J. KSIAM Vol.11, No.1, 85-93, 2007

# Existence Results for First Order Impulsive Functional Differential Equations in Banach Spaces

M.Mallika Arjunan and A.Anguraj
Department of Mathematics
P.S.G.College of Arts and Science
Coimbatore- 641 014
Tamil Nadu
INDIA.

ABSTRACT

In this paper we prove the existence of mild solutions of a first order impulsive initial value problems for functional differential equations in Banach spaces. The results are obtained by using the Lerey-Schauder nonlinear alternative fixed point theorem.

2000 Subject Classification: 34A37, 34G20, 34K25.

**Keywords:** Impulsive functional differential equations, fixed point, Banach space.

## 1.INTRODUCTION

This paper is concerned with the existence of mild solutions for the impulsive functional differential equations of the form

$$y' = Ay + f(t, y_t), \quad t \in J = [0, b], \ t \neq t_k, \ k = 1, 2, ..., m \tag{1.1}$$

$$y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, 2, ..., m \tag{1.2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0] \tag{1.3}$$

where $A$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \geq 0$, $f : J \times D \to E$ is a given function, where $D = \{\psi : [-r, 0] \to E | \psi$ is continuous everywhere except for a finite number of points $s$ at which $\psi(s^-)$ and $\psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, $\phi \in C([-r, 0], E)$ and $0 = t_0 < t_1 <, ..., < t_m < t_{m+1} = b, I_k \in C(E, E)(k = 1, 2, ..., m)$, are bounded functions, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$ , respectively.

For any continuous function $y$ defined on the interval $[-r, b] - \{t_1, t_2, ..., t_m\}$ and any $t \in J$, we denote by $y_t$ the element of $C([-r, 0], E)$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0]$$

Here $y_t(.)$ represents the history of the state from time $t-r$, upto the present time $t$.

Impulsive differential equations have become more important in recent years in some mathematical models of real world phenomena, especially in the biological or medical domain(See the monographs of Bainov and Simeonov[2], Lakshmikantham, Bainov and Simeonov[10], and Samoilenko and perestyuk[12], and the papers of Agur, Cojocaru, Mazur, Anderson and Danon[1], Goldbeter, Li and Dupont[6]).
Recently, an extension to functional differential equations with impulsive effects has been done by yujun[14] by using the coincidence degree theory. For other results on functional differential equations, we refer to the monograph of Erbe, Kong and Zhang[5], Hale[7], Henderson[8].

The fundamental tools used in the existence proofs of all above-mentioned works are essentially fixed-point arguments, nonlinear alternative, topological transversality[4], topological degree theory[11], or the monotone method combined with upper and lower solutions[9].

In this paper is an extension to impulsive functional differential equations of the results[3]. My approach is here based on the Lerey-Schauder nonlinear alternative fixed point theorem.

## 2.PRELIMINARIES

In this section, we give notations, definitions, and preliminary facts which are used throughout this paper.
$C([-r,0],E)$ is the Banach space of all continuous functions from $[-r,0]$ into $E$ with the norm

$$\|\phi\| = sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

By $C(J,E)$, we denote the Banach space of all continuous functions from $J$ into $E$ with the norm

$$\|y\|_J = sup\{|y(t)| : t \in J\}.$$

Let $B(E)$ be the Banach space of bounded linear operators from $E$ into $E$.
A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral, see Yosida [13].)
$L'(J,E)$ denotes the Banach space of function $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L'} = \int_0^b |y(t)|\mathrm{d}t \quad for\ all\ y \in L'(J,E).$$

**Definition 2.1**. A map $f : J \times D \longrightarrow E$ is said to be an $L' - Caratheodory$ if
(i) $t \longrightarrow f(t,u)$ is measurable for each $u \in D$;
(ii) $u \longrightarrow f(t,u)$ is continuous for almost all $t \in J$;
(iii) for each $k > 0$, there exists $g_k \in L'(J,R_+)$ such that

$$|f(t,u)| \leq g_k(t),\ for\ all\ \|u\| \leq k\ and\ for\ almost\ all\ t \in J.$$

In order to define the mild solution of (1.1)-(1.3) we shall consider the following space

