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Signed degree sequences in signed 3-partite graphs

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Abstract. A signed 3-partite graph is a 3-partite graph in which each edge is assigned a positive or a negative sign. Let G(U, V, W) be a signed 3-partite graph with $U = \{u_1, u_2, \cdots, u_p\}$,$V = \{v_1, v_2, \cdots, v_q\}$ and $W = \{w_1, w_2, \cdots, w_r\}$. Then, signed degree of $u_i(v_j$ and $w_k)$ is $sdeg(u_i) = d_i = d_i^+ - d_i^-$,$1 \leq i \leq p$ ($sdeg(v_j) = e_j = e_j^+ - e_j^-$,$1 \leq j \leq q$ and $sdeg(w_k) = f_k = f_k^+ - f_k^-$,$1 \leq k \leq r$) where $d_i^+(e_j^+$ and $f_k^+)$ is the number of positive edges incident with $u_i(v_j$ and $w_k)$ and $d_i^-(e_j^-$ and $f_k^-)$ is the number of negative edges incident with $u_i(v_j$ and $w_k)$. The sequences $\alpha = [d_1, d_2, \cdots, d_p]$, $\beta = [e_1, e_2, \cdots, e_q]$ and $\gamma = [f_1, f_2, \cdots, f_r]$ are called the signed degree sequences of G(U, V, W). In this paper, we characterize the signed degree sequences of signed 3-partite graphs.

1. Introduction

A signed graph is a graph in which each edge is assigned a positive or a negative sign. The concept of signed graphs is given by Harary [3]. Let G be a signed graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. Then, signed degree of v_i is $sdeg(v_i) = d_i = d_i^+ - d_i^-$, $1 \leq i \leq n$ where $d_i^+(d_i^-)$ is the number of positive(negative) edges incident with v_i . A signed degree sequence $\sigma = [d_1, d_2, ..., d_n]$ of a signed graph G is formed by listing the vertex signed degrees in non-increasing order. An integral sequence is s-graphical if it is the signed degree sequence of a signed graph. Also, a non-zero sequence $\sigma = [d_1, d_2, \cdots, d_n]$ is a standard sequence if σ is non-increasing, $\sum_{i=1}^{n} d_i$ is even, $d_1 > 0$, each $|d_i| < n$, and $|d_1| \geq |d_n|$.

The following result, due to Chartrand et.al. [1], gives a necessary and sufficient condition for an integral sequence to be s-graphical, which is similar to Hakimi's result for degree sequences in graphs [2].

Theorem 1.1. *A standard integral sequence $\sigma = [d_1, d_2, \cdots, d_n]$ is s-graphical if and only if*

$$\sigma' = [d_2 - 1, d_3 - 1, ..., d_{d_1+s+1} - 1, d_{d_1+s+2}, \cdots, d_{n-s}, d_{n-s+1} + 1, ..., d_n + 1]$$

is s-graphical for some s, $0 \leq s \leq \frac{n-1-d_1}{2}$.

The next result [5] provides a good candidate for parameter s in Theorem 1.1.

Theorem 1.2. *A standard integral sequence $\sigma = [d_1, d_2, \cdots, d_n]$ is s-graphical if and only if*

$$\sigma'_m = [d_2 - 1, d_3 - 1, \cdots, d_{d_1+m+1} - 1, d_{d_1+m+2}, \cdots, d_{n-m}, d_{n-m+1} + 1, \cdots, d_n + 1]$$

is s-graphical, where m is the maximum non negative integer such that $d_{d_1+m+1} > d_{n-m+1}$.

Some results for signed degree sets in signed graphs are given by Pirzada et al. [4].

