Bull. Korean Math. Soc. **44** (2007), No. 4, pp. 677–682

REPRESENTATION OF SOME BINOMIAL COEFFICIENTS BY POLYNOMIALS

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ABSTRACT. The unique positive zero of $F_m(z) := z^{2m} - z^{m+1} - z^{m-1} - 1$ leads to analogues of $2\binom{2n}{k}$ (k even) by using hypergeometric functions. The minimal polynomials of these analogues are related to Chebyshev polynomials, and the minimal polynomial of an analogue of $2\binom{2n}{k}$ (k even > 2) can be computed by using an analogue of $2\binom{2n}{2}$. In this paper we show that the analogue of $2\binom{2n}{2}$ is the only real zero of its minimal polynomial, and has a different representation, by using a polynomial of smaller degree than $F_m(z)$.

1. Introduction

To what extent can a sum and its factorization both be known? More precisely, if A and B belong to a ring, to what extent can we simultaneously know the factorizations of A, B and C where $A + B = C$? There is an inverse problem: Given C, find A and B with factorizations of a specific type such that $A + B = C$. Cases in which the complete factorizations of each of A, B and C are known we refer to as cases of "complete" information. For various results and examples in case of polynomials about this, see [1].

An example of complete information is that

$$\phi(q) = \prod_{k=1}^{\infty} \left(\frac{1 + q^{2k-1}}{1 - q^{2k-1}} \right)^2$$

satisfies

$$\left(\phi(q^2)\right)^2 = \frac{1}{2} \left(\phi(q) + \frac{1}{\phi(q)} \right).$$

A simpler result of this nature is

$$e^{iz} + e^{-iz} = 2 \cos z. \tag{1.1}$$

Received July 27, 2006.

2000 *Mathematics Subject Classification.* Primary 11B65; Secondary 05A10.

Key words and phrases. binomial coefficients, analogues, minimal polynomial, Chebyshev polynomial.

This Research was supported by the Sookmyung Women's University Research Grants 1-0703-0178.

Here each summand on the left has no zeros, while the right is

$$2\prod_{n=0}^{\infty}\left(1-\frac{4}{\pi^2}\left(\frac{z}{2n+1}\right)^2\right),$$

which has infinitely many zeros. The sum in (1.1) and its companion for the sine function leads us to consider sums such as

$$\left(1+\frac{z}{n}\right)^n \pm \left(1-\frac{z}{n}\right)^n,$$

since $e^z = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n$. Upon rescaling z this suggests the study of determining the factorization of polynomials of the form $f(z+1) \pm f(z-1)$. The simplest form of this seems to be

$$(z+1)^n + (z-1)^n. \tag{1.2}$$

It is natural to ask a question how well the coefficients of (1.2) can be represented by some polynomials? As an answer of this, Author [2] introduced a new analogue (not a $q-$analogue) of the doubled binomial coefficient $2\binom{2n}{k}$ (k even) by using a unique positive zero r_m of $F_m(z) = z^{2m} - z^{m+1} - z^{m-1} - 1$. Here the modulus of r_m is the biggest among all zeros of $F_m(z)$, and, as $m \to \infty$,

$$\begin{cases} r_m \to 1, \\ r_m^m \to 1+\sqrt{2}. \end{cases}$$

For a positive even integer k ($\leq n$), author [2] defined the (m,k)-analogue of $2\binom{2n}{k}$ by

$$\begin{aligned} a(m,k) &= 2\binom{n}{\frac{k}{2}}{}_2F_1\left(-\frac{k}{2}, -\frac{1}{2}(2n-k); \frac{1}{2}; \frac{1}{4}\left(r_m + r_m^{-1}\right)^2\right) \\ &= 2\binom{2n}{k} + f(m,k), \text{ say} \end{aligned} \tag{1.3}$$

by using hypergeometric functions. Here, as $m \to \infty$, we have $f(m,k) \to 0$. Author [2] defined $P_{m,n,k}(x)$ to be the minimal polynomial of $a(m,k)$ that is related to Chebyshev polynomials. In fact, author [2] first studied the case $k = 2$ and showed how to compute the minimal polynomial of an analogue of $2\binom{2n}{k}$ ($k > 2$) by using this. For the case $k = 2$, author [2] proved

