

## Finite Type Invariants and Virtual Twist Moves of Virtual Knots

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ABSTRACT. Generalizing twist moves of classical knots, we introduce  $t(a_1, \dots, a_m)$ -moves of virtual knots for an  $m$ -tuple  $(a_1, \dots, a_m)$  of nonzero integers. In [4], M. Goussarov, M. Polyak and O. Viro introduced finite type invariants of virtual knots and Gauss diagram formulae giving combinatorial presentations of finite type invariants. By using the Gauss diagram formulae for the finite type invariants of degree 2, we give a necessary condition for a virtual long knot  $K$  to be transformed to a virtual long knot  $K'$  by a finite sequence of  $t(a_1, \dots, a_m)$ -moves for an  $m$ -tuple  $(a_1, \dots, a_m)$  of nonzero integers with the same sign.

### 1. Introduction

In [10] L. Kauffman introduced virtual knots, a generalization of classical knots motivated from knots in thickened surfaces and abstract properties of Gauss code. Then he generalized many classical knot invariants to virtual ones. For example he extended the Jones polynomials  $V_K(t)$  of classical links in [5] to the  $f$ -polynomials  $f_K(A)$  of virtual links by using bracket polynomials ([9]). In 2000, M. Goussarov, M. Polyak and O. Viro introduced finite type invariants of virtual knots by using semi-virtual crossings instead of singular points in Vassiliev invariants theory ([4]).

We introduce  $t(a_1, \dots, a_m)$ -moves of virtual knots, a generalization of twist moves of classical knots. Then by using the Gauss diagram formulae for the finite type invariants of virtual knots of degree 2, we find a necessary condition for a virtual long knot  $K$  to be transformed to a virtual long knot  $K'$  by using finite times of  $t(a_1, \dots, a_m)$ -moves.

A knot diagram can be characterized by using a diagram of a circle with some signed arrows as following. For a diagram of a knot  $K$ , consider a diagram of a circle connected by chords whose endpoints are the double points of the projection of the knot diagram. Each chord in the diagram is oriented from the upper strand

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to the lower strand in the crossing and it is also equipped with the sign of the corresponding crossing. We call the diagram of the circle with signed chords the *Gauss diagram* of the knot diagram  $K$  and denote it by  $G(K)$ . We consider a Gauss diagram up to orientation preserving homeomorphism of the embedding of the underlying circle. See Figure 1 for the Gauss diagram of the figure eight knot diagram.

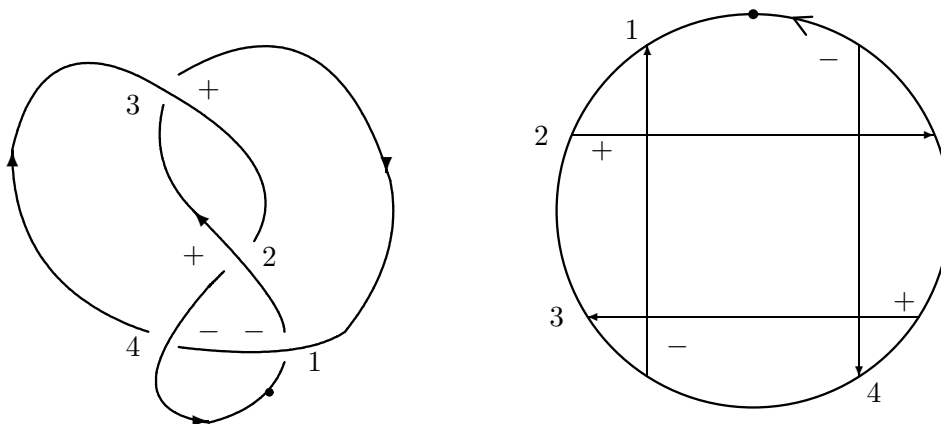


Figure 1:

A *virtual knot diagram* is a knot diagram with classical crossings and virtual crossings. We denote virtual crossings by singular points surrounded by a small circle as shown in Figure 2. For a virtual knot diagram  $K$ , we define its *Gauss diagram*  $G(K)$  to be the diagram with signed and oriented chords following the construction of a Gauss diagram for a classical one, disregarding virtual crossings. See Figure 2 for the Gauss diagram of a virtual knot.

Throughout this paper we denote the sets of all nonnegative integers, integers and real numbers by  $\mathbb{N}$ ,  $\mathbb{Z}$  and  $\mathbb{R}$  respectively.

In Section 2, we introduce finite type invariants of virtual knots and virtual long knots by using semi-virtual crossings ([4]). In Section 3, we find a necessary condition for a diagram  $K$  of a virtual long knot to be transformed to a diagram  $K'$  of a virtual long knot by using finite type invariants of long virtual knots of degree 2. This generalizes a criterion for two classical knots to be  $t_{2i}$ -equivalent, obtained by using the Vassiliev invariant of degree 2 ([8]).

