# On Strongly Nonlinear Implicit Complementarity Problems in Hilbert Spaces

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Abstract. In this paper, we study a class of strongly nonlinear implicit complementarity problems in the setting of Hilbert spaces $H$ (not necessarily Hilbert lattices). By using the property of the projection and a suitable change of variables, we establish the equivalence between the strongly nonlinear implicit complementarity problem and the fixed point problem in $H$.

Moreover, we use this equivalence and the fixed point theorem of Boyd and Wong to prove the existence and uniqueness of solutions for the strongly nonlinear implicit complementarity problem in $H$.

## 1. Introduction

Complementarity problem theory, introduced and studied by Lemke [12] and Cottle and Dantzig [7] in the 1960s and later developed by others, plays an important and fundamental role in the study of a wide class of problems arising in mechanics, physics, nonlinear programming, optimization and control, economics and transportation equilibrium, contact problems in elasticity, fluid flow through porous media, and many other branches of mathematical and engineering sciences (see [1], [5], [6], [8]-[11], [13]-[16] and the references therein).

In 1988, Noor [14] used the technique of change of variables to study some classes of complementarity problems in finite-dimensional space $R^n$. Recently, Ahmad, Kazmi and Rehman [1] were first to use the concept of change of variables to study a class of implicit complementarity problems in the setting of Hilbert lattices.

Motivated and inspired by recent research work in this field, in this paper,

we study a class of strongly nonlinear implicit complementarity problems in the setting of Hilbert spaces $H$ (not necessarily Hilbert lattices). In Section 2, by using the property of the projection and a suitable change of variables, we establish the equivalence between the strongly nonlinear implicit complementarity problem and the fixed point problem in $H$. In Section 3, we use this equivalence and the fixed point theorem of Boyd and Wong to prove the existence and uniqueness of solutions for the strongly nonlinear implicit complementarity problem in $H$.

## 2. Preliminaries

Let $H$ be a real Hilbert space endowed with the norm $\| \cdot \|$ and inner product $(\cdot, \cdot)$, respectively. If $K$ is a closed convex cone in $H$, we denote by $K^*$ the polar cone of $K$, i.e.,
$$K^* = \{u \in H : (u, v) \geq 0 \quad \text{for all } v \in K\}.$$

Given nonlinear mappings $f, h, g : D \to H$ and $N : H \times H \to H$, where $D$ is a nonempty subset of $H$, we consider the following problems:

The strongly nonlinear explicit complementarity problem (in short, SNECP) consists in finding $z \in H$ such that

$$(1) \qquad z \in K, \quad N(f(z), h(z)) \in K^*, \quad (z, N(f(z), h(z)) = 0.$$

The strongly nonlinear implicit complementarity problem (in short, SNICP) consists in finding $z \in H$ such that

$$(2) \qquad z \in D, \quad g(z) \in K, \quad N(f(z), h(z)) \in K^*, \quad (g(z), N(f(z), h(z)) = 0.$$

If $N(s, t) = s + t$ for all $s, t \in H$ and $h = 0$ in (1), then the problem (1) is reduced to the problem finding $z \in H$ such that

$$(3) \qquad z \in K, \quad f(z) \in K^*, \quad (z, f(z)) = 0,$$

and the problem (2) is reduced to the problem finding

$$(4) \qquad z \in D, \quad g(z) \in K, \quad f(z) \in K^*, \quad (g(z), f(z)) = 0,$$

respectively.

The problem (4) is called the implicit complementarity problem considered by Ahmad, Kazmi and Rehman [1] by using the concept of change of variables and the fixed point technique in the setting of Hilbert lattices.

When it is necessary to point out some data of the problems mentioned above, we will write the SNECP$(f, h, N, K)$ and the SNICP$(f, h, g, N, K)$ instead of the SNECP and the SNICP, respectively.

It is well known that the implicit complementarity problem arises in stochastic impulse optimal control problems and it was first considered by Bensoussan and Lions [2] and studied by Capuzzo Dolcetta and Mosco [4], and Isac [11].

The following results are important for our results.

