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ON A FAMILY OF BALANCED GROUPS

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Abstract. A family of balanced groups is introduced. We describe some geometric approach to find these groups in terms of the (orientable) closed 3-manifolds and its fundamental groups.

1. Introduction

Given a group presentation $G = \langle X|R\rangle$, where X is a set of generators and R relators, we may have a question: *if G is geometric.* In other words, if the canonical 2-dimensional complex associated to G is *a spine* of a closed connected orientable 3-manifold. In the case of G geometric we can get more information on the structure of the group G. For this problem we usually investigate whether G can be realized as the fundamental group of a manifold. As is well known, for $n \geq 4$ every finitely presented group $G = \langle X|R\rangle$ is the fundamental group of n-dimensional manifold ([2]). However, it is very difficult (generally impossible) to find such groups in the case of $n = 3$ ([9], [11]). In recent years Helling, Kim and Mennicke showed a very interesting example, *A geometric study of Fibonacci groups* ([5]). They constructed a family of 2-dimensional complexes on the 2-sphere S^2, whose fundamental groups are isomorphic to $F(2, 2n)$ and stated that these groups are "the link between certain objects in 3-dimensional topology, in 3-dimensional hyperbolic geometry, in the theory of discontinuous transformation groups; in rank one Lie groups". The similar works can be found in the various papers and its references ([1], [4], [5], [9], [12]). In this paper we study a family of balanced groups with equal number of generators and relators, denoted by

$$G(n) = \langle x_1, x_2, \ldots, x_n, y \mid x_1x_2\cdots x_n = 1, yx_{i+1}x_i = 1, \text{indices} \mod n\rangle,$$

which are arising from the identification of opposite faces of a polyhedron. The polyhedron consists of n-gons ($n \geq 3$) in the top and bottom faces, and of $2n$ triangles in the side faces. In particular, for $n = 3$ we have a combinatorial octahedral space which is the manifold M_3 obtained by opposite face identification of octahedron by a right-helix turn of angle $\pi/3$, corresponding

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to a $2\pi/3$ rotation of a tetrahedron. This manifold is a 2-fold covering of the truncated-cube space $S^3/\langle 432\rangle$, where $\langle 432\rangle$ is the binary octahedral group of order 48 ([8], p.120). The recognition problem of 3-manifold groups is of much interesting in both algebraic topology and combinatorial group theory. The main theorem of this paper is to show a family of balanced groups $G(n)$ is coincided with the fundamental groups of some closed orientable 3-dimensional manifolds, and is to find its some topological properties related to these groups.

2. The balanced groups $G(n)$

We shall consider the following group presentations

$$G(n) = \langle x_1, x_2, \ldots, x_n, y \mid R_0, R_i, n \geq 3\rangle,$$

where $R_0 = x_1x_2\cdots x_n, R_i = yx_{i+1}x_i$ (indices mod n). According to the definition of $G(n)$, we have $x_1x_2x_3\cdots x_n = 1$. $y = x_n^{-1}x_1^{-1}$, $y = x_i^{-1}x_{i+1}^{-1}, 1 \leq i \leq n-1, n \geq 3$. So, $x_i^{-1}x_{i+1}^{-1} = x_1^{-1}x_2^{-1}$, or equivalently $x_{i+1}x_i = x_2x_1$, whence $x_{2t} = (x_2x_1)^{t-1}x_2(x_2x_1)^{-(t-1)}, x_{2t+1} = (x_2x_1)^t x_2^{-1}(x_2x_1)^{-(t-1)}$ for $t = 1, 2, 3, \ldots$. Now, the relations $x_2x_1 = x_1x_n$ and $x_1x_2x_3\cdots x_n = 1$ lead to the ones between $a := x_1, b := x_2$. At last we eventually obtain the following desirable presentations:

(P1) $G(2m) = \langle a, b \mid (ba)^m = (ab)^m, a(b^2a^2)^{m-1}b = (ba)^{m-1}\rangle$ for $m \geq 2$,

(P2) $G(2m+1) = \langle a, b \mid b(ab)^m = (ab)^m a, a(b^2a^2)^{m-1}b^2a = (ba)^{m-1}b\rangle$ for $m \geq 1$. Now we come to the point where the difference between even and odd values of n is crucial. From (P1) and (P2) it is easily seen that the derived quotients $\overline{G(2m)} = G(2m)/G'(2m)$ and $\overline{G(2m+1)} = G(2m+1)/G'(2m+1)$, written in additive form, have the following:

$$\overline{G(2m)} = Z \oplus Z_m, \qquad \overline{G(2m+1)} = Z_{2m+1}\ .$$

Theorem 2.1. *The groups $G(n)$ are infinite for all $n = 2m$ $(m \geq 2)$. There exist an isomorphism ξ of $G(3)$ to the semidirect product $*$ of the quaternion group Q_8 by the cyclic group of order 3, Z_3.*