## 2. Finite type invariants of virtual knots

If a knot  $K$  is isotopic to another knot  $K'$  then there is a sequence of moves

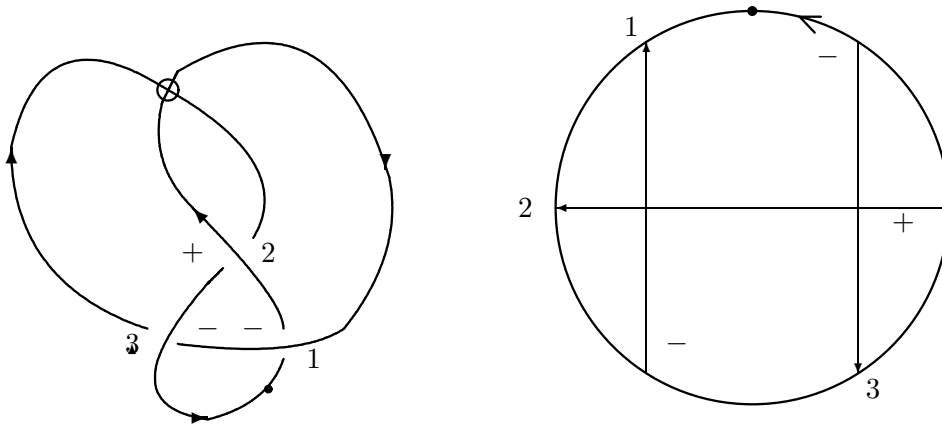


Figure 2:

from a diagram of  $K$  to a diagram of  $K'$  as shown in Figure 3 ([12]). We call these moves *Reidemeister moves*. We define *virtual moves* to be the moves of knot diagrams shown in Figure 4. A sequence of Reidemeister moves and virtual moves are called a *virtual isotopy*. A *virtual knot* is defined to be the virtual isotopy class of a virtual knot diagram.

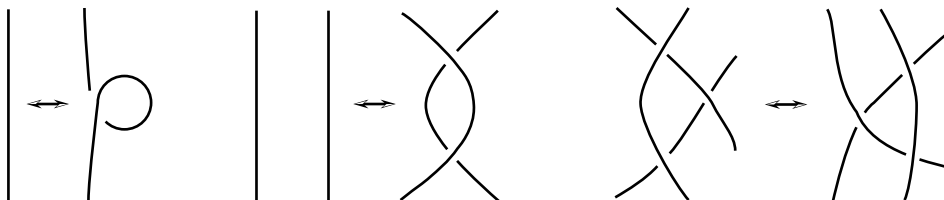


Figure 3:

Since a Gauss diagram defines a virtual knot diagram up to virtual moves ([4]), a virtual knot modulo virtual isotopy is equivalent to its Gauss diagram up to the moves on Gauss diagrams corresponding to the Reidemeister moves and virtual moves.

In [10], L. H. Kauffman showed that classical knot invariants such as fundamental groups, crystals, quandles and the bracket polynomials can be extended to virtual ones with the topological information of these invariants. L. Kauffman and Goussarov, Polyak and Viro independently showed that two classical knots are

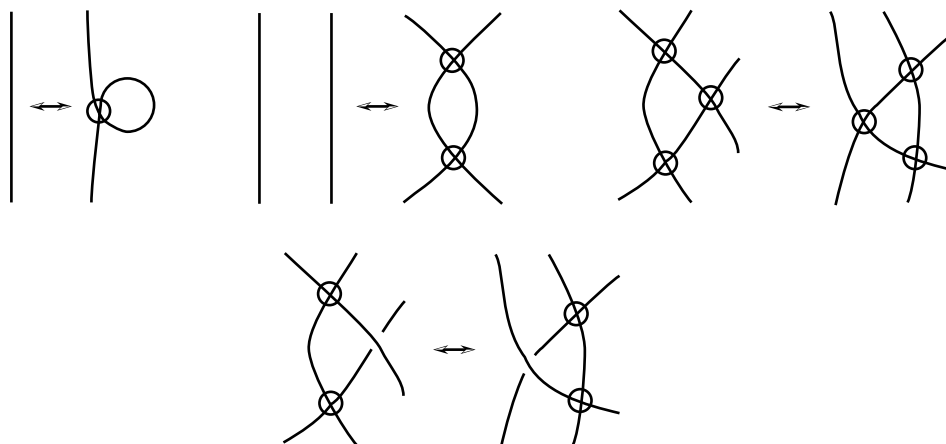


Figure 4:

isotopic if they are virtually isotopic ([10], [4]).

