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A STUDY OF BRAMBLE-HILBERT LEMMA AND ITS RELATION TO POINCARÉ' S INEQUALITY

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Abstract. This paper is concerned with the proof of so-called Bramble-Hilbert Lemma. We present that Poincaré' s inequality in [3] implies one of results of Morrey which is crucial in the proof. In this point of view, we recognize that removing the average term in Poincaré's inequality fulfills a crucial role in the proof of Bramble-Hilbert Lemma. It is accomplished by adding some polynomial of degree one less than the degree of the Sobolev space in the outset. So, the condition annihilating the set of polynomials P_{k-1} of degree $k-1$ is required necessarily in Bramble-Hilbert Lemma.

Key words.

AMS subject classifications.

1. Introduction. In the paper [1] of so-called Bramble-Hilbert Lemma, the authors gave estimates for a certain class of linear functionals on Sobolev spaces. These functionals have the property that they annihilate the set of polynomials P_{k-1} of degree $k-1$. The bounds were given in terms of the L_p norms of all k-th order partial derivatives. In this paper, one of the results of Morrey, Lemma 3.3 is deduced from Poincaré' s inequality in [3]

$$\|u-(u)_U\|_{L^p(U)} \leq Cr\|Du\|_{L^p(U)}, \tag{1.1}$$

where $(u)_U$ is the average of u over U, and r is a diameter of U. In other words, generalization of Poincaré' s inequality in [3] implies exactly Lemma 3.3. Another result of Morrey, Lemma 3.2 carry out a role removing the term $(u)_U$ in Poincaré' s inequality. Lemma 3.2 fulfills the role by adding some polynomial of degree $k-1$ with u. So, Bramble-Hilbert Lemma need the condition annihilating the set of polynomials P_{k-1} of degree $k-1$. The form removed the term $(u)_U$ in Poincaré' s inequality

$$\|u\|_{L^p(U)} \leq Cr\|Du\|_{L^p(U)} \tag{1.2}$$

is essentially crucial in the proof of Bramble and Hilbert. If we obtain the form of (1.2), then the conclusions of Bramble-Hilbert Lemma is derived naturally. For example, in $W_0^{k,p}$, the Sobolev space vanishing on the boundary, we get (1.2) by so-call Poincaré-Friedrichs inequality, and so the condition anninilating the set of polynomials P_{k-1} is unnecessary in applying Bramble-Hilbert Lemma in $W_0^{k,p}$.

2. Notation and Preliminaries. Let R with boundary ∂R be a bounded domain in Euclidean n-space, $\mathbb{R}^n$. Let ρ be the diameter of R. We shall assume that R satisfies a strong cone property; that is, there exists a finite open covering $\{O_i\}$, $i=1,\ldots,N$ of ∂R and corresponding cones $\{C_i\}$ with vertices at the origin such that $x+C_i$ is contained in R for any $x \in R\bigcap O_i$ (Figure 2.1). We shall consider complex-valued functions defined on R. As usual we define $L_p(R)$ as the set of all functions u such that

$$\|f\|_{p,R} = \left(\frac{1}{\rho^n}\int_R |f(x)|^p dx\right)^{\frac{1}{p}} \tag{2.1}$$

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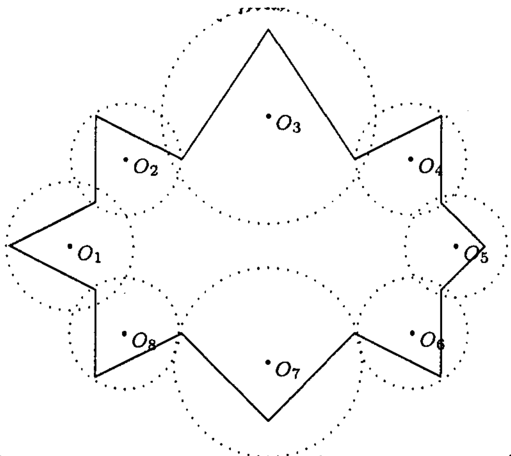

