)}(r,\vartheta)$,Eq. (4-23), is independent of the parameters of the fluid which means that this velocity is being the same as for all types of fluids (Ferry, 1980; Kearsley, 1962). The solution $V_\varphi^{(0)}$ as a function of r and ϑ in ρz- plane and in 3-dim configuration are shown in Figs. (1) and (2) in case of a=1.25, and a=2; respectively.

These figures show that, the geometrical ratio "a" doesn't effect the general behavior of the velocity field in zero order approximation. Moreover, there is no secondary flow, *i.e.* $\Psi^{(0)}(r,\vartheta)=0$. For the temperature equation the solution shows that the fluid behaves like a solid conductor in which the temperature profile is in the redial direction.

For the first order approximation, the solution for the stream lines given in Eq. (4-32) ; *i.e.* $\Psi^{(1)}(r,\vartheta)$=const. in ρz- plane as well as in 3-dim. configuration for a=1.25, and a=2, of the two fluids are shown in Figs. (3) to (6). It is well known that, this secondary flow is a normal stress-induced phenomena. The flow field of the stream function $\Psi^{(1)}$ divides the annular region between the two spheres into four similar parts. In fact, the fluid moves toward the inner sphere near the equator and away from it near the axis of rotation for the two fluids in case of a=2 as shown in Figs. (4) and (6) but the test fluids change their directions to the opposite side in the case of small gap widths, Figs. (3) and (5). In the same manner we notice that the stream function decreases as the gap width between the two spheres decreases and then change its direction depending on the value of 'a'. Also from Fig. (4), an appearance of another loops in the vicinity of the inner sphere that move in opposite direction of their behind loops. The existence of these loops can be attributed to the highly relaxation time value of Boger fluid. It is more interested to notice that the change of direction of the secondary flow is due to the effect of the temperature gradient in the zero order solution (Hassan, 2007b), which is not

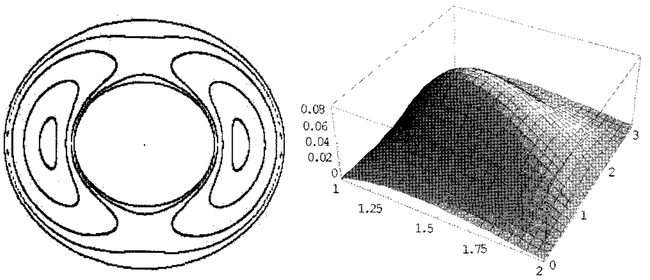

Fig. 8. Fluid I. The temperature field $\Theta^{(1)}$, a=2 in ρz- plane and in 3- dim. configuration.

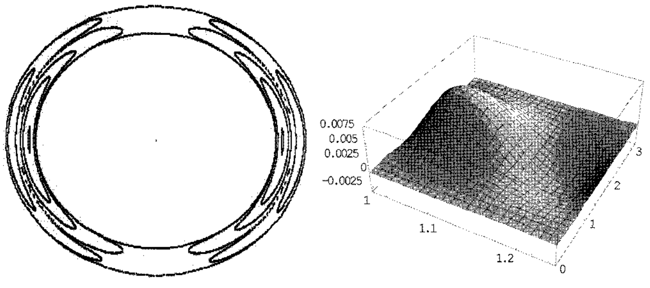

Fig. 9. Fluid I. The velocity field $V_\varphi^{(0)}$, a=1.25 in ρz- plane and in 3- dim. configuration.

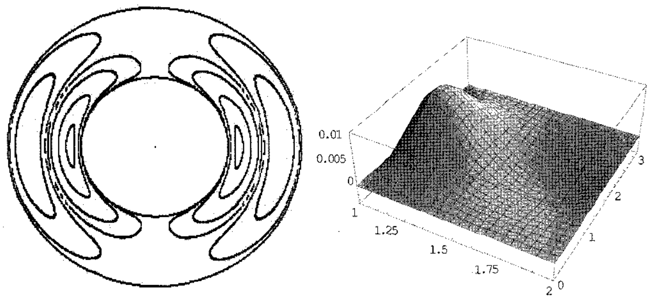

Fig. 10. Fluid I. The velocity field $V_\varphi^{(0)}$, a=2 in ρz- plane and in 3- dim. configuration.

observed in case of non-external temperature source.

The First order temperature distribution $\Theta^{(1)}(r,\vartheta)$ as function of r and ϑ in ρz- plane and in 3-dim. configuration, for a=1.25, and a=2 of the Fluid I are shown; respectively in Figs. (7) and (8).

The first order temperature distribution of the two fluids is the same. This solution is an effect of the viscous heating. Moreover, The temperature due to external sources is delivered in the zero order solution only while the solutions in higher orders are suffering from this effect beside viscous heating. The large value of the contours $\Theta^{(1)}$=const. appear at the equator such that the temperature decreases as moving towards the spherical boundaries. It seams like sources of temperature at that points and flow away from them. The generation of the heat in this order of approximation tends to be near the inner sphere. In addition, the geometrical ratio 'a' doesn't effect the general behavior of the temperature distribution.