A classical knot is an embedding of a circle  $S^1$  into the 3-dimensional Euclidean space  $\mathbb{R}^3$ . We define a *long knot* to be an embedding of an oriented line  $\mathbb{R}$  into the Euclidean space  $\mathbb{R}^3$ . Two long knots are said to be *isotopic* if there is a smooth isotopy between them in the class of embeddings. A *virtual long knot diagram* is an immersion of the oriented line  $\mathbb{R}$  into the plane whose double points are real and virtual crossings as in a virtual knot diagram. Two virtual long knot diagrams  $K$  and  $K'$  are said to be *isotopic* if there is a sequence of Reidemeister moves and virtual moves from  $K$  to  $K'$ . The isotopy class of a virtual long knot diagram is called a *virtual long knot*. We define the *Gauss diagram* of a virtual long knot as that of a virtual knot, disregarding all of the virtual crossings.

As a classical link is an embedding of circles  $S^1$  into the 3-dimensional Euclidean space  $\mathbb{R}^3$ , we can define a virtual link diagram, a virtual link and the Gauss diagram of a virtual link similarly.

In [15], Vassiliev introduced finite type invariants of knots and in [2] Birman and Lin redefined it by using the Vassiliev skein relation. A knot or link invariant  $v$  taking values in an abelian group can be extended to a singular knot or link invariant by using the Vassiliev skein relation:  $v(K_\times) = v(K_+) - v(K_-)$ , where  $K_\times$ ,  $K_+$  and  $K_-$  are singular knot or link diagrams which are identical except the indicated local parts in Figure 5 ([1], [2]).

A knot or link invariant  $v$  is called a *Vassiliev invariant of degree  $n$*  if  $n$  is the smallest nonnegative integer such that  $v$  vanishes on singular knots or links with more than  $n$  double points. A knot or link invariant  $v$  is called a *Vassiliev invariant* if  $v$  is a Vassiliev invariant of degree  $n$  for some nonnegative integer  $n$ . There are infinitely many Vassiliev invariants obtained from the coefficients of polynomial invariants of knots ([2]).

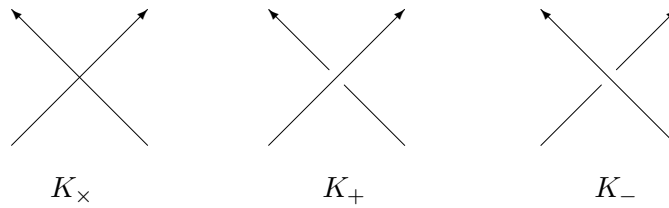


Figure 5:

We introduce a virtual version of finite type invariants by using semi-virtual crossings and Gauss diagram formulae of finite type invariants of long virtual knots.

**Definition 2.1.** Consider a diagram of an oriented circle with pairs of distinct points. The diagram obtained by connecting each pair of points, with signed dashed arrows is called an *arrow diagram*. We define the algebra  $\mathcal{A}$  of arrow diagrams to be the free  $\mathbb{Z}$ -module generated by all arrow diagrams.

We call  $\mathcal{A}$  an algebra since there is a multiplication making it to an associative algebra ([4], [13]). We denote the set of virtual knots by  $\mathcal{K}$  and the set of Gauss diagrams by  $\mathcal{D}$ . For a given Gauss diagram, we can associate an arrow diagram by turning chords of the diagram to be dashed. We extend the map to the free abelian group  $\mathbb{Z}\mathcal{D}$  and denote it by  $i : \mathbb{Z}\mathcal{D} \rightarrow \mathcal{A}$ . For Gauss diagrams  $D$  and  $D'$ , if  $D$  and  $D'$  are the same or  $D'$  can be obtained from  $D$  by removing some chords, then  $D'$  is said to be a *subdiagram* of  $D$  and we denote it by  $D' \subset D$ . A Gauss diagram  $D$  with  $n$  chords has  $2^n$  subdiagrams. We define another map  $I : \mathcal{D} \rightarrow \mathcal{A}$  defined by setting  $I(D)$  to be the sum of  $i(D')$  for all subdiagrams  $D'$  of  $D$ . We abuse  $I$  to denote its  $\mathbb{Z}$ -linear extension  $I : \mathbb{Z}\mathcal{D} \rightarrow \mathcal{A}$  defined by  $I(D) = \sum_{D' \subset D} i(D')$  for any Gauss diagram  $D' \in \mathcal{D}$ . We describe  $i$  and  $I$  as shown in Figure 6. We can easily see that the two maps  $i : \mathbb{Z}\mathcal{D} \rightarrow \mathcal{A}$  and  $I : \mathbb{Z}\mathcal{D} \rightarrow \mathcal{A}$  are isomorphisms.