**Lemma 2.1 ([17]).** *If $K$ is a closed convex cone of $H$ and $P_K$ denotes the projection of $H$ onto $K$, then, for each $x \in H$, $P_K(x)$ is characterized by the following properties:*

(i) $(P_K(x) - x, y) \geq 0$  *for all* $y \in K$,

(ii) $(P_K(x) - x, P_K(x)) = 0$.

**Lemma 2.2 ([5], [8]).** *The mapping $P_K$ is nonexpansive, i.e.,*

$$\|P_K u - P_K v\| \leq \|u - v\|$$

*for all $u, v \in H$.*

Let $H$ be a real Hilbert space (not necessarily a Hilbert lattice) and $D$ be a nonempty subset of $H$. Let $f, h, g : D \to H$ and $N : H \times H \to H$ be the nonlinear mappings such that $K \subset g(D)$. Consider two mappings $T, S : K \to H$ defined by $T(u) = f(z)$ and $S(u) = h(z)$, respectively, where $u \in K$ and $z$ is an arbitrary element of

$$g^{-1}(u) = \{z \in D : g(z) = u\}.$$

From the assumption $K \subset g(D)$, we know that $T$ and $S$ are both well defined. It is easy to see that the SNICP$(f, h, g, N, K)$ is now equivalent to the SNECP$(T, S, N, K)$ or to the following:

Find $u \in H$ such that

$$u \in K, \quad v = N(Tu, Su) \in K^*, \quad (u, v) = 0.$$

From Lemma 2.1, we now consider the following change of variables. For all $x \in H$, set

(5) $$u = P_K(x), \quad v = \frac{1}{\rho}(P_K(x) - x),$$

where $\rho > 0$ is a constant. It follows from (5) and Lemma 2.1 that $u \in K$, $v \in K^*$ and $(u, v) = 0$. Therefore, we have the following:

**Lemma 2.3.** *Let $H$ be a real Hilbert space (not necessarily a Hilbert lattice) and $D$ be a nonempty subset of $H$. Let $f, h, g : D \to H$ and $N : H \times H \to H$ be the nonlinear mappings such that $K \subset g(D)$. Consider two mappngs $T, S : K \to H$ defined by $T(u) = f(z)$ and $S(u) = h(z)$, respectively, where $u \in K$ and $z$ is an arbitrary element of*

$$g^{-1}(u) = \{z \in D : g(z) = u\}.$$

*Then the SNICP$(f, h, g, N, K)$ is equivalent to the fixed point problem*

(6) $$x = F(x),$$

*where*

(7)                    $$F(x) = P_K(x) - \rho N(T(P_K(x)), S(P_K(x)))$$

*and $\rho > 0$ is a constant.*

## 3. The existence and uniqueness theorems

We first give some definitions for our results.

**Definition 3.1.** Let $H$ be a real Hilbert space and $D$ be a nonempty subset of $H$. Let $f, h, g : D \to H$; $N : H \times H \to H$ and $\phi, \psi, \varphi : [0, \infty) \to [0, \infty)$ be the nonlinear mappings. We say that:

(i) The mapping $x \mapsto N(f(x), h(y))$ is a $\phi$-Lipschitz continuous with respect to $g$ if

$$\|N(f(x_1), h(y)) - N(f(x_2), h(y))\|$$
$$\leq \quad \|g(x_1) - g(x_2)\|\phi(\|g(x_1) - g(x_2)\|)$$

for all $x_1, x_2, y \in D$,

(ii) The mapping $y \mapsto N(f(x), h(y))$ is a $\varphi$-Lipschitz continuous with respect to $g$ if

$$\|N(f(x), h(y_1)) - N(f(x), h(y_2))\|$$
$$\leq \quad \|g(y_1) - g(y_2)\|\varphi(\|g(y_1) - g(y_2)\|)$$

for all $y_1, y_2, x \in D$,

(iii) The mapping $x \mapsto N(f(x), h(y))$ is a $\psi$-strongly monotone with respect to $g$ if

$$(N(f(x_1), h(y)) - N(f(x_2), h(y)), g(x_1) - g(x_2))$$
$$\geq \quad \|g(x_1) - g(x_2)\|^2 \psi(\|g(x_1) - g(x_2)\|)$$

for all $x_1, x_2, y \in D$.