Theorem 1.1. *Let u be an integer ≥ 1. Define*

$$W_{u,n}(x) := 4(n(n-1))^{2u+1}(-1+x^2)U_{u-1}(x)U_u(x),$$
$$Y_{u,n}(x) := 2(n(n-1))^{2u-1}(-1+x)U_{u-1}^2(x),$$

where $U_u(x)$ is the Chebyshev polynomial of the second kind of degree u. Then the polynomials $P_{2u+1,n,2}(x)$ and $P_{2u,n,2}(x)$ divide the integral polynomials

$$A_{2u+1,n}(x) := W_{u,n}\left(\frac{x}{2n(n-1)} - \frac{n}{n-1}\right) - 4(n(n-1))^{2u+1}, \tag{1.4}$$

and

$$B_{2u,n}(x) := Y_{u,n}\left(\frac{x}{2n(n-1)} - \frac{n}{n-1}\right) - (n(n-1))^{2u-1}, \tag{1.5}$$

respectively.

The main purpose of this paper is to show that $a(m,2)$, the analogue of $2\binom{2n}{2}$, is the only real zero of its minimal polynomial, and has a different representation, by using a polynomial of smaller degree than $F_m(z)$. While showing this, we will see zero distributions of some interesting polynomials. In addition, many related applications about the topics in this paper can be seen in [3].

2. Results and proofs

In this section we show that

$$a(m,2) = 2n^2 + n(n-1)(r_m^2 + r_m^{-2})$$

in (1.3) is the only real zero of $P_{m,n,2}$ and obtain another representation of $a(m,2)$.

Theorem 2.1. *Let m be an integer ≥ 2. Then $a(m,2)$ is the only real zero of $P_{m,n,2}$. Moreover, for any positive integer u,*

$$a(2u+1,2) = 2n^2 + n(n-1)(g_u + g_u^{-1}),$$
$$a(2u,2) = 2n^2 + n(n-1)(h_u + h_u^{-1}),$$

where g_u is the (positive) real zero of $z^{2u+1} - z^{u+1} - z^u - 1 = 0$, and h_u is the real zero > 1 of $\frac{z^{4u} - z^{2u+1} - 4z^{2u} - z^{2u-1} + 1}{(z+1)^2} = 0$.

To prove this, we will investigate zero distributions of some polynomials. We first consider the case $m = 2u+1$.

For the proof of following Lemma, we will need the theorem of Cauchy (for the proof of this, see p. 122 of [4]).

Theorem 2.2 (Cauchy). *All the zeros of the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_n \neq 0$, lie in the circle $|x| \leq r$, where r is the positive zero of the equation*

$$-|a_n|x^n + |a_{n-1}|x^{n-1} + \cdots + |a_1|x + |a_0| = 0.$$

Lemma 2.3. *Let u be an integer ≥ 1. The polynomial*

$$G_u(z) = z^{2u+1} - z^{u+1} - z^u - 1$$

has only one real zero > 1 and this is a zero of maximum modulus.

Proof. By Descartes' rule of signs, $G_u(z)$ has at most one positive zero. Since simple calculations yield $G_u(1) < 0$ and $G_u'(z) > 0$ on $(1, \infty)$, we have only one positive zero $g_u > 1$. Now we show that $G_u(z)$ has no negative zero. Suppose that u is even. If $-1 \le z < 0$, then

$$G_u(z) = z^{2u+1} - 1 - z^u(z+1) < 0. \tag{2.1}$$

If $z < -1$ is a zero of $G_u(z)$, it contradicts the equality

$$z^{u+1} = \frac{z^u + 1}{z^u - 1}, \tag{2.2}$$

since its left side is negative and its right side is positive. Suppose that u is odd. If $-1 < z < 0$, then it contradicts (2.2), since its left side is positive and its right side is negative. If $z \le -1$, then, by (2.1), $G_u(z) < 0$. Hence there are no negative zeros of $G_u(z)$. Moreover, by Theorem 2.2, all zeros of $G_u(z)$ lie in the disk $|z| \le g_u$. □

We use Lemma 2.3 to prove the following proposition.