FIG. 2.1. *a domain satisfying strong cone property with a finite open covering* $\{O_i\}$ *in* $\mathbb{R}^2$

exist and is finite, where dx denotes Lebesque measure. The above norm is equivalent to the ordinary usual norm $\|f\|_{L^p(R)} = (\int_R |f(x)|^p dx)^{\frac{1}{p}}$ since ρ is finite and R is bounded. We shall need the following seminorms :

$$|u|_{p,k,R} = \sum_{|\alpha|=k} \|D^\alpha u\|_{p,R} \tag{2.2}$$

and

$$|u|_{\infty,k,R} = \sum_{|\alpha|=k} |D^\alpha u|_{\infty,R}, \tag{2.3}$$

where $|u|_{\infty,R} = ess.sup_{x\in R}|u(x)|$. The above semi norm (2.2) is a little different from the usual semi norm $(\sum_{|\alpha|=k} \int_R |D^\alpha u|^p dx)^{\frac{1}{p}}$, but is equivalent to the usual semi norm since

$$C\sum_{|\alpha|=k}(\int_R |D^\alpha u|^p dx)^{\frac{1}{p}} \leq (\sum_{|\alpha|=k}\int_R |D^\alpha u|^p dx)^{\frac{1}{p}} \leq \sum_{|\alpha|=k}(\int_R |D^\alpha u|^p dx)^{\frac{1}{p}}, \tag{2.4}$$

where N_k is the number of multi-index α with $|\alpha| = k$, and C is some constant less than or equal to $N_k^{\frac{1}{p}-1}$. In (2.4), the first inequality is induced by Jensen's inequality, and the second inequality is easy to be shown by Theorem 4.1. In (2.2) and the sequel, α is a multi-index;

$$\alpha = (\alpha_1, \ldots, \alpha_n) \text{ and } |\alpha| = \sum_{i=1}^n \alpha_i, \quad D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1}\cdots(\frac{\partial}{\partial x_n})^{\alpha_n}.$$

Now we shall consider to introduce the Sovolev space and the weak derivative in [3]. Assume that $U \subset \mathbb{R}^n$ is open. Fix $1 \leq p \leq \infty$ and let k be a nonnegative integer.

Notation.
(i) $C^k(U) = \{u : U \to \mathbb{C} \mid u \text{ is k-times continuously differentiable }\}$
(ii) $C^\infty(U) = \{u : U \to \mathbb{C} \mid u \text{ is infinitely differentiable }\} = \bigcap_{k=0}^{\infty} C^k(U)$

(iii) $C_c(U)$, $C_c^k(U)$, etc. denote these functions in $C(U)$, $C^k(U)$, etc. with compact support.
(iv) $L^p(U) = \{u : U \to \mathbb{C} \mid u$ is Lebesque measurable, $\|u\|_{L^p(U)} < \infty\}$ $(1 \leq p < \infty)$
(v) $L^1_{loc}(U) = \{u : U \to \mathbb{C} \mid u \in L^p(V)$ for each $V \subset\subset U\}$, where $V \subset\subset U$ denotes that V is compactly embedded in U.

We define the Sobolev space $W^{k,p}(U)$ consists of all locally summable functions $u : U \to \mathbb{C}$, that is, $u \in L^1_{loc}(U)$ such that for each multi index α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.
In this paper we take the norm on $W^{m,p}(R)$ to be

$$\|u\|^p_{p,m,R} = \sum_{k=0}^{m} \rho^{kp} |u|^p_{p,k,R}. \tag{2.5}$$

It is trivial that this is equivalent to the usual norm for $W^{m,p}(R)$.
We shall also consider the space of functions which have continuous derivatives of order up to and including m in R; this space will be denoted by $C^m(R)$. For the purpose of this paper we take the norm on $C^m(R)$ to be:

$$\|u\|_{\infty,m,R} = \sum_{k=0}^{m} \rho^k |u|_{\infty,k,R}. \tag{2.6}$$

Again, the usual norm on $C^m(R)$ is equivalent to (2.6). We shall denote by P_k the set of polynomials of degree less than to equal to k, restricted to R. Throughout this paper we shall use C to denote a generic constant not necessarily the same in any two places.

