Computing the Average Symbol Error Probability of the MPSK System Having Quadrature Error

Seungkeun Park and Sung Ho Cho

ABSTRACT—When quadrature error exists, the shape of the M-ary phase shift keying (MPSK) signal constellation becomes skewed-elliptic. Each MPSK symbol takes on a different symbol error probability (SEP) value. The analytical results presented thus far have been derived from studies which examined the SEP problem assuming that the SEP of each MPSK symbol is equally likely; therefore, those results should not be treated as offering a complete solution. In this letter, we present a new and more complete solution to the SEP problem of MPSK by relaxing the above assumption and finding the expressions for the average as well as individual SEP in the presence of quadrature error.

Keywords—M-ary phase shift keying, symbol error probability, quadrature error.

I. Introduction

The symbol error probability (SEP) performance of M-ary phase shift keying (MPSK) in the presence of quadrature error has been studied in [1]-[3]. It has been shown that the quadrature error is generated by phase shifts other than 90^{o} between the I and Q paths due to the imperfect local ocsillator in the radio frequency circuits. A quadrature error thus results in the incorrect positioning of a received signal vector at the demodulator.

The first analytical result was presented in [1] by Simon and Divsalar, where the SEP performance was investigated only for the particular MPSK symbol having the phase angle 0 or π. Using the same framework as in [1], an alternative expression for the SEP of MPSK was developed in [2]. The joint SEP performance of MPSK in the presence of phase/quadrature error and I-Q gain mismatch were examined in [3].

The results in [1]-[3] were obtained using the receiver model in which the shape of the MPSK signal constellation is viewed as a perfect circle even in the presence of quadrature error. It is, however, important to note that the signal constellation of MPSK changes its shape to a skewed ellipse whenever the system experiences quadrature error (see Fig. 1 for the case of M=8 as an example). Therefore, the results in [1]-[3] are valid only for cases in which the transmitted MSPK signal has the phase angle 0 or π, and should not be accepted as a complete solution to the SEP problem of MPSK. More recently, an expression for the average SEP of MPSK in the presence of I/Q phase unbalance was presented in [4].

In this letter, we focus on finding a complete solution to the SEP problem of MPSK in the presence of quadrature error. Employing the two-dimensional (2-D) Gaussian *Q*-function representation of the probability of an arbitrary wedge-shaped region developed in [4], we present new expressions for the average SEP of MPSK as well as the individual SEP of each MPSK signal. Computer simulations show very good agreement between the analytical and empirical results.

II. The Average SEP of MPSK in the Presence of Quadrature Error

Consider the *i*-th received signal vector $\mathbf{S_i}=(X_i, Y_i)$, $i=1,\cdots, M$, of MPSK over an additive white Gaussian noise (AWGN) channel given in [1] as

$$X_i = \overbrace{\sqrt{E_s}\cos\frac{2\pi(i-1)}{M}}^{\mu_{X_i}} + N_X,$$
$$Y_i = \overbrace{\sqrt{E_s}\sin\left[\frac{2\pi(i-1)}{M} + \phi_u\right]}^{\mu_{Y_i}} + N_Y, \tag{1}$$

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where E_s denotes the signal energy, ϕ_u denotes an quadrature error, and the two noise random variables N_X and N_Y are jointly Gaussian with zero mean and equal variance σ^2 such that $\sigma^2 = N_0/2$. Note that the noise contour of equal probability density for (N_X, N_Y) becomes skewed-elliptic in the presence of quadrature error and the correlation coefficient between N_X and N_Y is given by $\sin\phi_u$.

The mean of the i-th received signal vector, $\bar{\mathbf{S}}_i = (\mu_{X_i}, \mu_{Y_i})$, is characterized by the magnitude μ_i and the phase angle ϕ_i such that

$$\mu_i = \sqrt{\mu_{X_i}^2 + \mu_{Y_i}^2} = \sqrt{E_S \cos^2 \frac{2\pi(i-1)}{M} + E_S \sin^2\left[\frac{2\pi(i-1)}{M} + \phi_u\right]}, \quad (2)$$

$$\phi_i = \tan^{-1}\frac{\mu_{Y_i}}{\mu_{X_i}} = \tan^{-1}\left\{\frac{\sin\left[\frac{2\pi(i-1)}{M} + \phi_u\right]}{\cos\frac{2\pi(i-1)}{M}}\right\}. \quad (3)$$

Figure 1 illustrates an example of the signal constellation of $\bar{\mathbf{S}}_i = (\mu_{X_i}, \mu_{Y_i})$ for 8-PSK.