$\Omega = \{y : [-r, b] \to E : y_k \in C(J_k, E), k = 0, 1, ..., m$ and there exists $y(t_k^-)$ and $y(t_k^+)$ with $y(t_k^-) = y(t_k), k = 1, 2, ..., m, y(t) = \phi(t)$, for all $t \in [-r, 0]\}$
Which is a Banach space with the norm

$$\|y\|_\Omega = max\{\|y_k\|_\infty, k = 0, 1, ..., m\},$$

where $y_k$ is the restriction of $y$ to $J_k = [t_k, t_{k+1}], k = 0, 1, ..., m$.
Next we define the mild solution of problem (1.1)-(1.3).
**Definition 2.2**. A function $y \in \Omega$ is said to be a mild solution of (1.1)-(1.3) if $y(t) = \phi(t)$, on $[-r, 0]$, and the impulsive integral equation

$$y(t) = \begin{cases} T(t)\phi(0) + \int_0^t T(t-s)f(s, y_s)ds, & t \in [0, t_1], \\ I_k(y(t_k^-)) + \int_{t_k}^t T(t-s)f(s, y_s)ds, & t \in J_k, \ k = 1, 2, ..., m. \end{cases}$$

is satisfied.
Our main result is based on the following .
**Lemma 2.3**.(Nonlinear Alternative[4]). Let $X$ be a Banach space with $C \subset X$ closed and convex. Assume $U$ is a relatively open subset of $C$ with $0 \in U$ and $G : U^* \to C$ is a compact map.Then either
(i) $G$ has a fixed point in $U^*$; or
(ii) there is a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda G(u)$.
**Remark 2.4**. By $U^*$ and $\partial U$, we denote the closure of $U$ and the boundary of $U$, respectively.
We are now in a position to state and prove our existence result for IVP (1.1)-(1.3). For the study of this problem we first list the following hypotheses.

(H1) $A$ is the infinitesimal generator of a linear bounded compact semigroup $T(t)$, $t \geq 0$ and there exists $M \geq 1$ such that $|T(t)|_{B(E)} \leq M$;

(H2) $f : J \times D \to E$ is an $L'$ - Caratheodory map ;

(H3) there exists a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and $p \in L'(J, R_+)$ such that

$$|f(t, u)| \leq p(t)\psi(\|u\|),$$

for a.e $t \in J$ and each $u \in D$ with

$$\int_{t_{k-1}}^{t_k} p(s)ds < \int_{N_{k-1}}^{\infty} \frac{d\tau}{\psi(\tau)}, \ k = 1, 2, ..., m+1,$$

where $N_0 = M\|\phi\|$, and for k=2,3,...,m+1, we have

$$N_{k-1} = \sup_{y \in [-M_{k-2}, M_{k-2}]} |I_{k-1}(y)|, \quad M_{k-2} = \Gamma_{k-1}^{-1}\left(M\int_{t_{k-2}}^{t_{k-1}} p(s)ds\right)$$

with

$$\Gamma_l(z) = \int_{N_{l-1}}^{z} \frac{d\tau}{\psi(\tau)}, \quad z \geq N_{l-1}, \ l \in \{1, 2, ..., m+1\}$$

(H4) for each bounded $B \subseteq C([-r, b], E)$ and the set

$$\begin{cases} T(t)\phi(0) + \int_0^t T(t-s)f(s, y_s)\mathrm{d}s, & t \in [0, t_1], \\ I_k(y(t_k^-)) + \int_{t_k}^t T(t-s)f(s, y_s)\mathrm{d}s, & t \in J_k, \ k = 1, 2, ..., m, : y \in B. \end{cases}$$

is relatively compact in $E$.

## 3.MAIN RESULT

**Theorem 3.1**. Assume that hypotheses (H1)-(H4) hold. Then the IVP (1.1)-(1.3) has atleast one mild solution $y \in \Omega$.

**proof** . The proof is given in several steps.
**Step I**. Consider the problem (1.1)-(1.3) on $[-r, t_1]$

$$y' = Ay + f(t, y_t), \quad a.e \quad t \in J_0 \tag{3.1}$$

$$y_0 = \phi \tag{3.2}$$

We will show that the possible mild solutions of (3.1)-(3.2) and a *priori* bounded, that is, there exists a constant $B_0$ such that, if $y \in \Omega$ is a mild solution on (3.1)-(3.2), then

$$sup\{|y(t)| : t \in [-r, 0] \cup [0, t_1]\} \leq B_0.$$

Let $y$ be a mild solution to $(3.1) - (3.2)$. Then for each $t \in [0, t_1]$

$$y(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s, y_s)\mathrm{d}s.$$

From (H3) we get

$$|y(t)| \leq M\|\phi\| + M \int_0^t p(s)\psi(\|y_s\|)\mathrm{d}s, \quad t \in [0, t_1] \tag{3.3}$$

we consider the function $\mu_0$ defined by

$$\mu_0(t) = sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq t_1.$$

Let $t^* \in [-r, t]$ be such that $\mu_0(t) = |y(t^*)|$. If $t^* \in [0, t_1]$, by the previous inequality (3.3), we have for $t \in [0, t_1]$