3. Estimation of linear functionals. Let us consider B a Banach space with norm $\|\cdot\|_B$ and let B_1 be a closed linear subspace of B. We define Q by the quotient or factor space of B with respect to B_1, denoted by B/B_1. The elements of Q are equivalence classes $[u]$, where $[u]$ is the class containing u. The equivalence relation is given by $\sim$ where for $u, v \in B, u \sim v$ if and only if $u - v \in B_1$. The usual norm on Q is given by $\|[u]\|_Q = inf_{v\in B_1}\|u + v\|_B$. Under the assumptions we have made for B and B_1 , it is well known that Q is a Banach space with norm $\|\cdot\|_Q$.
Now consider the (closed) finite-dimensional subspace of $W^k_p(R)$ given by P_{k-1}.(We know that a finite-dimensional subspace of a normed space is closed.) Here, $p(x) \in P_{k-1}$ if and only if $p(x) = \sum_{|\gamma|\leq k-1} a_\gamma x^\gamma$ for $x \in R$, where a_γ are complex numbers and γ is a multi-index.

THEOREM 3.1. *Let $Q = W^{k,p}(R)/P_{k-1}$. Then $|u|_{k,p,R}$ is a norm on Q equivalent to $\|[u]\|_Q$. Further, there exist C independent of ρ and u such that for any $u \in W^{k,p}(R)$*

$$\rho^k |u|_{k,p,R} \leq \|[u]\|_Q \leq C\rho^k |u|_{k,p,R}. \tag{3.1}$$

We shall make use of two lemmas which can be found in Morrey [4], p.85 . In this paper, we give each lemma its proof which is not cited from Morrey.

LEMMA 3.2. *For any $u \in W^{k,p}(R)$ there is a unique polynomial p of degree less than or equal to $k-1$ (or 0) such that*

$$\int_R D^\alpha(u + p) = 0 \tag{3.2}$$

for all α *with* $0 \le |\alpha| \le k-1$.

Proof. of Lemma 3.2. Let p be an element in P_{k-1}. Then $p(x)$ is written by $\sum_{|\gamma|\le k-1} a_\gamma x^\gamma$ for all $x \in R$, where a_γ are complex numbers and γ is a multi-index. Now, we shall show that $p(x)$ satisfying (3.2) is unique in P_{k-1}. Since $u \in W_p^k(R)$, weak derivatives of order less than or equal to k of u satisfies $\int_R |D^\alpha u|^p < \infty$ for all α with $|\alpha| \le k$. we note that each $\int_R D^\alpha u$ is bounded since

$$\int_R D^\alpha u \le \int_R |D^\alpha u| \le \int_R |D^\alpha u|^p.$$

The above latter inequality is easily shown by Jensen inequality with using convex function x^p and bounded domain. Even though the above inequalities is not considered, weak derivative may be defined conventionally in a set of locally summable functions. We see that (3.2) means

$$\int_R D^\alpha(p(x)) = -\int_R D^\alpha u, \tag{3.3}$$

for all α with $0 \le |\alpha| \le k-1$. Here, the left term of (3.3) stands for

$$\int_R D^\alpha(p(x)) = \int_R D^\alpha\Big(\sum_{|\gamma|\le k-1} a_\gamma x^\gamma\Big) = \sum_{|\gamma|\le k-1} a_\gamma \int_R D^\alpha(x^\gamma).$$

Hence, (3.3) is represented with the system of linear equations

$$\left\{\begin{array}{l} a_{\gamma_1}\int_R D^{\alpha_1}(x^{\gamma_1}) + a_{\gamma_2}\int_R D^{\alpha_1}(x^{\gamma_2}) + \cdots + a_{\gamma_N}\int_R D^{\alpha_1}(x^{\gamma_N}) = -\int_R D^{\alpha_1}u \\ a_{\gamma_1}\int_R D^{\alpha_2}(x^{\gamma_1}) + a_{\gamma_2}\int_R D^{\alpha_2}(x^{\gamma_2}) + \cdots + a_{\gamma_N}\int_R D^{\alpha_2}(x^{\gamma_N}) = -\int_R D^{\alpha_2}u \\ \quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ a_{\gamma_1}\int_R D^{\alpha_N}(x^{\gamma_1}) + a_{\gamma_2}\int_R D^{\alpha_N}(x^{\gamma_2}) + \cdots + a_{\gamma_N}\int_R D^{\alpha_1}(x^{\gamma_N}) = -\int_R D^{\alpha_N}u \end{array}\right., \tag{3.4}$$