Proof. Since $\overline{G(2m)} = Z \oplus Z_m$, for every $n = 2m$, $m \geq 2$, $G(n)$ is infinite. Secondly, by definition

$$G(3) = \langle a, b | aba = bab,\ ab^2a = b\rangle$$

is a balanced group. An obvious epimorphism $\xi : G(3) \longrightarrow SL_2(3) \cong Q_8 * Z_3$ such that

$$\xi(a) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \xi(b) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

force to think of the kernel ξ_{Ker} of ξ . Certainly, it has been known for a long time that the kernel ξ_{Ker} is trivial. Anyhow we may consider an implication $\{aba = bab, ab^2a = b\} \Rightarrow \{aba^2 = b^2ab, bab^2 = a^2ba, ba^2b = a, b^2 = a^2ba^2, a^2 = b^2ab^2, a^2b^2a^2 = aba, b^3 = (ab)^3 = (ba)^3 = a^3, a^6 = 1\}$.

So, $G(3)$, as a set, is the following one:

$G(3) = \{1,\ a^3,\ a^2b,\ ba^2,\ aba,\ a^4ba,\ a^3ba^2,\ a^5b,\ a^2,\ a^4,\ a,\ a^5,\ b,\ ab,\ ba,\ a^3b,\ a^2ba,\ aba^2,\ a^4b,\ a^3ba,\ a^2ba^2\ ,\ a^5ba,\ a^4ba^2,\ a^5ba^2\}$. Thus $|G(3)| = 24$, and the epimorphism ξ should be an isomorphism, and we can put $Q_8 = \{1, a^3, a^2b, ba^2, aba, a^4ba, a^3ba^2, a^5b\} = \langle a^2b, ba^2\rangle, Z_3 = \{1, a^2, a^4\}$.

Remark 2.2. (i) It is an ordinary method in combinatorial group theory to investigate an epimorphism $\xi : G \to SL_2(F)$, when we like to verify that $|G| = \infty$ or at least to obtain an information on G ([7]). But this is far from being the case that such an epimorphism does exist at all. In particular $G(2m+1), m \geq 1$, do not possess this property.

(ii) The group of outer automorphisms

$$OutG = AutG/InnG,\ InnG \cong G/Z(G)$$

is an important characteristic of any group G. Certainly, the determination of $OutG(n)$ is a deep matter beyond the scope of this paper. So we only pay our attention to some special automorphisms. The group $G(n)$ has an outer automorphism

$$\psi = (1, 2, 3, \ldots, n)(n+1).$$

That is easily seen from some geometric considerations. From the former presentations (P1) and (P2) we have $\psi(a) = b$, $\psi(b) = bab^{-1}$. So it is obvious that $\psi(ba) = ba$, therefore a cyclic subgroup $\langle ba\rangle$ is fixed under the action of ψ. If we put $c = ba, b = ca^{-1}$, then the presentation of the questionable group $G(5)$ can be rewritten as follows:

$$G(5) = \langle a, c; ac^2a = c^3, caca^{-1}cac = aca\rangle.$$

So the action of the automorphism ψ of order 5 will be expressed as $\psi(c) = c, \psi(a) = ca^{-1}$. Generally, the second relation has a form

$$aca^{-1}caca^{-1}ca = c^2.$$

But applying ψ we will come to the form, indicated above.

3. Some geometric realizations of $G(n)$

In this paragraph we want to construct a 3-manifold M_n from a polyhedron P_n such that

$$\pi_1(M_n) \cong G(n).$$

For this we shall consider a combinatorial polyhedron P_n consisting of two n-gons F, F^* in the north and south hemisphere, and of $2n$ triangles F_i, F_i^* in the equator zone, shown in the figure 1.

The above polyhedron P_n has $4n$ edges, $2n+2$ faces, and $2n$ vertices. The oriented edges can be labeled in the following manner. The oriented edges fall into $n+1$ classes, the verticle edges P_iQ_i consisting of n edges and the others of 3 edges. The oriented edges in the same class carry the same label, say, $x_1, x_2, x_3, \ldots, x_n, y$. The n-gons in the north and south hemisphere have a boundary $x_1x_2\cdots x_n$ for all $n \geq 3$, and each triangle $yx_{i+1}x_i$ (indices mod n).