A signed 3-partite graph is a 3-partite graph in which each edge is assigned a positive or a negative sign. Let G(U, V, W) be a signed 3-partite graph with $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$ and $W = \{w_1, w_2, \cdots, w_r\}$. Then,
signed degree of u_i is $sdeg(u_i) = d_i = d_i^+ - d_i^-$,
degree of u_i is $deg(u_i) = d_i = d_i^+ + d_i^-$,
where $1 \leq i \leq p$ and $d_i^+(d_i^-)$ is the number of positive(negative) edges incident with u_i,
signed degree of v_j is $sdeg(v_j) = e_j = e_j^+ - e_j^-$,
degree of v_j is $\deg(v_j) = e_j = e_j^+ + e_j^-$,
where $1 \leq j \leq q$ and $e_j^+(e_j^-)$ is the number of positive(negative) edges incident with v_j. signed degree of w_k is $sdeg(w_k) = f = f_k^+ - f_k^-$, degree of w_k is $\deg(w_k) = f_k = f_k^+ + f_k^-$, where $1 \leq k \leq r$ and $f_k^+(f_k^-)$ is the number of positive(negative) edges incident with w_k. Clearly, $|d_i| \leq q+r$,$|e_j| \leq r+p$ and $|f_k| \leq p+q$. Then, the sequences $\alpha = [d_1, d_2, \cdots, d_p]$, $\beta = [e_1, e_2, \cdots, e_q]$ and $\gamma = [f_1, f_2, \cdots, f_r]$ are called the signed degree sequences of the signed 3-partite graph G(U, V, W). Three sequences $\alpha = [d_1, d_2, \cdots, d_p]$, $\beta = [e_1, e_2, \cdots, e_q]$ and $\gamma = [f_1, f_2, \cdots, f_r]$ of integers are s-graphical if α,β and γ are the signed degree sequences of some signed 3-partite graph .We denote a positive edge xy by xy^+ and a negative edge xy by xy^-.

2. Characterizations of signed degree sequences in signed 3-partite graphs

First we obtain the following result.

Theorem 2.1. *Let G(U, V, W) be a signed 3-partite graph with m edges. Then,*$g = \sum_{u \in U} sdeg(u) + \sum_{v \in V} sdeg(v) + \sum_{w \in W} sdeg(w) \equiv 2m(mod4)$ *and the number of positive edges and negative edges of G(U, V, W) are respectively* $\frac{2m+g}{4}$ *and* $\frac{2m-g}{4}$

Proof. Let G(U, V, W) be a signed 3-partite graph with $U = \{u_1, u_2, \cdots, u_p\}$, $V = \{v_1, v_2, \cdots v_q\}$,$W = \{w_1, w_2, \cdots w_r\}$. Let $v_i (1 \leq i \leq p)$ be incident with d_i^+ positive edges and d_i^- negative edges, $v_j(1 \leq j \leq q)$ be incident with e_j^+positve edges and e_j^- negative edges and $w_k(1 \leq k \leq r)$ be incident with f_k^+ positive edges and f_k^- negative edges so that

$sdeg(u_i) = d_i^+ - d_i^-$ while $deg(u_i) = d_i^+ + d_i^-$, for $1 \leq i \leq p$,
$sdeg(v_j) = e_j^+ - e_j^-$ while $deg(v_j) = e_j^+ + e_j^-$, for $1 \leq j \leq q$,
and $sdeg(w_k) = f_k^+ - f_k^-$, while $deg(w_k) = f_k^+ + f_k^-$, for $1 \leq k \leq r$.
Clearly, $\sum_{i=1}^{p} \deg(u_i) + \sum_{j=1}^{q} \deg(v_j) + \sum_{k=1}^{r} \deg(w_k) = 2m$.
Let $G(U, V, W)$ have x positive edges and y negative edges.
Then $m = x + y, \sum_{i=1}^{p} d_i^+ + \sum_{j=1}^{q} e_j^+ + \sum_{k=1}^{r} f_k^+ = 2x$ and $\sum_{i=1}^{p} d_i^- + \sum_{j=1}^{q} e_j^- + \sum_{k=1}^{r} f_k^- = 2y$.
Further, $\sum_{i=1}^{p} sdeg(u_i) + \sum_{j=1}^{q} sdeg(v_j) + \sum_{k=1}^{r} s\deg(w_k) = \sum_{i=1}^{p}(d_i^+ - d_i^-) + \sum_{j=1}^{q}(e_j^+ - e_j^-) + \sum_{k=1}^{r}(f_k^+ - f_k^-) = \left(\sum_{i=1}^{p} d_i^+ + \sum_{j=1}^{q} e_j^+ + \sum_{k=1}^{r} f_k^+\right) - \left(\sum_{i=1}^{p} d_i^- + \sum_{j=1}^{q} e_j^- + \sum_{k=1}^{r} f_k^-\right) = 2x - 2y$. Hence, $g = \sum_{i=1}^{p} sdeg(u_i) + \sum_{j=1}^{q} sdeg(v_j) + \sum_{k=1}^{r} sdeg(w_k) = 2x - 2y = 2(m - y) - 2y = 2m - 4y$,
so that $g \equiv 2m(mod4)$.
Also, from $x + y = m$ and $2x - 2y = g$, we have $x = \frac{2m+g}{4}$ and $y = \frac{2m-g}{4}$. □