where α_i and γ_i are attached to index i for α and γ, respectively, and N is the number of all multi indices α with $|\alpha| \le k-1$. Here, we note that the number of all multi indices α with $|\alpha| \le k-1$ is the same with the number of all multi indices γ with $|\gamma| \le k-1$, since α and γ have the same dimension. We may assume that α and γ are the same each other, and that$\{\alpha_i\}_{i=1}^N$; $\{\alpha_1, \alpha_2, \cdots, \alpha_N\}$ is arranged in the order that satisfies some entry of α_i is lager than or equal to all entries of α_{i-1},(i.e. there exist an entry $\alpha_{i,j}$ in $\alpha_i = (\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n})$ such that $\alpha_{i,j} \ge \alpha_{i-1,k}$, $\forall\, 1 \le k \le n$). Example of the case of dimension $n = 2$ and $k-1 = 2$ illustrates

$$\{\alpha_1 = (0,0), \alpha_2 = (1,0), \alpha_3 = (0,1), \alpha_4 = (1,1), \alpha_5 = (2,0), \alpha_6 = (0,2)\}.$$

Then, we observe that

$$D^{\alpha_i}(x^{\gamma_j}) = \begin{cases} \text{is constant,} & \text{if } i = j, \\ 0, & \text{if } i > j, \end{cases} \tag{3.5}$$

because $D^{\alpha_i}(x^{\gamma_j}) = D^{(\alpha_i-\alpha_j)}(D^{\alpha_j}x^{\gamma_j})$, where $\alpha_i - \alpha_j = (\alpha_{i,1}-\alpha_{j,1}, \alpha_{i,2}-\alpha_{j,2}, \ldots, \alpha_{i,n}-\alpha_{j,n})$, and obviously $\alpha_i - \alpha_j$ has some positive entry. Consequently, the coefficient matrix of (3.4) is

$$\begin{bmatrix} C & \times & \times & \dots & \dots \\ 0 & C & \times & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \times \\ 0 & 0 & 0 & \dots & C \end{bmatrix}, \tag{3.6}$$

where C denotes a constant not necessary the same each other. (3.6) is in echelon form whose diagonal entries is not vanishing. Thus, (3.6) is invertible. In conclusion, (3.4) has the only solution. Thus, a polynomial p is uniquely determined in P_{k-1}. □

LEMMA 3.3. *Let R satisfy a strong cone condition. Then (since R is contained in a sphere of radius ρ)*

$$|u|_{j,p,R} \le C\rho^{k-j}|u|_{k,p,R} \tag{3.7}$$

for $0 \le j \le k-1$ for all $u \in W^{k,p}(R)$ such that the average over R of each $D^\alpha u$ is 0 for $0 \le |\alpha| \le k-1$, where C is a constant independent of ρ and u .

Note. Morrey assumes that his domain is strongly Lipschitz, but the proof is exactly the same if the domain satisfies a strong cone condition. Hence, we may assume that R satisfies strongly Lipschitz. Before proof of Lemma 3.3, we shall state Poincare's inequality which is introduced in Evans [3], pp.275 .

Notation.
(i) $(u)_U = \frac{1}{meas(U)}\int_U u$. (i.e. $(u)_U$ means the average of u over U.)
(ii) Du denotes the gradient of u; that is, $Du = [\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]$

THEOREM 3.4 (Poincaré' s inequality). *Let U be a bounded, connected, open subset of $\mathbb{R}^n$, with a C^1 boundary ∂U. Assume $1 \le p \le \infty$. Then there exists a constant C, depending only on n, p and U, such that*

$$\|u - (u)_U\|_{L^p(U)} \le C\|Du\|_{L^p(U)} \tag{3.8}$$

for each function $u \in W^{1,p}(U)$. In particular, there exists a constant C, depending only on n and p, such that

$$\|u - (u)_U\|_{L^p(U)} \le Cr\|Du\|_{L^p(U)} \tag{3.9}$$

for each function $u \in W^{1,p}(B^0(x,r))$, where $B^0(x,r) = \{y \in \mathbb{R}^n \mid |x-y| < r\}$.