Consider the free abelian group  $\mathbb{Z}\mathcal{K}$  generated by the space  $\mathcal{K}$  of virtual knots. We define a *semi-virtual crossing* by a formal relation as shown in Figure 7 and denote it by a real crossing surrounded by a small circle.

In the free abelian group generated by the set of Gauss diagrams we denote a semi-virtual crossing by a dashed arrow as in Figure 8.

**Definition 2.2.** Let  $\mathcal{K}$  be the space of virtual knots and  $\mathbb{Z}\mathcal{K}$  be the free abelian group generated by  $\mathcal{K}$ . For an abelian group  $G$  and for a virtual knot invariant  $v : \mathcal{K} \rightarrow G$ , we extend the invariant  $v$  to  $\mathbb{Z}\mathcal{K}$  by linearity. For a natural number  $n$ , if  $v$  vanishes for all virtual knots with more than  $n$  semi-virtual crossings, then it is called an *invariant of finite type*. The smallest such integer  $n$  is called the *degree*

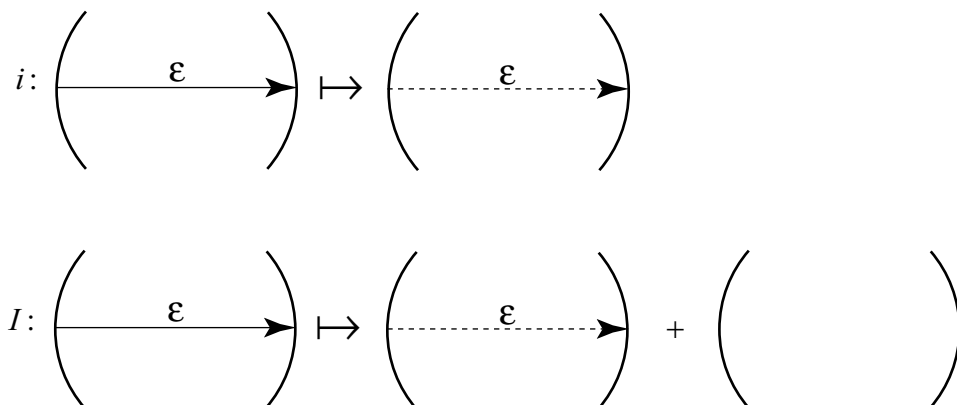


Figure 6:

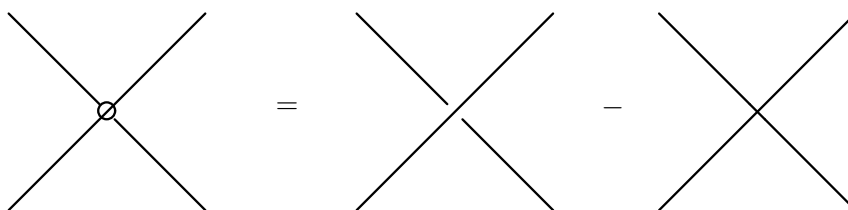


Figure 7:

of  $v$ .

Since  $\mathcal{A}$  is the free  $\mathbb{Z}$ -module generated by all arrow diagrams, we have an orthonormal scalar product  $(\cdot, \cdot)$  on  $\mathcal{A}$  defined by setting

$$(D_1, D_2) = \begin{cases} 1 & \text{if } D_1 = D_2, \\ 0 & \text{if } D_1 \neq D_2, \end{cases}$$

for arrow diagrams  $D_1$  and  $D_2$  and extend it linearly. Now we define the pairing  $\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{D} \rightarrow \mathbb{Z}$  as following. For any  $A \times D \in \mathcal{A} \times \mathcal{D}$ , we put

$$\langle A, D \rangle = (A, I(D)) = (A, \sum_{D' \subset D} i(D')).$$

In the case of Gauss diagrams of classical knots, this pairing was given in [14] to describe Gauss diagram formulae for knot invariants. Goussarov, Polyak and Viro

$$\left( \begin{array}{c} \text{---} \varepsilon \text{---} \\ \curvearrowright \end{array} \right) = \left( \begin{array}{c} \text{---} \varepsilon \text{---} \\ \curvearrowright \end{array} \right) - \left( \begin{array}{c} \text{---} \\ \curvearrowright \end{array} \right)$$

Figure 8:

proved that any invariant of virtual knots of finite type taking values in the ring  $\mathbb{Z}$  of the integers can be obtained by a Gauss diagram formula  $\langle A, D \rangle : \mathcal{K} \rightarrow \mathbb{Z}$  for some  $A \in \mathcal{A}$  ([4]). The degree of the invariant is less than or equal to the maximal number of the arrows of the diagrams in the linear combination giving  $A$ .