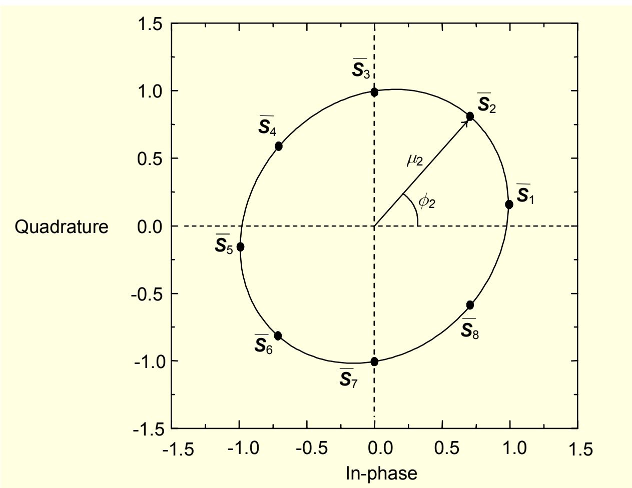

Fig. 1. Example of a skewed-elliptic constellation for $\bar{\mathbf{S}}_i = (\mu_{X_i}, \mu_{Y_i})$ for $i=1, 2, \cdots, 8$, when $E_s/N_0=1$ and quadrature error $\phi_u = \pi/20$.

Now, the i-th wedge-shaped correct region C_i, $i = 1, 2, \cdots, M$, for $\mathbf{S}_i=(X_i, Y_i)$ is given by

$$C_i \triangleq \left\{(X_i Y_i) \middle| \frac{\pi(2i-3)}{M} < \tan^{-1}\frac{Y_i}{X_i} < \frac{\pi(2i-1)}{M}\right\}. \quad (4)$$

We are able to compute the probabilities for C_i, $\Pr\{(X_i, Y_i) \in C_i\}$, $i = 1, 2, \cdots, M$, by making use of the 2-D Gaussian Q-function representation for the probability on an arbitrary wedge-shaped region of MPSK [5] as

$$\Pr\{(X_i, Y_i) \in C_i\} = Q\left(a_i\sqrt{\frac{2E_S}{N_0}}, b_i\sqrt{\frac{2E_S}{N_0}}; \rho_i\right), \quad (5)$$

where

$$a_i = -\sin\left(\phi_i - \frac{\pi(2i-3)}{M}\right) \cdot \sqrt{\frac{\cos^2\frac{2\pi(i-1)}{M} + \sin^2\left(\frac{2\pi(i-1)}{M} + \phi_u\right)}{1 - \sin\phi_u \sin\frac{2\pi(2i-3)}{M}}}, \quad (6)$$

$$b_i = -\sin\left(\phi_i - \frac{\pi(2i-1)}{M}\right) \cdot \sqrt{\frac{\cos^2\frac{2\pi(i-1)}{M} + \sin^2\left(\frac{2\pi(i-1)}{M} + \phi_u\right)}{1 - \sin\phi_u \sin\frac{2\pi(2i-1)}{M}}}, \quad (7)$$

$$\rho_i = \frac{-\left(\cos\frac{2\pi}{M} - \sin\phi_u \sin\frac{2\pi(i-2)}{M}\right)}{\sqrt{\left(1 - \sin\phi_u \sin\frac{2\pi(2i-1)}{M}\right)\left(1 - \sin\phi_u \sin\frac{2\pi(2i-3)}{M}\right)}}. \quad (8)$$

Note that each ϕ_i in (6) and (7) takes on the value in $(0, 2\pi)$, and thus the range of the arctangent function in (3) should be modified from $(-\pi/2, \pi/2)$ to $(0, 2\pi)$. For this, we develop a formula for the arctangent function as

$$\tan^{-1}\left(\frac{y}{x}\right) \triangleq \frac{\pi}{2}[1 - \mathrm{sgn}(x)] + \mathrm{sgn}(x)\tan^{-1}\left(\frac{y}{|x|}\right) + \frac{\pi}{2}[1 - \mathrm{sgn}(x)\mathrm{sgn}(xy)][1 + \mathrm{sgn}(y)\mathrm{sgn}(xy)]. \quad (9)$$

After modifying (3) using (9), and substituting the result into (6) and (7), we are ready to compute the individual SEP for each MPSK signal. The SEP of the i-th signal $\mathbf{S}_i$ becomes

$$P(E|\mathbf{S}_i) = 1 - Q\left(a_i\sqrt{\frac{2E_S}{N_0}}, b_i\sqrt{\frac{2E_S}{N_0}}; \rho_i\right). \quad (10)$$

When the MPSK signals are equally likely to take place, the overall average SEP, $P(E)$, of MPSK in the presence of quadrature error can finally be written as

$$P(E) = 1 - \frac{1}{M}\sum_{i=1}^{M} Q\left(a_i\sqrt{\frac{2E_S}{N_0}}, b_i\sqrt{\frac{2E_S}{N_0}}; \rho_i\right). \qquad (11)$$

Note that the average bit error rate of MPSK in the presence of quadrature error can also be obtained by combining our analytical result (5) with (2) in [6].