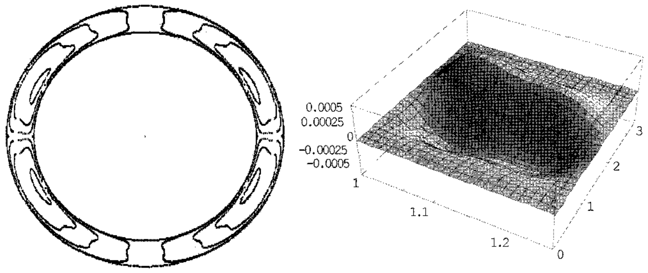

Fig. 11. Fluid I, The stream function $\Psi^{(2)}$, a=1.25 in ρz- plane and in 3- dim. configuration.

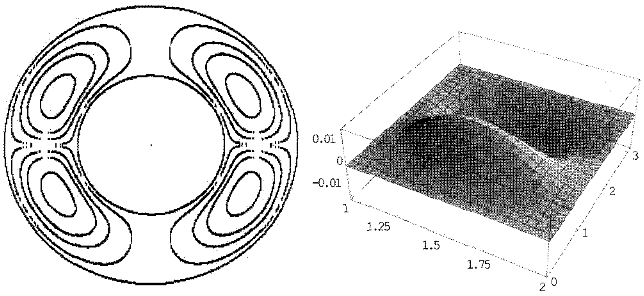

Fig. 12. Fluid I, The stream function $\Psi^{(2)}$, a=2 in ρz- plane and in 3- dim. configuration.

The first order solution of the axial velocity $V_\varphi^{(1)}(r,\vartheta)$ as a function of r and ϑ in ρz- plane and in 3-dim. configuration for Fluid I in case of a=1.25, and a=2, are shown in Figs. (9) and (10).

Also in this case, the distribution for the two cases is the same as for the test Fluid II. This solution which doesn't appear in isothermal case is an effect of presence of viscous heating. The contribution of the external source appears in the velocity value and not in its distribution. As shown in these figures, the velocity $V_\varphi^{(1)}(r,\vartheta$ divides the gap width between the two spheres into two similar parts. The eddy loops that found in the vicinity of the inner sphere move in the same direction as the primary velocity, but that in the nearest of the outer sphere move in the opposite side. Hence, there is a fluid-stagnant layer in this order of approximation between these two kind of loops with zero velocity, *i.e.* a stationary layer. The maxima of the velocity is being at the center of these two eddy loops on the equator and the velocity slow down in a direction far away from that center tends to zero at the stationary layer as well as on the two spherical boundaries. Moreover, the velocity distribution in this case is independent of the geometrical ratio 'a'.

In the second order approximation, the solution for the stream function $\Psi^{(2)}(r,\vartheta)$ is presented. The stream lines $\Psi^{(2)}$=const. in ρz- plane and in 3-dim. configuration, in case of a=1.25, and a=2 for the two test fluids are shown in Figs. (11) to (14).

At present, these figures show that the situation is mainly different. For Fluid I, the flow field is in the opposite of

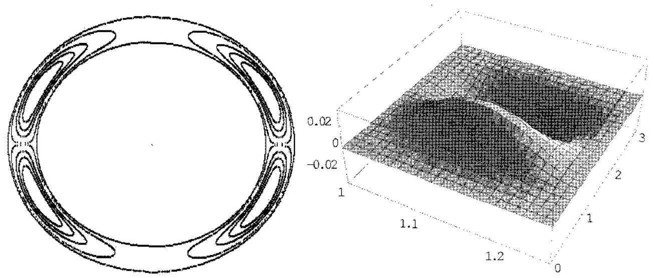

Fig. 13. Fluid II, The stream function $\Psi^{(1)}$, a=1.25 in ρz- plane and in 3- dim. configuration.

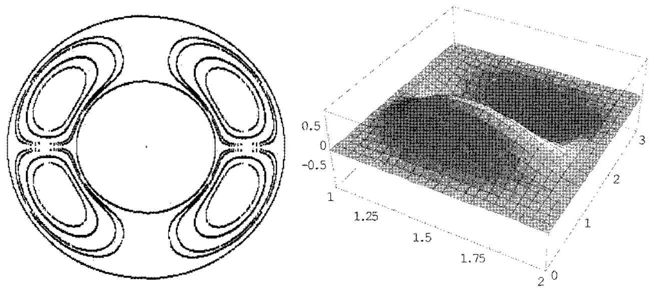

Fig. 14. Fluid II, The stream function $\Psi^{(2)}$, a=2 in ρz- plane and in 3- dim. configuration.