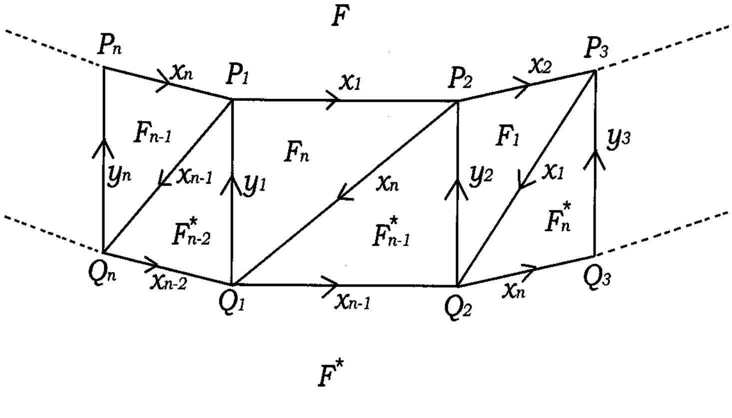


Figure 1: The polyhedral schema of the closed 3-manifold M_n

For the pair of n-gons (F, F^*) and the pairs of triangles $(F_i\ ,\ F_i^*)$ define identifications α, β. The resulting 3-dimensional complex K is an orientable pseudo manifold. In this case there is a simple criterion, due to H. Seifert and W. Threlfall [10], for K to be a manifold.

Theorem 3.1 (Seifert and Thelfall). *Let K be an orientable, closed, 3-dimensional pseudo manifold arising from a (simply connected) polyhedron by identifying pairs of faces on the boundary of K. K is a manifold if and only if its Euler characteristic vanishes.*

So we now define identifications α and β on the boundary of the above polyhedron P_n as follows:

$$\alpha : \begin{array}{ccc} F & \to & F^* \\ P_i & \to & Q_{i+2} \end{array} \qquad \beta : \begin{array}{ccc} F_i & \to & F_i^* \\ P_i & \to & P_{i+2} \\ P_{i+1} & \to & Q_{i+1} \\ Q_i & \to & Q_{i+2} \end{array}$$

(all indices mod n).

These α and β make all edges of the polyhedron labeled as follows :

$$Q_iP_i = y, P_iP_{i+1} = P_{i+2}Q_{i+1} = Q_{i+2}Q_{i+3} = x_i \text{ (all indices mod } n).$$

These identifications α and β produce a complex M_n, say, with

$$\begin{array}{rcll} \alpha^0 & = & 1, & \text{vertex} \\ \alpha^1 & = & n+1, & \text{edges} \\ \alpha^2 & = & n+1, & 2-\text{cells} \\ \alpha^3 & = & 1, & 3-\text{cell}, \end{array}$$
$$\chi(M_n) = 1 - (n+1) + (n+1) - 1 = 0.$$

Hence we can read off the fundamental groups $\pi_1(M_n)$ of M_n with generators $\{x_1, x_2, \ldots, x_n, y\}$ and relators, $\{x_1x_2\cdots x_n = 1, yx_{i+1}x_i = 1, yx_1x_n = 1, \text{indices mod } n\}$, which is coincided with the balanced groups $G(n)$. So applying theorem 3.1, we have the following

Theorem 3.2. *The complex M_n constructed above is a closed, connected, orientable 3-manifold. The fundamental group $\pi_1(M_n)$ admits the following group presentations $\pi_1(M_n) = \langle x_1, x_2, \ldots, x_n, y | x_1x_2\cdots x_n = 1, yx_{i+1}x_i = 1$, indices mod $n, n \geq 3\rangle$, which is coincided with $G(n)$, corresponding to a spine of the manifold M_n (hence it is geometric).*

Remark 3.3. (i) In the above, $G(n)$ has n distinct group presentations, denoted by $G_k(n) = \langle x_1, x_2, \ldots, x_n, y | x_1x_2\cdots x_n = 1, yx_ix_{i+k-1} = 1, 1 \leq k \leq n, \text{indices mod } n\rangle$. From these presentations a natural question arises: *whether these groups are different or not.* So we here leave the classification problems of $G_k(n)$ for all $n \geq 3$.