PROPOSITION 3.5. *Under the assumptions of Lemma 3.3 and Theorem 3.4, the above inequality (3.9) implies just the same as (3.7) in Lemma 3.3 for $j = 0,\ k = 1$, in particular, if $u \in W^{2,p}(U)$ then (3.9) implies just the same as (3.7) in Lemma 3.3 for $j = 0, 1$, $k = 2$.*

Proof. We observe at first that $\|Du\|_{L^p(U)}$ is less than $|u|_{p,1,U}$ because

$$\|Du\|_{L^p(U)} = \||Du|\|_{L^p(U)}([3], p.618) = \| \left(\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2\right)^{\frac{1}{2}} \|_{L^p(U)}$$

$$\leq \| \sum_{i=1}^{n} |\frac{\partial u}{\partial x_i}| \|_{L^p(U)} \qquad \text{(by Theorem 4.1 in the following section.)}$$

$$= \| \sum_{|\alpha|=1} |D^\alpha u| \|_{L^p(U)} \qquad \text{(by the notation of partial derivative } D^\alpha. \text{)}$$

$$\leq \sum_{|\alpha|=1} \|D^\alpha u\|_{L^p(U)} = |u|_{p,1,U} \qquad \text{(by (2.2))}$$

Thus, In Theorem 3.4, (3.9) implies

$$\|u - (u)_U\|_{L^p(U)} \leq Cr|u|_{p,1,U}. \tag{3.10}$$

Since $\| \cdot \|_{L^p(U)}$ is equivalent to $| \cdot |_{p,0,U}$ by (2.1) and (2.2), it follows that

$$|u - (u)_U|_{p,0,U} \leq Cr|u|_{p,1,U}. \tag{3.11}$$

Thus, the above inequality (3.11) is just the same as (3.7) in Lemma 3.3 for $j = 0,\ k = 1$ since $(u)_U = 0$ from assumption of Lemma 3.3.

Now, let us consider to suppose $u \in W^{2,p}(U)$. Then $D^\alpha u \in W^{1,p}(U)$ for some α with $|\alpha| = 1$ ([3], p.247). We get from (3.10)

$$\|D^\alpha u - (D^\alpha u)_U\|_{L^p(U)} \leq Cr|D^\alpha u|_{p,1,U}, \tag{3.12}$$

and from assumption of Lemma 3.3 so that

$$\|D^\alpha u\|_{L^p(U)} \leq Cr|D^\alpha u|_{p,1,U}. \tag{3.13}$$

Taking a summation over $|\alpha| = 1$ for (3.13),

$$\sum_{|\alpha|=1} \|D^\alpha u\|_{L^p(U)} \leq Cr \sum_{|\alpha|=1} |D^\alpha u|_{p,1,U}. \tag{3.14}$$

The left side of the above (3.14) is equivalent to $|u|_{p,1,U}$ by (2.1) and (2.2), and a part of the right side of (3.14) is

$$\sum_{|\alpha|=1} |D^\alpha u|_{p,1,U} = \sum_{|\alpha|=1} (\sum_{|\beta|=1} \|D^\beta (D^\alpha u)\|_{p,U}) = \sum_{|\gamma|=2} \|D^\gamma u\|_{p,U} = |u|_{p,2,U}. \tag{3.15}$$

Hence, we obtain that for $u \in W^{2,p}(U)$ satisfying assumptions of Lemma 3.3 and Theorem 3.4 ,

$$|u|_{p,1,U} \leq Cr|u|_{p,2,U}, \tag{3.16}$$

With considering (3.10) and multiplying Cr in both sides of (3.16), consequently

$$|u|_{p,0,U} \leq Cr|u|_{p,1,U} \leq C^2 r^2 |u|_{p,2,U}. \tag{3.17}$$

This completes the proof. □

Thus we are motivated to verify that a generalization of Theorem 3.4 (Poincaré's inequality) implies exactly the Lemma 3.3. It is shown below.