III. Numerical Results

We first want to show that the SEP analysis presented in [1] and [2] is valid only for the particular MPSK signal having the phase angle 0 or π. For this, we compute the empirical values of the SEP for the received signals $\mathbf{S}_2$ and $\mathbf{S}_5$ in 8-PSK, where the modulated phase angles are given by π/4 and π, respectively (see Fig. 1). The quadrature error is chosen to be $\phi_u = \pi/20$. Figure 2 illustrates the experimental SEP curves for the signals $\mathbf{S}_2$ and $\mathbf{S}_5$, and the theoretical SEP curve obtained using (70) in [1] and (19) in [2]. We can see from the figure that the analytical SEP results presented in [1] and [2] agree only with the experimental SEP result for the signal $\mathbf{S}_5$, but not for the signal $\mathbf{S}_2$. In fact, in 8-PSK, the results in [1] and [2] match with the SEP performance only for the signals $\mathbf{S}_1$ and $\mathbf{S}_5$.

Next, in order to demonstrate the validity of our analytical results, we compute the numerical values of the SEP using (10) and (11) for 8-PSK, and compare them with those obtained by the simulations. The quadrature error is once again chosen to be $\phi_u = \pi/20$. Figure 3 illustrates the theoretical SEP curves for the signals $\mathbf{S}_2$ and $\mathbf{S}_5$ using (10) and their simulation counterparts. Also plotted in the figure is the theoretical average SEP curve for all the signals using (11) and its simulation counterpart. It is observed that our analytical results presented in (10) and (11) agree very well with those obtained by simulation.

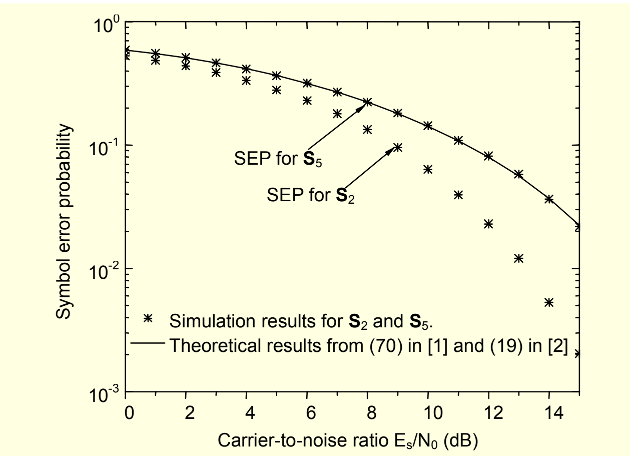

Fig. 2. SEP results for the signals $\mathbf{S}_2$ and $\mathbf{S}_5$ in 8-PSK when $\phi_u = \pi/20$.

IV. Conclusion

In this letter, we have presented a new and complete solution to the SEP problem of MPSK in the presence of quadrature error. We have shown that when quadrature error exists, the signal constellation of MPSK becomes skewed-elliptic, and each MPSK signal takes on a different SEP value. Analytical expressions for the average SEP of MPSK as well as the individual SEP of each MPSK signal have been presented. Our analytical results have been verified by computer simulations.

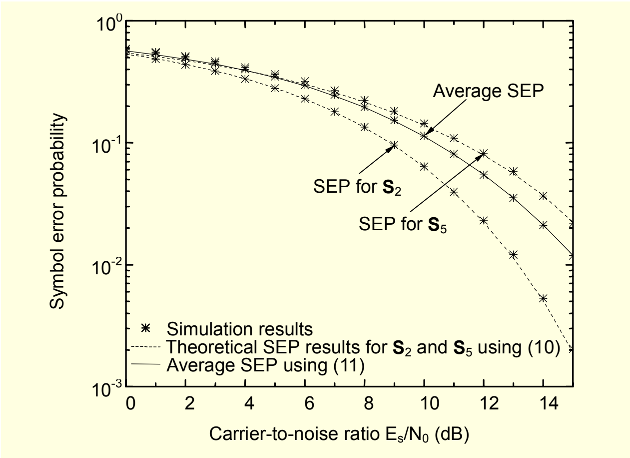

Fig. 3. Various SEP results of 8-PSK when $\phi_u = \pi/20$.

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