$$\mu_0(t) \leq M\|\phi\| + M \int_0^t p(s)\psi(\mu_0(s))\mathrm{d}s.$$

If $t^* \in [-r, 0]$ then $\mu_0(t) = \|\phi\|$ and the previous inequality holds if $M \geq 1$.
Let us take the right-hand side of the above inequality as $v_0(t)$, then we have

$$v_0(0) = M\|\phi\| = N_0, \qquad \mu_0(t) \leq v_0(t), \quad t \in [0, t_1]$$

and

$$v_0'(t) = Mp(t)\psi(\mu_0(t)), \quad t \in [0, t_1].$$

Using the nondecreasing character of $\psi$ we get

$$v_0'(t) \leq Mp(t)\psi(v_0(t)), \quad t \in [0, t_1].$$

This implies that for each $t \in [0, t_1]$ that

$$\int_{N_0}^{v_0(t)} \frac{d\tau}{\psi(\tau)} \leq M \int_0^{t_1} p(s)\mathrm{d}s.$$

In view of (H3), we get

$$|v_0(t^*)| \leq \Gamma_1^{-1}\left(M \int_0^{t_1} p(s)\mathrm{d}s\right) = M_0$$

Since for every $t \in [0, t_1]$, $\|y_t\| \leq \mu_0(t)$, we have

$$\sup_{t \in [-r, t_1]} |y(t)| \leq max\{\|\phi\|, M_0\} = B_0.$$

We transform the problem into a fixed point problem. Consider the map $G : C([-r, t_1], E) \to C([-r, t_1], E)$ defined by

$$(Gy)(t) = \begin{cases} \phi(t) & t \in [-r, 0] \\ T(t)\phi(0) + \int_0^t T(t-s)f(s, y_s)\mathrm{d}s, & t \in J_0. \end{cases}$$

We shall show that $G$ satisfies the assumptions of Lemma 2.3. The proof will be given in several steps.

**Step 1**. $G$ maps bounded sets into bounded sets in $C(J_0, E)$.

Let $B_q = \{y \in C(J_0, E) : \|y\|_\infty \leq q\}$ be a bounded set in $C(J_0, E)$ and $y \in B_q$, then for each $t \in J_0$, we have

$$(Gy)(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s, y_s)\mathrm{d}s, \quad t \in J_0.$$

Thus, we have for each $t \in J_0$

$$\begin{aligned} |(Gy)(t)| &\leq M\|\phi\| + M\int_0^t |f(s, y_s)|\mathrm{d}s \\ &\leq M\|\phi\| + M\int_0^t |g_q(s)|\mathrm{d}s \\ &\leq M\|\phi\| + M\|g_q\|_{L'}. \end{aligned}$$

**Step 2**. $G$ maps bounded sets in $C(J_0, E)$ into equicontinuous sets.

Let $r_1, r_2 \in J_0, r_1 < r_2$, and $B_q = \{y \in C(J_0, E) : \|y\|_\infty \leq q\}$ be a bounded set in $C(J_0, E)$ . Let $y \in B_q$. Then

$$(Gy)(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s, y_s)\mathrm{d}s, \quad t \in J_0.$$

Hence,

$$\begin{aligned} |(Gy)(r_2) - (Gy)(r_1)| &\leq |T(r_2) - T(r_1)||\phi(0)| \\ &\quad + |\int_0^{r_2} [T(r_2 - s) - T(r_1 - s)]f(s, y_s)\mathrm{d}s| \\ &\quad + |\int_{r_1}^{r_2} T(r_1 - s)f(s, y_s)\mathrm{d}s| \\ &\leq |T(r_2) - T(r_1)||\phi(0)| \\ &\quad + \int_0^{r_2} [T(r_2 - s) - T(r_1 - s)]f(s, y_s)\mathrm{d}s \\ &\quad + M\int_{r_1}^{r_2} |g_q(s)|\mathrm{d}s \end{aligned}$$

As $r_2 \to r_1$, the right hand side of the above inequality tends to zero.
The equicontinuity for the cases $r_1 < r_2 \leq 0$ and $r_1 \leq 0 \leq r_2$ are obvious.