Proposition 2.4. *Let u be an integer ≥ 1. The polynomial*

$$\bar{G}_u(y) = (-1 + y^2)U_{u-1}(y)U_u(y) - 1$$

has only one real zero $1/2(g_u + 1/g_u)$, where g_u is the real (positive) zero of $G_u(z) = z^{2u+1} - z^{u+1} - z^u - 1$.

Proof. The formula

$$U_u(y) = \frac{(y + \sqrt{y^2-1})^{u+1} - (y - \sqrt{y^2-1})^{u+1}}{2\sqrt{y^2-1}}, \tag{2.3}$$

yields

$$z^{2u+1}\bar{G}_u(y) = \frac{1}{4}(z^{2u+1} - z^{u+1} - z^u - 1)(z^{2u+1} + z^{u+1} + z^u - 1), \tag{2.4}$$

where

$$z = y + \sqrt{y^2 - 1}, \text{ i.e., } y = \frac{1}{2}\left(z + \frac{1}{z}\right). \tag{2.5}$$

In (2.4), $G_u(z) = z^{2u+1} - z^{u+1} - z^u - 1$ is the reciprocal polynomial of $-(z^{2u+1} + z^{u+1} + z^u - 1)$, and, by (2.5), y is real when $z = g_u$. Next we show that y is nonreal when z is a nonreal zero of $G_u(z)$. We observe that, for $z = a + ib\,(b \ne 0)$, we have

$$\Im\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right) = \frac{b}{2}\left(\frac{a^2 + b^2 - 1}{a^2 + b^2}\right). \tag{2.6}$$

So in order to show that the left side of (2.6) is not equal to zero, we need to show that the polynomial $G_u(z)$ has no zero on the unit circle. Suppose that z_0 is a zero of $G_u(z) = 0$ and $|z_0| = 1$. Then we have $|z_0^u - 1| = |z_0^u + 1|$, so z_0^u must lie on the imaginary axis and $z_0^u = i$ or $-i$. Then $z_0 = -1$. In fact, if $z_0^u = i$, then

$$0 = G_u(z_0) = -z_0 - iz_0 - i - 1 = -(z_0 + 1)(i + 1),$$

and, if $z_0^u = -i$, then, by a similar calculation, $0 = -(z_0 + 1)(i + 1)$. But $G_u(-1) \neq 0$ which leads a contradiction. Hence we conclude that y is nonreal when z is a nonreal zero of $G_u(z)$. Since $G_u(z)$ has only one real (positive) zero, it follows from (2.4) and (2.5) that the result holds. □

Next, we consider the case $m = 2u$ of Theorem 2.1.

Lemma 2.5. *Let u be an integer ≥ 1. The integral polynomial*

$$H_u(z) = \frac{z^{4u} - z^{2u+1} - 4z^{2u} - z^{2u-1} + 1}{(z+1)^2}$$

has only two real (positive) zeros h_u and $1/h_u$ for some $h_u > 1$, and no zero on the unit circle.

Proof. By Descartes' rule of signs, $(z + 1)^2 H_u(z)$ has at most two positive zeros. Since $(z + 1)^2 H_u(z)$ is self-reciprocal and $H_u(1) < 0$, $(z + 1)^2 H_u(z)$ has exactly two positive zeros. Use Descartes' rule of signs to deduce that $(z + 1)^2 H_u(z)$ has either 0 or 2 negative zeros counting multiplicity. Since -1 is a zero with multiplicity 2, we conclude that H_u has no negative zeros. Also one can use the triangle inequality for complex numbers to deduce immediately that $(z + 1)^2 H_u(z)$ can only have zeros on the unit circle that are real. More specifically, we use that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

with equality if and only if all nonzero z_j have the same argument. Letting $z_1 = (z + 1)^2 H_u(z)$, $z_2 = -z^{4u}$, $z_3 = z^{2u+1}$, $z_4 = z^{2u-1}$ and $z_5 = -1$ (and consider what it means if z^{2u+1} and z^{2u-1} have the same argument) completes the proof. □

We use Lemma 2.5 to prove the following proposition.