Corollary 2.2. *A necessary condition for the sequences* $\alpha = [d_1, d_2, \cdots, d_p]$, $\beta = [e_1, e_2, \cdots, e_q]$ *and* $\gamma = [f_1, f_2, \cdots, f_r]$ *of integers to be s-graphical is that* $\sum_{i=1}^{p} d_i^+ + \sum_{j=1}^{q} e_j^+ + \sum_{k=1}^{r} f_k^+$ *is even.*

A zero sequence is a finite sequence each term of which is 0. Clearly, every three finite zero sequences are the signed degree sequences of a signed 3-partite graph. If $\delta = [a_1, a_2, \cdots, a_n]$ is a sequences of integers , then the negative of δ is the sequence $-\delta = [-a_1, -a_2, \cdots, -a_n]$.

The following result follows by interchanging positive edges with negative edges.

Lemma 2.3. *The sequences* $\alpha = [d_1, d_2, \cdots, d_p]$, $\beta = [e_1, e_2, \cdots, e_q]$ *and* $\gamma = [f_1, f_2, \cdots, f_r]$ *are the signed degree sequences of some signed 3-partite graph if and only if*

$$-\alpha = [-d_1, -d_2, \cdots, -d_p], -\beta = [-e_1, -e_2, \cdots, -e_q] \text{ and } -\gamma = [-f_1, -f_2, \cdots, -f_r]$$

are the signed degree sequences of some signed 3-partite graph.

We may assume without loss of generality, that a non-zero sequence $\delta = [a_1, a_2, \cdots, a_n]$ is non-increasing and $|a_1| \geq |a_n|$, for we may always replace δ by $-\delta$ if necessary .The sequences of integers $\alpha = [d_1, d_2, \cdots, d_p]$ $\beta = [e_1, e_2, \cdots, e_q]$ and $\gamma = [f_1, f_2, \cdots, f_r]$ are said to be standard sequences if α is non-zero and non-increasing, $\sum_{i=1}^{p} d_i^+ + \sum_{j=1}^{q} e_j^+ + \sum_{k=1}^{r} f_k^+$ is even , $d_1 > 0$, each $|d_i| \leq q + r$, each $|e_j| \leq r + p$ each $|f_k| \leq p + q$,$|d_1| \geq |d_p|$,$|d_1| \geq |e_j|$ and $|d_1| \geq |f_k|$ for each j and k.

The following result provides a useful recursive test whether the three sequences of integers form the signed degree sequences of some complete signed 3-partite graph.