For the classical knots, the second coefficient of the Conway polynomial of a knot is the unique Vassiliev invariant of degree 2 up to multiplication and addition of constants ([1]). Let  $A_1$  and  $A_2$  be the linear combination of arrow diagrams as shown in Figure 9. There are two independent finite type invariants  $v_{2,1}$  and  $v_{2,2}$  of degree 2 for virtual long knots defined by setting  $v_{2,1}(K) = \langle A_1, G(K) \rangle$  and  $v_{2,2}(K) = \langle A_2, G(K) \rangle$  respectively for any virtual long knot  $K$ , where  $G(K)$  is the Gauss diagram of  $K$  ([4]).

$$A_1 = \begin{array}{cccc} \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} \\ \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow \\ + & + & - & - \end{array} - \begin{array}{cccc} \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} \\ \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow \\ + & + & - & - \end{array} - \begin{array}{cccc} \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} \\ \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow \\ - & - & + & + \end{array} + \begin{array}{cccc} \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} \\ \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow \\ - & - & + & + \end{array}$$

$$A_2 = \begin{array}{cccc} \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} \\ \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow \\ + & + & - & - \end{array} - \begin{array}{cccc} \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} \\ \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow \\ + & + & - & - \end{array} - \begin{array}{cccc} \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} \\ \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow \\ - & - & + & + \end{array} + \begin{array}{cccc} \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} & \text{---} \times \text{---} \\ \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow \\ - & - & + & + \end{array}$$

Figure 9:

### 3. Twist of virtual knots

The  $k$ -move on knots for an integer  $k$  which is the local transform of two parallel strands to the  $k$  times half twist as shown in Figure 10, have been first considered by S. Kinoshita ([11]) in 1957.

Depending on the orientation of the two strings, the  $k$ -move is said to be the  $t_k$ -move or  $\bar{t}_k$ -move as shown in Figure 11 and Figure 12. Note that the  $\bar{t}_k$ -moves are defined only for  $k$  even because of the orientations of the two strands.

Two links  $L$  and  $L'$  are said to be  $k$ -equivalent if one can transform  $L$  to  $L'$  by using the  $k^{\pm 1}$ -moves and two oriented links  $L$  and  $L'$  are said to be  $t_k$ -equivalent and  $\bar{t}_k$ -equivalent if one can transform  $L$  to  $L'$  by using the  $t_k^{\pm 1}$ -moves and  $\bar{t}_k^{\pm 1}$ -moves respectively.

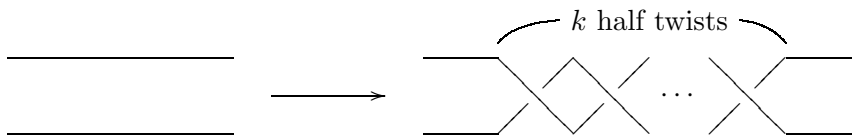


Figure 10:  $k$ -move

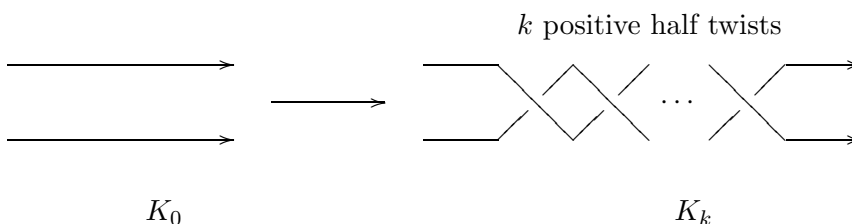


Figure 11:  $t_k$ -move

**Definition 3.1.** We fix  $k$  distinct points in the closed interval  $[0, 1]$ , say  $x_1, \dots, x_k$ . A *virtual  $(k, k)$ -tangle diagram* is an immersion of  $k$  oriented curves and some oriented circles into the square  $[0, 1]^2$  with the set of endpoints of the curves to be  $\{(x_i, y) \mid i = 1, \dots, k \text{ and } y = 0, 1\}$ . The double points of the generic immersion are real crossings and virtual crossings. Two virtual  $(k, k)$ -tangle diagrams  $T$  and  $S$  are said to be *isotopic* if there is a sequence of Reidemeister moves and virtual moves from  $T$  to  $S$ . The isotopy class of a virtual  $(k, k)$ -tangle diagram is called a *virtual  $(k, k)$ -tangle*. We define the *Gauss diagram* of a virtual tangle  $T$  as that of a virtual knot, disregarding all of the virtual crossings, and denote it by  $G(T)$ .