(ii) For $n = 3$, M_3 is the octahedral space $S^3/\langle 332\rangle$, where $\langle 332\rangle$ is the binary tetrahedral group of order 24. This space is the manifold obtained by an identification of opposite faces of an octahedron by a right-helix turn of angle $3/\pi$, which corresponds to a $2\pi/3$ rotations of a tetrahedron. This manifold M_3 is a 2-fold coverings of the truncated-cube space $S^3/\langle 432\rangle$, where $\langle 432\rangle$ is the binary octahedral group of order 48 (see: [8], p.122). The fundamental group $\pi_1(M_3)(= G_3)$ admits the following presentations

$$\pi_1(M_3) = \langle x_1, x_2, \ldots, x_n, y | x_1x_2x_3 = 1, x_ix_{i+2}y = 1, \text{indices mod } 3\rangle.$$

Let H_3 be *the split extension group* of G_3 by $Z_3 = \langle\theta; \theta^3 = 1\rangle$, where θ is the automorphism of G_3 defined by $\theta(x_i) = x_{i+1}(\bmod 3)$, and $\theta(y) = y$. Then H_3 has the following presentations:

$$\begin{aligned} H_3 &= \langle x, y, \theta \mid \theta^3 = 1, (\theta^{-1})^3 = 1, \theta^2x\theta^{-2}x = y, \theta y\theta^{-1} = y\rangle \\ &\cong \langle x, \theta \mid \theta^3 = 1, (\theta^{-1}x)^3 = 1, \theta(x\theta x) = (x\theta x)\theta\rangle \\ &\cong \langle \theta, \tau \mid \theta^3 = 1, \tau^3 = 1, \theta(\tau\theta^{-1}\tau) = (\tau\theta^{-1}\tau)\theta\rangle \end{aligned}$$

where $x_1^{-1} = x, x_{i+1}^{-1} = \theta^ix\theta^{-i}, \theta^{-1}x = \tau$. In this case the presentation $\langle\theta, \Omega | \Omega\theta = \theta\Omega\rangle$ for $\Omega = \tau\theta^{-1}\tau$ defines the fundamental group of the resulting link L which is arising from the manifold M_3, where θ, τ are meridians around its components. In fact, the link L is the 3-fold cyclic covering of the 3-sphere S^3 branched over the link L with branching index 3.

We now study some topological properties of the manifold M_n. As is well known, every closed orientable 3-manifold is a branched covering of the 3-sphere S^3 over some knots/links. So we describe the manifold M_n as the branched coverings of the 3-sphere. In the Theorem 3.2. the identification α, β determine the edges and faces identifications for the polyhedron P_n as follows: $Q_iP_i = y, P_iP_{i+1} = P_{i+2}Q_{i+1} = Q_{i+2}Q_{i+3} = x_i, P_1P_2\cdots P_n \longrightarrow Q_3Q_4\cdots Q_2$, $P_iP_{i+1}Q_i \longrightarrow P_{i+2}Q_{i+1}Q_{i+2}$(all indices mod n).

We now define an automorphism ϕ of $G(n)$ by $\phi(x_i) = x_{i+1}$(indices mod n) and $\phi(y) = y$. Denote the corresponding homeomorphism of M_n also by ϕ. As the ϕ corresponds to n-rotational symmetry of the polyhedron P_n , the $1/nP_n$ in the figure 2 is *the fundamental domain* for the quotient space $M_n/\langle\phi\rangle$. A Heegaard diagram for the quotient space $M_n/\langle\phi\rangle(\cong S^3)$ obtained from $1/nP_n$ by side Pairing of its boundary faces is shown in the Figure 3.

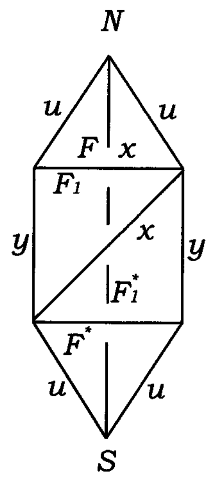


Figure 2: The $1/n$ P_n with side pairing

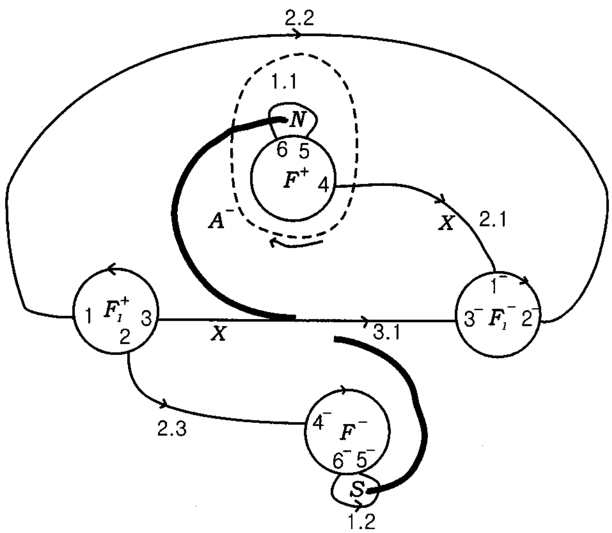


Figure 3: A Heegaard diagram of the quotient space ($\cong S^3$) obtained from $1/nP_n$