**Step 3**. $G : C(J_0, E) \to C(J_0, E)$ is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J_0, E)$. Then there is an integer $q$ such that $\|y_n\|_\infty \leq q$ for all $n \in N$ and $\|y\|_\infty \leq q$, so $y_n \in B_q$ and $y \in B_q$.
From Dominated convergence theorem,

$$\|Gy_n - Gy\|_\infty \leq \sup_{t \in J_0} \left[\int_0^t |T(t-s)||f(s, y_{ns}) - f(s, y_s)|\mathrm{d}s\right] \to 0 \quad as \quad n \to \infty.$$

Thus $G$ is continuous.
Set

$$U = \{y \in C([-r, t_1], E) : \|y\|_\infty < B_0 + 1\}$$

As a consequence of Step 2, Step 3, and in the view of (H4) together with the Arzela-Ascoli theorem, we conclude that the map $G : U^* \to C([-r, t_1], E)$ is compact. From the choice of $U$ there is no $y \in \partial U$ such that $y = \lambda G(y)$ for any $\lambda \in (0, 1)$. As a consequence of Lemma 2.3, we deduce that $G$ has a fixed point $y_0 \in U^*$ which is a mild solution of (3.1)-(3.2).

**Step II**. Now consider the problem on $J_1 = [t_1, t_2]$.

$$y' = Ay + f(t, y_t), \quad a.e\ t \in J_1, \tag{3.4}$$

$$y(t_1^+) = I_1(y(t_1^-)) \tag{3.5}$$

Let $y$ be a mild solution to (3.4)-(3.5). Then for each $t \in [t_1, t_2]$

$$y(t) = I_1(y(t_1^-)) + \int_{t_1}^{t} T(t-s)f(s, y_s)ds.$$

Note that

$$|y(t_1^+)| \leq \sup_{t \in [-M_0, M_0]} |I_1(y_0(t^-))| = N_1$$

From (H3), we get

$$|y(t)| \leq N_1 + M \int_{t_1}^{t} p(s)\psi(\|y_s\|)ds, \quad t \in [t_1, t_2].$$

we consider the function $\mu_1$ defined by

$$\mu_1(t) = sup\{|y(s)| : t_1 \leq s \leq t\}, \quad t_1 \leq t \leq t_2.$$

Let $t^* \in [t_1, t]$ be such that $\mu_1(t) = |y(t^*)|$. Then we have for each $t \in [t_1, t_2]$

$$\mu_1(t) \leq N_1 + M \int_{t_1}^{t} p(s)\psi(\mu_1(s))ds. \tag{3.6}$$

Let us take the right hand side of the above inequality as $v_1(t)$, then we have

$$v_1(t_1) = N_1, \quad \mu_1(t) \leq v_1(t), \quad t \in [t_1, t_2]$$

and

$$v_1'(t) = Mp(t)\psi(\mu_1(t)), \quad t \in [t_1, t_2].$$

Using the nondecreasing character of $\psi$ we get

$$v_1'(t) \leq Mp(t)\psi(v_1(t)), \quad t \in [t_1, t_2].$$

This implies for each $t \in [t_1, t_2]$ that

$$\int_{N_1}^{v_1(t)} \frac{d\tau}{\psi(\tau)} \leq M \int_{t_1}^{t_2} p(s)ds.$$

In view of (H3), we obtain

$$|v_1(t^*)| \leq \Gamma_2^{-1}\left(M\int_{t_1}^{t_2} p(s)\mathrm{d}s\right) = M_1$$

Since for every $t \in [t_1, t_2]$, $\|y_t\| \leq \mu_1(t)$, we have

$$\sup_{t\in[t_1,t_2]} |y(t)| \leq M_1.$$

A mild solution to (3.4)-(3.5) is a fixed point of the operator $G : C(J_1, E) \to C(J_1, E)$ defined by

$$G(y) = I_1(y(t_1^-)) + \int_{t_1}^{t} T(t-s)f(s, y_s)\mathrm{d}s$$

Set

$$U = \{y \in C(J_1, E) : \|y\|_\infty < M_1 + 1\}.$$

As in Step I, we can show that(with obvious modifications) $G : U^* \to C(J_1, E)$ is compact.
From the choice of $U$ there is no $y \in \partial U$ such that $y = \lambda G(y)$ for any $\lambda \in (0, 1)$.
As a consequence of Lemma 2.3, we deduce that $G$ has a fixed point $y_1 \in U^*$ which is a mild solution of (3.4)-(3.5).

**Step III**. We continue this process and taking account that $y_k \in C(J_k, E)$, k= 2,3,...,m to

$$y'(t) = Ay(t) + f(t, y_t), \quad a.e\ t \in J_k, \tag{3.7}$$

$$y(t_k^+) = I_k(y(t_k^-)). \tag{3.8}$$

Then

$$y(t) = \begin{cases} y_0(t), & t \in [-r, t_1] \\ y_1(t), & t \in (t_1, t_2] \\ y_2(t), & t \in (t_2, t_3] \\ . & \\ . & \\ . & \\ y_{m-1}(t), & t \in (t_{m-1}, t_m] \\ y_m(t), & t \in (t_m, b] \end{cases}$$

is a mild solution to the problem (1.1)-(1.3).

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