For two  $(k, k)$ -virtual tangles  $S$  and  $T$  such that the orientations of curves of  $S$  near the upper closed interval  $[0, 1] \times 1$  and those of the curves of  $T$  near the lower closed interval  $[0, 1] \times 0$  are matched, we define the *product  $TS$*  of the two tangles  $T$  and  $S$  to be the virtual tangle obtained by gluing the lower part of the square containing  $S$  and the upper part of the square containing  $T$ .

For a virtual  $(k, k)$ -tangle  $T$  such that the orientations of the curves in the upper part of the square and those of the curves in the lower part of the square are parallel, we define the *closure  $\bar{T}$*  to be the virtual link obtained from  $T$  by attaching  $k$  parallel strands to the endpoints of the curves in the exterior of the square.

We extend the twist moves on classical knots to virtual knots and virtual long knots by using a tangle  $T(a_1, \dots, a_m)$  with virtual crossings.

**Definition 3.2.** Let  $\sigma, \sigma^{-1}$  and  $\tau$  be the two virtual tangle as shown in Figure 13.



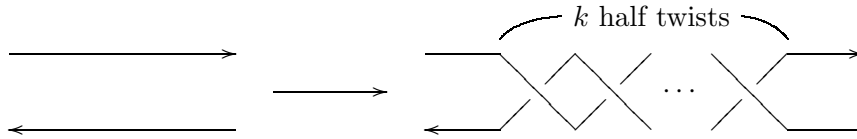


Figure 12:  $\bar{t}_k$ -move

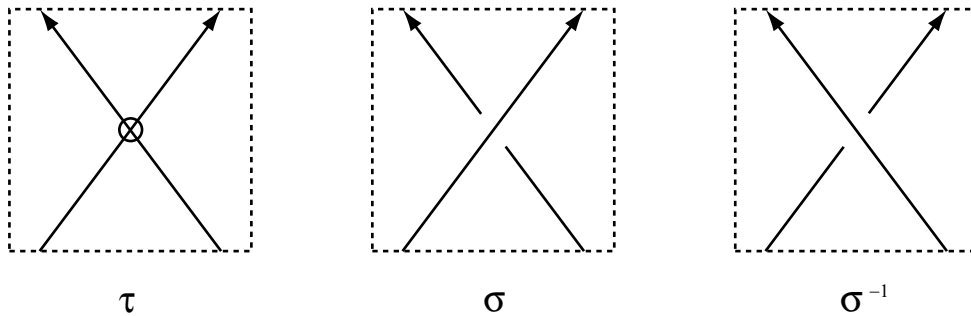


Figure 13:

For nonzero integers  $a_1, \dots, a_m$ , we define the virtual tangle  $T(a_1, \dots, a_m)$  by

$$T(a_1, \dots, a_m) = \begin{cases} \sigma^{a_1} \tau \sigma^{a_2} \tau \dots \tau \sigma^{a_m} & \text{if } \overline{\sigma^{a_1} \tau \sigma^{a_2} \tau \dots \tau \sigma^{a_m}} \text{ is a two components} \\ & \text{virtual link,} \\ \sigma^{a_1} \tau \sigma^{a_2} \tau \dots \tau \sigma^{a_m} \tau & \text{if } \overline{\sigma^{a_1} \tau \sigma^{a_2} \tau \dots \tau \sigma^{a_m}} \text{ is a virtual knot.} \end{cases}$$

**Definition 3.3.** The  $t(a_1, \dots, a_m)$ -move on virtual (long) knots for an  $m$ -tuple  $(a_1, \dots, a_m)$  of nonzero integers  $a_1, \dots, a_m$  is the local transform of two parallel strands to the tangle  $T(a_1, \dots, a_m)$ . If there is a sequence of  $t(a_1, \dots, a_m)$ -moves from a virtual (long) knot  $K$  to a virtual (long) knot  $K'$  then  $K$  and  $K'$  are said to be  $t(a_1, \dots, a_m)$ -equivalent.

In particular,  $t(a_1)$ -move with  $a_1 = i > 0$  for classical knots is exactly the  $t_i$ -move.

Let  $\mathbf{a} = (a_1, \dots, a_m)$  be an  $m$ -tuple of nonzero integers with the same sign. Then we introduce notations for  $d_1(\mathbf{a})$ ,  $d_2(\mathbf{a})$ ,  $d_3(\mathbf{a})$  and  $d_4(\mathbf{a})$ . For a real number  $r$ ,  $\lceil r \rceil$  and  $\lfloor r \rfloor$  denote the least integer greater than or equal to  $r$ , and the greatest integer less than or equal to  $r$  respectively. For each  $i = 1, \dots, m$ , we set  $x_i$ ,  $y_i$  and  $z_i$  as following. If  $a_i > 0$  then we set  $x_i = \lceil \frac{a_i}{2} \rceil$  and  $y_i = \lfloor \frac{a_i}{2} \rfloor$ . If  $a_i < 0$  then we set  $x_i = \lfloor \frac{a_i}{2} \rfloor$  and  $y_i = \lceil \frac{a_i}{2} \rceil$ . Set  $z_1 = 1$  if  $a_1$  is even, and  $z_1 = 0$  otherwise. Put  $E$  to be the set  $\{j \in \mathbb{N} \mid a_j \text{ is even, and } 1 \leq j < i\}$ .