Proof. of lemma 3.3. It is trivial that strongly Lipschitz condition of R satisfies C^1 boundary condition. Since $u \in W^{k,p}(R)$, $D^\alpha u \in W^{1,p}(R)$ for some α with $|\alpha| = j$. Hence we can apply (3.9) in Theorem 3.4 (Poincaré's inequality) since R is contained in a sphere of radius ρ. From (3.10) and (2.1) and the assumption that the average over R of each $D^\alpha u$ is 0,

$$\|D^\alpha u\|_{p,R} \le C\rho |D^\alpha u|_{p,1,U}. \tag{3.18}$$

Taking a summation over all α with $|\alpha| = j$ for the above inequality,

$$\sum_{|\alpha|=j} \|D^\alpha u\|_{p,R} \le C\rho \sum_{|\alpha|=j} |D^\alpha u|_{p,1,U}. \tag{3.19}$$

The left side of (3.19) is the just $|u|_{p,j,R}$ by (2.2), and a part of right side of (3.19) is

$$\sum_{|\alpha|=j} |D^\alpha u|_{p,1,U} = \sum_{|\alpha|=j} \Big(\sum_{|\beta|=1} \|D^\beta (D^\alpha u)\|_{p,U} \Big) = \sum_{|\gamma|=j+1} \|D^\gamma u\|_{p,U} = |u|_{p,j+1,U}. \tag{3.20}$$

Thus, we get

$$|u|_{p,j,R} \le C\rho |u|_{p,j+1,R}. \tag{3.21}$$

Similarly, since $D^\alpha u \in W^{p,1}(R)$ for each of α with $|\alpha| = j+1, \ldots, k-1$ ([3], p.247), we obtain

$$\begin{aligned} |u|_{p,j+1,R} &\le C\rho |u|_{p,j+2,R} \\ |u|_{p,j+2,R} &\le C\rho |u|_{p,j+3,R} \\ &\vdots \\ |u|_{p,k-1,R} &\le C\rho |u|_{p,k,R}. \end{aligned}$$

Thus, in all,

$$|u|_{p,j,R} \le C\rho |u|_{p,j+1,R} \le C^2\rho^2 |u|_{p,j+2,R} \le \cdots \le C^{(k-j)}\rho^{(k-j)} |u|_{p,k,R}.$$

This completes the proof. □

Proof. of Theorem 3.1. we shall now prove the right hand inequality of (3.1) in Theorem 3.1. By Lemma 1 we can choose $\bar{p} \in P_{k-1}$ such that $\int_R D^\gamma (u + \bar{p}) = 0$ for $|\gamma| \le k-1$. Hence using Lemma 2 it follows that

$$\|u + \bar{p}\|_{k,p,R} = \sum_{i=0}^{k} \rho^i |u + \bar{p}|_{p,i,R} \le C\rho^k |u + \bar{p}|_{k,p,R} = C\rho^k |u|_{p,k,R}. \tag{3.22}$$

In the above, the first inequality is derived by our notation, and the second inequality is shown by using Lemma 2, and the last equality holds since $D^\alpha p = 0$ for each $|\alpha| = k$. However, since $\bar{p} \in P_{k-1}$ we have that $\|[u]\|_Q \le \|u + \bar{p}\|_{k,p,R}$. Hence $\|[u]\|_Q \le C\rho^k |u|_{k,p,R}$ for each $u \in W^{p,k}(R)$. The other inequality of (3.1) is easily seen from the observation that $\rho^k|u+p|_{k,p,R} = \rho^k|u|_{k,p,R}$ for any $p \in P_{k-1}$ from which we immediately obtain

$$\rho^k|u|_{k,p,R} \le inf_{p\in P_{k-1}}\|u+p\|_{k,p,R} = \|[u]\|_Q. \qquad \square$$

Now, the main result of this section is the following theorem.

THEOREM 3.6 (Bramble-Hilbert Lemma). *Let F be a linear functional on $W^{p,k}(R)$ which satisfies*
(i) $|F(u)| \le C\|u\|_{k,p,R}$ for all $u \in W^{p,k}(R)$ with C independent of ρ and u and
(ii) $F(p) = 0$ for all $p \in P_{k-1}$.
Then $|F(u)| \le C_1\rho^k|u|_{k,p,R}$ for any $u \in W^{p,k}(R)$ with C_1 independent of ρ and u.