Theorem 2.4. *Let* $\alpha = [d_1, d_2, \cdots, d_p]$, $\beta = [e_1, e_2, \cdots, e_q]$ *and* $\gamma = [f_1, f_2, \cdots, f_r]$ *be standard sequences and let* $g = \frac{d_1+q+r}{2}$. *let* α' *be obtained from* α *by deleting* d_1 , *and* β' *and* γ' *be obtained from* β *and* γ *by reducing g greatest entries of* β *and* γ *by 1 each and adding remaining entries of* β *and* γ *by 1 each. Then,* α,β *and* γ *are the signed degree sequences of some complete signed 3-partite graph if and if* α', β' *and* γ' *are also.*

Proof. Let $G'(U', V', W')$ be a complete signed 3-partite graph with signed degree sequences α', β' and γ'. Let $U' = \{u_1, u_2, \cdots, u_p\}, V' = \{v_1, v_2, \cdots, v_q\}$, and $W' = \{w_1, w_2, \cdots, w_r\}$. Then, a complete signed 3-partite graph with signed degree sequences α,β and γ can be obtained by adding a vertex u_1 to U' so that there are g positive edges from u_1 to those g vertices of V' and W' , whose signed degrees were reduced by 1 in going from α,β and γ to α',β' and γ', and there are negative edges from u_1 to the remaining vertices of V' and W' , whose signed degrees were increased by 1 in going from α,β γ to α',β' and γ'. Note that the signed degree of u_1 is $g - (q + r - g) = 2g - (q + r) = d_1$

Conversely, let α,β and γ be the signed degree sequences of a complete signed 3-partite graph. Let the vertex sets of the complete signed 3-partite graph be $U' = \{u_1, u_2, \cdots, u_p\}$,$V' = \{v_1, v_2, \cdots, v_q\}$, and $W' = \{w_1, w_2, \cdots, w_r\}$ such that $sdeg(u_i) = d_i$,$1 \leq i \leq p$, $sdeg(v_j) = e_j$,$1 \leq j \leq q$ and $sdeg(w_k) = f_k$, $1 \leq k \leq r$. Among all the complete signed 3-partite graphs with α,β and γ as the signed degree sequences, let $G(U, V, W)$ be one with the property that the sum S of the signed degrees of the vertices of V and W joined to $u_1$by positive edges is maximum. Let d_1^+ and d_1^- be respectively the number of positive edges and the number of negative edges incident with u_1. Then,$sdeg(u_1) = d_1 = d_1^+ - d_1^-$, and $deg(u_1) = d_1^+ + d_1^- = q + r$. Therefore $d_1^+ = \frac{d_1+q+r}{2} = g$. Let X be the set of g vertices of V and W with highest signed degrees and let $Y = (V \cup W) - X$.We claim that u_1 must be joined by positive edges to the vertices of X.If this is not true, then there exist vertices $x \in X$ and $y \in Y$ such that the edge u_1x is negative and the edge u_1y is positive. Since

$sdeg(x) \geq sdeg(y)$, therefore there exists a vertex $u_n(\neq u_1)$ of U such that $u_n x$ is a positive edge and $u_n y$ is a negative edge.Now, change the signs of these edges so that $u_1 x$ and $u_n y$ are positive and $u_1 y$ and $u_n x$ are negative.

Hence, we obtain a complete signed 3-partite graph with signed degree sequences α,β and γ in which the sum of the signed degrees of the vertices of $V$and W joined to u_1 by positive edges exceeds S, a contradiction.

So, assume that u_1 is joined by positive edges to the vertices of X and by negative edges to the vertices of Y. Therefore $G(U, V, W) - u_1$ is a complete signed 3-partite graph with α',β' and γ' as the signed degree sequences.