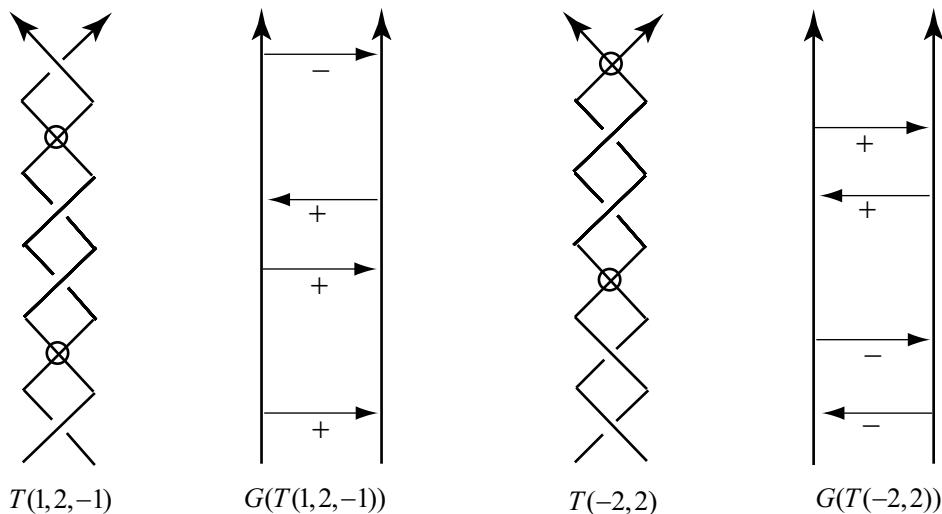


Figure 14:  $T(a_1, \dots, a_m)$  and  $G(T(a_1, \dots, a_m))$

For  $1 < i \leq m$ , we set  $z_i$  as following

$$z_i = \begin{cases} 1 & \text{if the cardinal number } |E| \text{ of the set } E \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Then we set  $d_1(\mathbf{a})$ ,  $d_2(\mathbf{a})$ ,  $d_3(\mathbf{a})$  and  $d_4(\mathbf{a})$  as following

$$\begin{aligned} d_1(\mathbf{a}) &= x_1 + \dots + x_m, \\ d_2(\mathbf{a}) &= y_1 + \dots + y_m, \\ d_3(\mathbf{a}) &= \sum_{k=1}^n \sum_{j=1}^{\lfloor \frac{|a_k| - z_k}{2} \rfloor} j + \sum_{1 \leq j < i \leq n} x_i y_j, \\ d_4(\mathbf{a}) &= \sum_{k=1}^n \sum_{j=1}^{|a_k| - 1 - \lfloor \frac{|a_k| - z_k}{2} \rfloor} j + \sum_{1 \leq i < j \leq n} x_i y_j. \end{aligned}$$

**Theorem 3.4.** *Let  $\mathbf{a} = (a_1, \dots, a_m)$  be an  $m$ -tuple of nonzero integers with the same sign. Let  $d(\mathbf{a})$  be the greatest common divisor of  $d_1(\mathbf{a})$ ,  $d_2(\mathbf{a})$ ,  $d_3(\mathbf{a})$  and  $d_4(\mathbf{a})$ . Assume that two virtual knots  $K$  and  $K'$  are  $t(a_1, \dots, a_m)$ -equivalent. Then we have*

$$\begin{cases} v_{2,1}(K) \equiv v_{2,1}(K') \pmod{d}, \\ v_{2,2}(K) \equiv v_{2,2}(K') \pmod{d}. \end{cases}$$

*Proof.* Consider the Gauss diagram  $G$  of the trivial long knot diagram obtained by attaching the five trivial Gauss diagram  $G_1, G_2, G_3, G_4$  and  $G_5$  as shown in Figure 15.