Proposition 2.6. *Let u be an integer ≥ 1. The polynomial*

$$\bar{H}_u(y) = 2(-1 + y)U_{u-1}^2(y) - 1$$

has only one real zero $1/2(h_u + 1/h_u)$, where h_u is the real zero > 1 of $(z + 1)^2 H_u(z) = z^{4u} - z^{2u+1} - 4z^{2u} - z^{2u-1} + 1$.

Proof. By using (2.3), we calculate

$$z^{2u+1}(z+1)^2 \bar{H}_u(y) = z^{4u} - z^{2u+1} - 4z^{2u} - z^{2u-1} + 1, \tag{2.7}$$

where

$$z = y + \sqrt{y^2 - 1}, \text{ i.e., } y = \frac{1}{2}\left(z + \frac{1}{z}\right). \tag{2.8}$$

We see that the polynomial $(z + 1)^2 H_u(z) = z^{4u} - z^{2u+1} - 4z^{2u} - z^{2u-1} + 1$ is self-reciprocal, and, by Lemma 2.5, $H_u(z)$ has only two real zeros h_u and $1/h_u$ for some $h_u > 1$. By (2.8), y is real when $z = h_u$ or $z = 1/h_u$. Next we show that y is nonreal when z is a nonreal zero of $(z + 1)^2 H_u(z)$. By (2.6), we only need to show that the polynomial $(z+1)^2 H_u(z)$ has no zero on the unit circle. But this is true by Lemma 2.5. Hence we conclude that y is nonreal when z is a nonreal zero of $(z+1)^2 H_u(z)$. Since $H_u(z)$ has only one real (positive) zero, the result holds. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let

$$y = \frac{x}{2n(n-1)} - \frac{n}{n-1}. \tag{2.9}$$

Then, by (1.4) and (1.5),

$$A_{2u+1,n}(x) = 4(n(n-1))^{2u+1}\bar{G}_u(y),$$
$$B_{2u,n}(x) = (n(n-1))^{2u+1}\bar{H}_u(y),$$

where $\bar{G}_u(y) = (-1+y^2)U_{u-1}(y)U_u(y) - 1$ and $\bar{H}_u(y) = 2(-1+y)U_{u-1}^2(y) - 1$. By (2.9), $x = 2n((n-1)y+n)$. So, by Proposition 2.4 and Proposition 2.6, we have

$$\begin{aligned} &A_{2u+1,n}\left(2n((n-1)1/2(g_u + g_u^{-1}) + n)\right) \\ =&A_{2u+1,n}\left(2n^2 + n(n-1)(g_u + g_u^{-1})\right) = 0 \end{aligned}$$

and

$$\begin{aligned} &B_{2u,n}\left(2n((n-1)1/2(h_u + h_u^{-1}) + n)\right) \\ =&B_{2u,n}\left(2n^2 + n(n-1)(h_u + h_u^{-1})\right) = 0, \end{aligned}$$

where g_u is the real (positive) zero of $z^{2u+1} - z^{u+1} - z^u - 1 = 0$ and h_u is the real zero > 1 of $z^{4u} - z^{2u+1} - 4z^{2u} - z^{2u-1} + 1 = 0$. Here $1/2(g_u + g_u^{-1})$ and $1/2(h_u + h_u^{-1})$ are the only real zeros of $\bar{G}_u(y)$ and $\bar{H}_u(y)$, respectively. Since $P_{2u+1,n,2}$ and $P_{2u,n,2}$ divide $A_{2u+1,n}$ and $B_{2u,n}$, respectively, the result follows. □

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