Theorem 2.4 provides an algorithm for determining whether or not the standard sequences α,β and γ are the signed degree sequences, and for constructing a corresponding complete signed 3-partite graph. Suppose $\alpha = [d_1, d_2, \cdots, d_p]$, $\beta = [e_1, e_2, \cdots, e_q]$ and $\gamma = [f_1, f_2, \cdots, f_r]$ be the standard signed degree sequences of a complete signed 3-partite graph with parts $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$ and $W = \{w_1, w_2, \cdots, w_r\}$. Deleting d_1 and reducing $g = \frac{d_1+q+r}{2}$ greatest entries of V and W by 1 each and adding remaining entries of V and W by 1 each to form V' and W' . Then, edges are defined by $u_1 v_j^+ (u_1 w_k^+)$ if $e_j(f_k)$ is reduced by 1 and $u_1 v_j^- (u_1 w_k^-)$ if $e_j(f_k)$ is increased by 1. For $-\alpha$,$-\beta$ and $-\gamma$,the edges are defined by $u_1 v_j^- (u_1 w_k^-)$ if $e_j(f_k)$ is reduced by 1 and $u_1 v_j^+ (u_1 w_k^+)$ if $e_j(f_k)$ is increased by 1. If the conditions of the standard sequences do not hold, then we delete $e_1$or f_1 for which the conditions of the standard sequences get satisfied. If this method is applied recursively, then a complete signed 3-partite graph with signed degree sequences α,β and γ is constructed.

The next result gives a necessary and sufficient conditions for the threesequences of integers to be the signed dgree sequences of some signed 3-partite graph.

Theorem 2.5. *Let $\alpha = [d_1, d_2, \cdots, d_p]$, $\beta = [e_1, e_2, \cdots, e_q]$ and $\gamma = [f_1, f_2, \cdots, f_r]$ be standard sequences . Then, α,β and γ are the signed degree sequences of a signed 3-partite graph if and only if there exist integers g and h with $d_1 = g - h$ and $0 \leq h \leq \frac{q+r-d_1}{2}$ such that α',β' and γ' are the signed degree sequences of a signed 3-partite graph, where α' is obtained from α by deleting d_1 and β' and γ' are obtained from β and γ by reducing g greatest entries ofβ and γ by 1 each and adding h least entries of β and γ by 1 each.*

Proof. Let g and h be integers with $d_1 = g - h$ and $0 \leq h \leq \frac{q+r-d_1}{2}$ such that α',β' and γ' are the signed degree sequences of a signed 3-partite graph $G'(U', V', W')$. Let $U' = \{u_1, u_2, \cdots, u_p\}$, $V' = \{v_1, v_2, \cdots, v_q\}$ and $W' = \{w_1, w_2, \cdots, w_r\}$. Let X be the set of g vertices of V', and W' with highest signed degrees; Y be the set of h vertices of V' and W' with least signed degrees and let $Z = (V' \cup W') - X - Y$. Then, a signed 3-partite graph with signed degree sequences α,β and γ can be obtained by adding a vertex u_1 to U' so that there are g positive edges from u_1 to the vertices of X and h negative edges from u_1 to the vertices of Y. Note that the signed degree of u_1 is $g - h = d_1$

Conversely, let α,β and γ be the signed degree sequences of a signed 3-partite graph. Let the vertex sets of the signed 3-partite graph be $U = \{u_1, u_2, \cdots, u_p\}$, $V = \{v_1, v_2, \cdots, v_q\}$

and $W = \{w_1, w_2, \cdots, w_r\}$ such that $sdeg(u_i) = d_i, 1 \leq i \leq p$, $sdeg(v_j) = e_j, 1 \leq j \leq q$ and $sdeg(w_k) = f_k$, $1 \leq k \leq r$ Among all the signed 3-partite graphs with α,β and γ as the signed degree sequences, let $G(U,V,W)$ be one with the property that the sum S of the signed degrees of the vertices of V and W joined to u_1 by positive edges is maximum. Let $d_1^+ = g$ and $d_1^- = h$ be resoectively the number of positive edges and the number of negative edges incident with u_1. Then,$sdeg(u_1) = d_1 = d_1^+ - d_1^-$, and $deg(u_1) = d_1^+ + d_1^- = g + h \leq q + r$, and hence $0 \leq h \leq \frac{q+r-d_1}{2}$. Let X be the set of g vertices of V and W with highest signed degrees and let $Y = (V \cup W) - X$. We claim that u_1 must be joined by positive edges to the vertices of X. If this is not true, then there exist vertices $x \in X$ and $y \in Y$ such that the edge u_1y is positive and either (i) u_1x is negative or (ii) u_1 and x are not adjacent in G(U, V, W). As $sdeg(x) \geq sdeg(y)$, therefore we consider only case (i), while as case (ii) is similar to case (i).