**Definition 3.2.** A metric space $(X, d)$ is said to be metrically convex if, for any $x, y \in X$ $(x \neq y)$, there exists $z$ $(z \neq x, y)$ such that

$$d(x, y) = d(x, z) + d(z, y).$$

We will use the set

$$Q = \{d(x, y) : x, y \in X\}.$$

The following result is important for proving an existence theorem:

**Lemma 3.1 ([3]).** *Let $(X, d)$ be a complete metrically convex metric space and assume that, for any mapping $A : X \to X$, there exists a mapping $\Phi : Q \to [0, \infty)$ such that*

(i) $d(Ax, Ay) \leq \Phi(d(x, y))$   *for all*  $x, y \in X$,

(ii) $\Phi(t) < t$   *for all*  $t \in \overline{Q} \setminus \{0\}$.

*Then $A$ has a unique fixed point $x^*$ in $X$ and $A^n x_0 \to x^*$ for any $x_0 \in X$, where $\overline{Q}$ denotes the closure of $Q$ and $\{A^n x_0\}$ is the sequence of successive approximations.*

Now, we give our main theorem:

**Theorem 3.1.** *Let $H$ be a real Hilbert space (not necessarily a Hilbert lattice) and $D$ be a nonempty subset of $H$. Let $f, h, g : D \to H$ and $N : H \times H \to H$ be the nonlinear mappings such that $K \subset g(D)$ and*

(i)  *The mapping $x \mapsto N(f(x), h(y))$ is $\phi$-Lipschitz continuous with respect to $g$,*

(ii)  *The mapping $y \mapsto N(f(x), h(y))$ is $\varphi$-Lipschitz continuous with respect to $g$,*

(iii)  *The mapping $x \mapsto N(f(x), h(y))$ is $\psi$-strongly monotone with respect to $g$,*

(iv)  *$\phi$ and $\varphi$ are two increasing mappings and $\psi$ is a decreasing mapping.*

*If there exists a real number $\rho > 0$ such that*

$$(8) \qquad \begin{cases} \rho(\phi^2(t) - \varphi^2(t)) < 2(\psi(t) - \varphi(t)), \\ \varphi(t) < \psi(t) \leq \phi(t), \quad \rho\varphi(t) < 1 \end{cases}$$

*for all $t \in [0, \infty)$, then the SNICP$(f, h, g, N, K)$ has a solution. In addition, if $g$ is a one-to-one mapping, then the solution is unique.*

*Proof.* By the definitions of $T$ and $S$ and the conditions (i)$\sim$(iii), we know that

(a) The mapping $u \mapsto N(T(u), S(v))$ is $\phi$-Lipschitz continuous, i.e.,

$$\|N(T(u_1), S(v)) - N(T(u_2), S(v))\|$$
$$\leq \quad \|u_1 - u_2\|\phi(\|u_1 - u_2\|)$$

for all $u_1, u_2, v \in K$,

(b) The mapping $v \mapsto N(T(u), S(v))$ is $\varphi$-Lipschitz continuous, i.e.,

$$\|N(T(u), S(v_1)) - N(T(u), S(v_2))\|$$
$$\leq \quad \|v_1 - v_2\|\varphi(\|v_1 - v_2\|)$$

for all $v_1, v_2, u \in K$,

(c) The mapping $u \mapsto N(T(u), S(v))$ is $\psi$-strongly monotone, i.e.,

$$(N(T(u_1), S(v)) - N(T(u_2), S(v)), u_1 - u_2)$$
$$\geq \quad \|u_1 - u_2\|^2 \psi(\|u_1 - u_2\|)$$

for all $u_1, u_2, v \in K$.

From Lemma 2.3, it follows that the $\mathrm{SNICP}(f, h, g, N, K)$ has a solution if and only if the mapping $F$ defined by (7) has a fixed point. For all $x,\ y \in H$, we have

(9) $\quad \|F(x) - F(y)\|$

$= \quad \|P_K(x) - P_K(y) - \rho(N(T(P_K(x)), S(P_K(x))) - N(T(P_K(y)), S(P_K(y))))\|$

$\leq \quad \|P_K(x) - P_K(y) - \rho(N(T(P_K(x)), S(P_K(x))) - N(T(P_K(y)), S(P_K(x))))\|$
$\quad + \rho\|N(T(P_K(y)), S(P_K(x))) - N(T(P_K(y)), S(P_K(y)))\|.$