Proof. Since F is linear and satisfies condition (ii),

$$|F(u)| = |F(u+p)| \qquad \text{for all } p \in P_{k-1}. \tag{3.23}$$

By condition (i) and (3.23) we have

$$|F(u)| \le C\|u+p\|_{k,p,R}. \tag{3.24}$$

Taking the infimum over P_{k-1} in (3.24) we have

$$|F(u)| \le C\|[u]\|_Q. \tag{3.25}$$

The result now follows from Theorem 3.1. $\square$

4. Consideration on the preceding section. In the preceding section, we used the following theorem to prove the latter inequality about the equivalence of norm in (2.4). It seems to be analogous to Jensen's inequality, but is quite different from the inequality. It is introduced as an exercise of [6], p.15 .

Notation $\mathbb{R}^+ = \{x \in \mathbb{R} |\ x \ge 0\}$

THEOREM 4.1. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a strictly convex function with $f(0) \le 0$. Assume that $a_1, \ldots, a_n \ge 0$ and at least two a_i are non-zero. Then*

$$\sum_{i=1}^n f(a_i) < f(\sum_{i=1}^n a_i).$$

Proof. Since f is a strictly convex function on $\mathbb{R}^+$, f is satisfied with the requirement that

$$\frac{f(t)-f(s)}{t-s} < \frac{f(u)-f(t)}{u-t}, \tag{4.1}$$

whenever $0 \le s < t < u < \infty$ ([7], p.61). Furthermore, it follows from (4.1) that

$$\frac{f(t)-f(s)}{t-s} < \frac{f(v)-f(u)}{v-u}, \tag{4.2}$$

whenever $0 \le s < t < u < v < \infty$.
It is sufficient to prove the statement for the just any two numbers in $\{a_1, \ldots, a_n\}$. At first, it is trivial that for some non-zero a,

$$f(a) + f(0) \le f(a).$$

Next, we consider for two non-zero a and b in $\mathbb{R}^+$. By applying (4.2),

$$\frac{f(a)-f(0)}{a} < \frac{f(a+b)-f(b)}{a}. \tag{4.3}$$

We get

$$f(a)-f(0) < f(a+b)-f(b). \tag{4.4}$$

Since $f(0) \leq 0$,

$$f(a) \leq f(a)-f(0) < f(a+b)-f(b).$$

Thus, we obtain

$$f(a)+f(b) < f(a+b). \tag{4.5}$$

(We note that strictly inequality incurs here.) Since the preceding method is similarly used for n elements $a_1, \ldots, a_n$, this completes the proof. □

We need to consider the following corollary to apply the case of concave function as like that $x^{\frac{1}{p}}$, $p > 1$.

COROLLARY 4.2. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a strictly concave function with $f(0) \geq 0$. Assume that $a_1, \ldots, a_n \geq 0$ and at least two a_i are non-zero. Then*

$$\sum_{i=1}^{n} f(a_i) > f(\sum_{i=1}^{n} a_i).$$ □

In the preceding section, let us observe the process of method of proof in from (3.18) to (3.22). Then we recognize that (3.18) is essentially important, that is, if we assume (3.18), then the conclusion to (3.22) is deduced naturally. So, we get the following corollary from this point of observation.

COROLLARY 4.3. *Suppose that*

$$\|D^\alpha u\|_{p,U} \leq C\rho |D^\alpha u|_{p,1,U}, \tag{4.6}$$

for all $D^\alpha u \in W^{1,p}(U)$ with $|\alpha| = j$. Then

$$|u|_{p,j,U} \leq C\rho |u|_{p,j+1,U}.$$

In particular, supposing (4.6) for $j = 0, \ldots, k-1$,

$$|u|_{p,0,U} \leq C\rho |u|_{p,1,U} \leq C^2\rho^2 |u|_{p,2,U} \leq \cdots \leq C^k\rho^k |u|_{p,k,U},$$

Moreover,

$$\|u\|_{p,k,U} \leq C\rho^k |u|_{p,k,U},$$

So, $|\cdot|_{p,k,U}$ is norm-equivalent to $\|\cdot\|_{p,k,U}$. □

The following theorem is introduced as Poincaré-Friedrichs inequality in Braess [5], p.30. In this paper, however, its proof is not cited in [5], but almost analogous to the proof of Poincaré's inequality in [3], p.275 .