We note that if there exists a vertex $u_n(\neq u_1)$ such that u_nx is a positive edge and u_ny is a negative edge, then change the signs of these edges so that u_1x and u_ny are positive , and u_1y and u_nx are negative . Hence, we obtain a signed 3-partite graph with signed degree sequences α,β and γ in which the sum of the signed degrees of the vertices of V and W joined to u_1 by positive edges exceeds S, a contradiction. So, assume that no such vertex u_n exists.

Now, suppose that x is not incident to any positive edges. Since $sdeg(x) \geq sdeg(y)$, then there exists at least two verttices u_n and u_t (both distinct from u_1) such that u_ny and u_ty are negative edges and both u_n and u_t are not adjacent to x . Then, by changing the edges so that u_1x is a positive edge and u_1y , xu_n ,xu_t are negative edges, we again get a contradiction.Hence, x must be incident to at least one positive edge.

We claim that there exists at least one vertex u_l such that u_lx is a positive edge and u_l is not adjacent to y . Suppose on contrary that whenever x is joined to a vertex by a positive edge , then y is also joined to this vertex by a positive edge.Since $sdeg(x) \geq sdeg(y)$, then again we have the same situation as above, from which we get a contradiction. Thus, there exists a vertex u_l such that u_lx is a positive edge and u_l is not adjacent to y. Similalry, it can be shown that there exists a vertex u_n such that u_ny is a negative edge and u_n is not adjacent to x. By changing the edges so that u_1x, yu_l are positive edges and u_1y, xu_n are negative edges , we again get a contradiction. Thus,u_1 must be joined by positive edges to the vertex of X.

In a similar way, it can be shown that u_1 is joined by negative to the h vertices of Vand W with least signed degrees. Hence, $G(U,V,W) - u_1$ is a signed 3-partite graph with α',β' and γ' as the signed degree sequences. □

Theorem 2.5 provides an algorithm of checking whether the standard sequences α,β and γ are the signed degree sequences, and for constructing a corresponding signed 3-partite graph . Suppose $\alpha = [d_1, d_2, \cdots, d_p]$, $\beta = [e_1, e_2, \cdots, e_q]$ and $\gamma = [f_1, f_2, \cdots, f_r]$ be the standard signed degrees sequences of signed 3-partite graph with parts $U = \{u_1, u_2, \cdots, u_p\}$, $V = \{v_1, v_2, \cdots, v_q\}$ and $W = \{w_1, w_2, \cdots, w_r\}$. Let $d_1 = g - h$ and $0 \leq h \leq \frac{q+r-d_1}{2}$. Deleting d_1 and reducing g greatest entries of β and γ by 1 each and adding h least entries of β and γ by 1 each to form β' and γ'.Then, edges are defined by $u_1v_j^+(u_1w_k^+)$ if $e_j(f_k)$ is reduced by 1 ; $u_1v_j^-(u_1w_k^-)$ if $e_j(f_k)$ is increased by 1, and u_1 and $v_j(u_1$ and $w_k)$ are not adjacent if $e_j(f_k)$ are unchanged. For $-\alpha, -\beta$ and $-\gamma$, edges are defined by $u_1 v_j^-(u_1 w_k^-)$ if $e_j(f_k)$ is reduced by 1; $u_1 v_j^+(u_1 w_k^+)$ if $e_j(f_k)$ is increased by 1, and u_1 and $v_j(u_1$ and $w_k)$ are not adjacent if $e_j(f_k)$ are unchanged. If the cnditions of the standard sequences do not hold, then we delete e_1 or $f_1$for which the conditions of the standard sequences get satisfied. If this method is applied recursively, then a signed 3-partite graph with signed degrees sequences α, β and γ is constructed.

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