Then we apply Hilden-Lozano-Montesinos techniques for the figure-8-knot ([6], p.173) and Grunewald-Hirsch methods for link complements ([4], pp.355-362) to modify Fig.3 to Fig.6. The figure 4 is obtained from figure 3 by a simplication along the closed curve A, which surrounds the *"hole"* F^+. The figure 5 is obtained from figure 4 by canceling a *handle along* X between the "holes" F_1^+, F_1^-. The result is also a Heegaard diagram for the quotient space ($\cong S^3$) obtained from $1/nP_n$. Finally, we cancel the last handle between the "holes" A^+, A^-, and we get the pictured resulting link L in the figure 6, equivalent to the link in the figure 7 (see [3], p. 90).

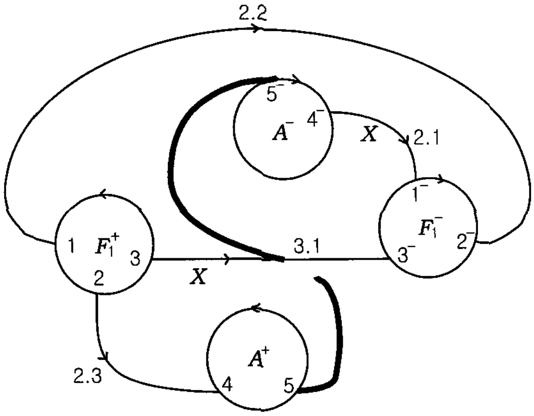


Figure 4: The simplication along the closed curve A

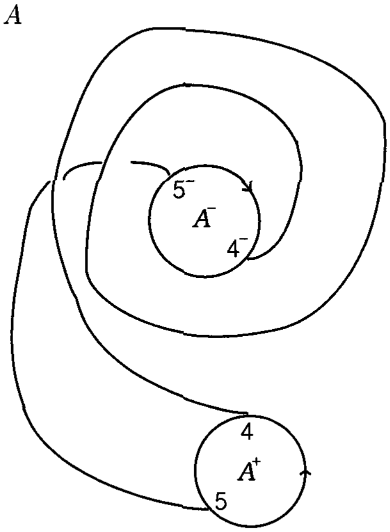
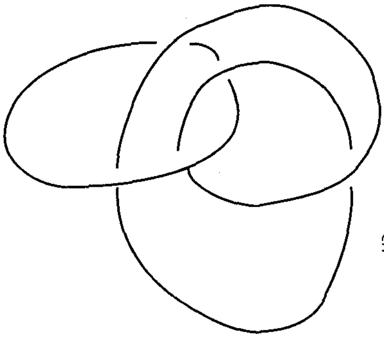


Figure 5: The canceling of a handle along X



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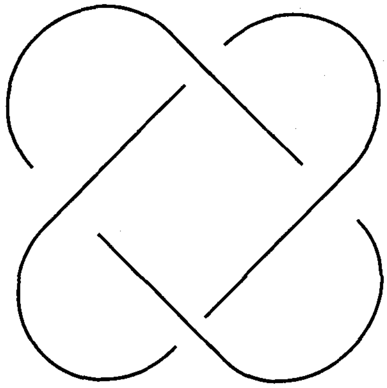

Figure 6: The resulting link L, fibering over S^3

Figure 7:

On the other hands, if we define the identifications α, β by $\alpha : F \to F^*$, $P_i \to Q_{i+n-2}$; $\beta : F_i \to F_i^*$, $P_iP_{i+1}Q_i \to P_iQ_{i+n-1}Q_i$, (all indices mod n), then the resulting link L will be the Hopf's link 2_1^2, shown in the figure 8 (see: [3], p.89). So we summarize all details above in the following Theorem and Corollary:

Theorem 3.4. *The 3-manifold M_n is n-fold cyclic covering of the 3-sphere S^3 branched over the link in figure 6. The branching indices on the components of L are equal to n.*

Corollary 3.5. *For some identifications α, β the link L will be the Hopf's link 2_1^2.*

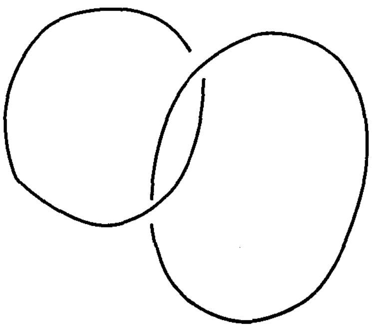

Figure 8: *Hopf's link* 2_1^2

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