Notation $W_0^{1,p}(U) = \{u \in W^{1,p}(U) \mid u = 0 \quad \text{on } \partial U\}$

THEOREM 4.4 (Poincaré-Friedrichs inequality). *Let U be a bounded, connected, open subset of $\mathbb{R}^n$, with a C^1 boundary ∂U. Assume $1 \leq p \leq \infty$. Then there exists a constant C, depending only on n, p and U, such that*

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)} \tag{4.7}$$

for each function $u \in W_0^{1,p}(U)$. In particular, there exists a constant C, depending only on n and p, such that

$$\|u\|_{L^p(B(x,r))} \leq Cr\|Du\|_{L^p(B(x,r))} \tag{4.8}$$

for each function $u \in W^{1,p}(B^0(x,r))$, where $B^0(x,r) = \{y \in \mathbb{R}^n \mid |x-y| < r\}$.

The proof of Thoerem 4.4 only requires zero boundary conditions on a part of the boundary. It suffices that the function vanishes on a part of a set V, where V is a set with positive measure (Braess [5], p.30). The following corollary is a generalization of Corollary 4.4 from this point of view.

Notation $W_{0,0}^{1,p}(U) = \{u \in W^{1,p}(U) \mid u = 0 \quad \text{on } V\}$, where V is a subset of U with $m(V) > 0$.

COROLLARY 4.5 (Poincaré-Friedrichs inequality). *Let U be a bounded, connected, open subset of $\mathbb{R}^n$, with a C^1 boundary ∂U. Assume $1 \leq p \leq \infty$. Then there exists a constant C, depending only on n, p and U, such that*

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)} \tag{4.9}$$

for each function $u \in W_{0,0}^{1,p}(U)$. □

Remark. In $W_{0,0}^{p,k}(R)$, the condition (ii) that $F(p) = 0$ for all $p \in P_{k-1}$ of Theorem 3.6 (Bramble-Hilbert Lemma) becomes unnecessary. Thus we get the following theorem.

THEOREM 4.6. *Let F be a linear functional on $W_{0,0}^{p,k}(R)$ which satisfies $|F(u)| \leq C\|u\|_{k,p,R}$ for all $u \in W_{0,0}^{p,k}(R)$ with C independent of ρ and u. Then $|F(u)| \leq C_1\rho^k|u|_{k,p,R}$ for any $u \in W_{0,0}^{p,k}(R)$ with C_1 independent of ρ and u.*

Proof. By Corollary 4.5, since $D^\alpha u \in W_{0,0}^{1,p}(U)$ for all α with $|\alpha| = 0, \ldots, k-1$, we obtain

$$\|D^\alpha u\|_{p,U} \leq C|D^\alpha u|_{p,1,U},$$

from (3.10) and (2.1). This is just the assumption of Corollary 4.3. So, we obtain the conclusion of the corollary. This completes the proof. □

5. Conclusion and Discussion. If we discuss Poincaré inequality in a Sobolev space nonvanishing on a set of measure of nonzero, the term of average and the degree of the Sobolev space are looked upon. In just the above space but vanishing on a set of measure of nonzero, it is not easy to remove the term of average and to raise the degree of the Sobolev space. Here, if the term of average is removed, then to raise the degree of the Sobolev space is obtained naturally. Otherwise, it is very difficult or impossible on the point of my observation without changing in itself. Bramble and Hilbert used the result of Morrey to remove the term of average in Poincaré inequality.

The result of Morrey is to make the term of average be zero by adding a polynomial of degree of less one than the Sobolev space in the outset. It is deduced originally from linearly independence of each term of a polynomial. If we consider linearly independence, we may discover a little application with Bramble-Hilbert Lemma, for example, to deal with trigonometric polynomial.

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