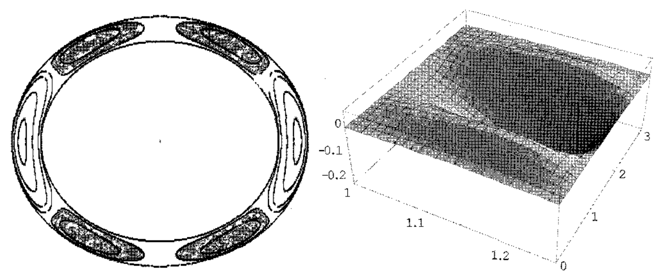

Fig. 15. Fluid I. The temperature field $\Theta^{(2)}$, a=1.25 in ρz- plane and in 3- dim. configuration.

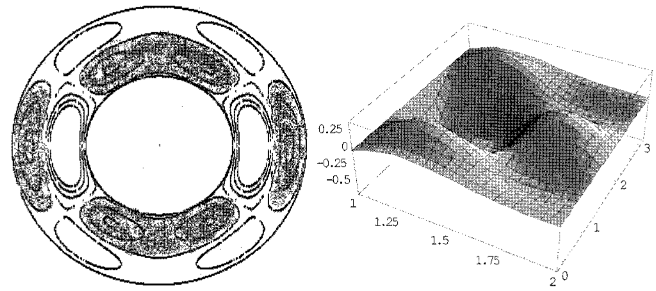

Fig. 16. Fluid I. The temperature field $\Theta^{(2)}$, a=2 in ρz- plane and in 3- dim. configuration.

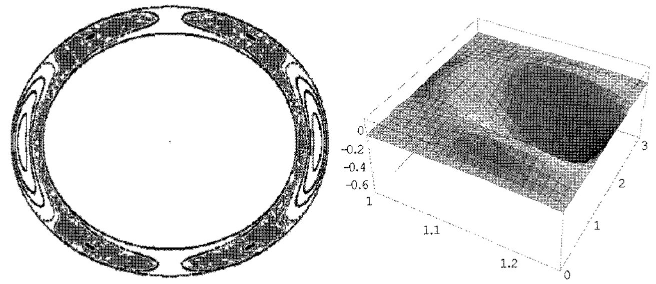

Fig. 17. Fluid II. The temperature field $\Theta^{(2)}$, a=1.25 in ρz- plane and in 3- dim. configuration.

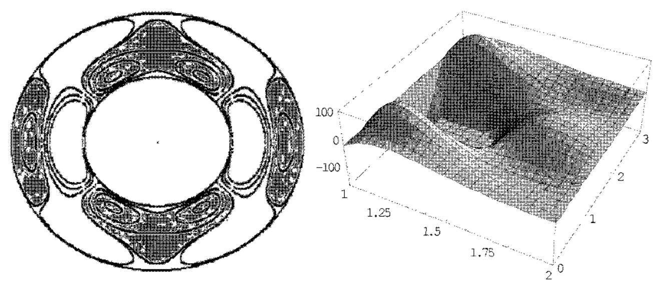

Fig. 18. Fluid II. The temperature field $\Theta^{(2)}$, a=2 in ρz- plane and in 3- dim. configuration.

that in the first order and there is an appearance of a dramatic change in the flow field in the small gap which dies for large 'a'. This means that the gap width has an effect in the motion of this fluid. For Fluid II, the flow is in the opposite direction of that in the cases of isothermal or in presence of viscous heating. Hence, the gap width doesn't effect the flow regime.

As a result of this analysis we can not expect an exact motion or a general behavior of the Oldroyd-B fluid in the presence of external heat source. This is attributed to the effect of temperature dependence of its relaxation time.

Finally, the second order temperature solution is presented. The distribution $\Theta^{(2)}(r,\vartheta)$ in ρz- plane and in 3-dim. configuration in case of a=1.25, and a=2 for the two considered fluids are shown in Figs. (15) to (18); respectively.

As shown in these figures, the temperature distribution depend on the gap width 'a' between the two spheres. Moreover, there exist a thermal distribution of positive (the shaded regions) and negative values depend on their positions in the domain region between the two spheres. The appearance of the negative value can be attributed to the expansion used in this solution.

6. Conclusion

The present work is concerned with non-isothermal spherical Couette flow of Oldroyd-B fluid in the annular region between two concentric spheres. The inner sphere rotates with a constant angular velocity while the outer sphere is kept at rest. The two spherical boundaries are maintained at different temperature values. Hence, the fluid is effected by two temperature sources; namely, viscous heating and external one. Using successive approximate method, a solution is obtained through the expansion of the dynamical fields in power series of Nahme number. Two viscoelastic fluids with different rheological properties are considered. The analysis show that a primary flow in zero order solution is independent of the fluid parameters, so it is the same for all fluids. In the same order, the solution for the extra temperature is also appears while the solution for viscous heating emerges in higher orders. In first order solution the direction of the secondary flow depend on the gap width between the two spheres. The axial velocity and temperature profile are the same for the two test fluids. The situation is mainly different in second order solution in which the secondary flow has no fixed behavior due to the complex dependence of the relaxation time on temperature. Also the temperature distribution depends on the fluid parameters as well as the gap width between the two spheres. Moreover, there is apparent regions of positive and negative values, the negative distribution seams like a correction for the first order temperature.

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