It follows from (a)∼(c) that

(10) $\quad \|P_K(x) - P_K(y) - \rho(N(T(P_K(x)), S(P_K(x))) - N(T(P_K(y)), S(P_K(x))))\|^2$

$\leq \quad \|P_K(x) - P_K(y)\|^2 - 2\rho\psi(\|P_K(x) - P_K(y)\|)\|P_K(x) - P_K(y)\|^2$
$\quad + \rho^2\phi^2(\|P_K(x) - P_K(y)\|)\|P_K(x) - P_K(y)\|^2$

and

(11) $\quad\quad\quad \|N(T(P_K(y)), S(P_K(x))) - N(T(P_K(y)), S(P_K(y)))\|$

$\leq \quad \varphi(\|P_K(x) - P_K(y)\|)\|P_K(x) - P_K(y)\|.$

Since $\phi$ and $\varphi$ are both the increasing mappings and $\psi$ is the decreasing mapping, from (9)∼(11) and Lemma 2.2, we have

(12) $\quad\quad\quad\quad \|F(x) - F(y)\| \leq \Psi(\|x - y\|)\|x - y\|$

for all $x, y \in H$, where

(13) $\quad\quad\quad\quad \Psi(t) = \sqrt{1 - 2\rho\psi(t) + \rho^2\phi^2(t)} + \rho\varphi(t)$

for all $t \in [0, \infty)$. By the condition (8), $\Psi(t) < t$ for all $t \in [0, \infty)$. Now, let $\Phi(t)$ in Lemma 3.1 be defined by $\Phi(t) = t\Psi(t)$ for all $t \in [0, \infty) \setminus \{0\}$. Then we have

(14) $\quad\quad\quad\quad\quad\quad \Phi(t) < t$

for all $t \in [0, \infty) \setminus \{0\}$. Moreover, the Hilbert space $H$ is a complete metrically convex metric space. It follows from (12)∼(14) and Lemma 3.1 that $F$ has a unique fixed point in $H$ and so the $\mathrm{SNICP}(f, h, g, N, K)$ has a solution. Furthermore, if $g$ is a one-to-one mapping, it is easy to see that the solution is unique. This completes the proof. $\quad\quad\square$

**Example 3.1.** Let $H = (-\infty, +\infty)$, $K = [0, +\infty)$ and $D = (0, +\infty)$. Let

$$f(x) = g(x) = \ln x, \quad h(x) = \cos \ln x$$

for all $x \in D$ and

$$N(u, v) = \alpha \sin u + \beta u + \gamma v$$

for all $u, v \in H$, where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ are all constants such that $\alpha < \beta$ and $\gamma < \beta - \alpha$. Then

$$N(f(x), h(y)) = \alpha \sin \ln x + \beta \ln x + \gamma \cos \ln y$$

for all $x, y \in D$. It is easy to see that $K^* = [0, +\infty)$ and $K \subset g(D)$. Setting $\phi = \alpha + \beta$, $\varphi = \gamma$ and $\psi = \beta - \alpha$, we know that the conditions (i)-(iv) and (8) of Theorem 3.1 are satisfied and the $\text{SNICP}(f, h, g, N, K)$ has a unique solution $z = 1$.

From Theorem 3.1, we have the following:

**Theorem 3.2.** *Let $H$ be a real Hilbert space* (*not necessarily a Hilbert lattice*) *and $D$ be a nonempty subset of $H$. Let $f$, $g : D \to H$ be two mappings such that $K \subset g(D)$ and*

   (i)  *The mapping $f$ is $\phi$-Lipschitz continuous with respect to $g$,*

   (ii)  *The mapping $f$ is $\psi$-strongly monotone with respect to $g$,*

   (iii)  *$\phi$ is an increasing mapping and $\psi$ is a decreasing mapping.*

*If there exists a real number $\rho > 0$ such that*

$$\rho \phi^2(t) < 2\psi(t), \quad \psi(t) \leq \phi(t)$$

*for all $t \in [0, \infty)$, then the implicit complementarity problem* (4) *has a solution. In addition, if $g$ is a one-to-one mapping, the solution is unique.*

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