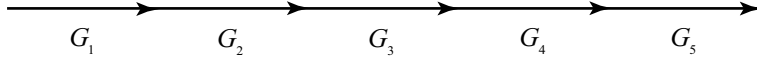


Figure 15:

Then we attach the endpoints of chords to  $G$  to get the Gauss diagram  $G(K)$  of  $K$  and the Gauss diagram  $G(K')$  of  $K'$  such that the Gauss diagram obtained from  $G(K')$  by removing the chords with endpoints in  $G_2 \cup G_4$  becomes  $G(K)$ . Let  $L_1$  and  $L_2$  be the two parallel strands of the tangle  $G(T(a_1, \dots, a_m))$ . Then in the Gauss diagram  $G(K')$ , either the endpoints of the chords of  $L_1$  and the endpoints of the chords of  $L_2$  are connected to  $G_2$  and  $G_4$  respectively, or the endpoints of the chords in  $L_1$  and the endpoints of the chords in  $L_2$  are connected to  $G_4$  and  $G_2$  respectively.

Then we will prove the statement for the invariant  $v_{2,1}$  as we can prove it for  $v_{2,2}$  similarly. To prove the statement it is sufficient to show that  $v_{2,1}(K') - v_{2,1}(K)$  is a  $\mathbb{Z}$ -linear combination of  $d_1(\mathbf{a}), d_2(\mathbf{a}), d_3(\mathbf{a})$  and  $d_4(\mathbf{a})$ . Now we use induction argument on the number  $|G(K)|$  of chords of  $G(K)$ . If the number  $|G(K)|$  of chords in  $G(K)$  is 0, then  $v_{2,1}(K') - v_{2,1}(K) = d_3(\mathbf{a})$  or  $d_4(\mathbf{a})$ .

Assume that  $v_{2,1}(L') - v_{2,1}(L)$  is a  $\mathbb{Z}$ -linear combination of  $d_1(\mathbf{a}), d_2(\mathbf{a}), d_3(\mathbf{a})$  and  $d_4(\mathbf{a})$  for any virtual knot  $L$  with  $G(L) < l$  and for any virtual knot  $L'$  obtained from  $L$  by a  $t(a_1, \dots, a_m)$ -move.

Let  $K'$  be a virtual knot obtained from a knot  $K$  by a  $t(a_1, \dots, a_m)$ -move with  $|G(K)| = l$ . Assume that  $K$  has a chord  $\overrightarrow{PQ}$  from a point  $P \in G_1 \cup G_3 \cup G_5$  to a point  $Q \in G_1 \cup G_3 \cup G_5$ . If  $(P, Q) \in (G_1 \times G_1) \cup (G_3 \times G_3) \cup (G_5 \times G_5) \cup (G_3 \times G_1) \cup (G_3 \times G_5) \cup (G_1 \times G_5) \cup (G_5 \times G_1)$ , then by the induction hypothesis we see that  $v_{2,1}(K') - v_{2,1}(K) = v_{2,1}(H') - v_{2,1}(H)$  is a linear combination of  $d_1(\mathbf{a}), d_2(\mathbf{a}), d_3(\mathbf{a})$  and  $d_4(\mathbf{a})$ , where  $H$  and  $H'$  are the virtual knot diagrams obtained from  $K$  and  $K'$  by making the real crossing corresponding to the chord  $\overrightarrow{PQ}$  to a virtual crossing respectively.

If  $(P, Q) \in (G_1 \times G_3)$ , then  $v_{2,1}(K') - v_{2,1}(K) = v_{2,1}(H') - v_{2,1}(H) \pm d_2(\mathbf{a})$  or  $v_{2,1}(K') - v_{2,1}(K) = v_{2,1}(H') - v_{2,1}(H) \pm d_1(\mathbf{a})$  depending on the attaching of the endpoints of the chords in  $L_1 \cup L_2$  to  $G_2 \cup G_4$  in  $G(K')$  and on the sign of the chord  $\overrightarrow{PQ}$ , where  $H$  and  $H'$  are the virtual knots obtained from  $K$  and  $K'$  by making the real crossing corresponding to the chord  $\overrightarrow{PQ}$  to a virtual crossing respectively. Thus  $v_{2,1}(K') - v_{2,1}(K)$  is a  $\mathbb{Z}$ -linear combination of  $d_1(\mathbf{a}), d_2(\mathbf{a}), d_3(\mathbf{a})$  and  $d_4(\mathbf{a})$  by the induction hypothesis.

If  $(P, Q) \in (G_5 \times G_3)$ , then  $v_{2,1}(K') - v_{2,1}(K) = v_{2,1}(H') - v_{2,1}(H) \pm d_1(\mathbf{a})$  or  $v_{2,1}(K') - v_{2,1}(K) = v_{2,1}(H') - v_{2,1}(H) \pm d_2(\mathbf{a})$  depending on the attaching of the

endpoints of chords in  $L_1 \cup L_2$  to  $G_2 \cup G_4$  in  $G(K')$  and on the sign of the chord  $\overrightarrow{PQ}$ , where  $H$  and  $H'$  are the virtual knots obtained from  $K$  and  $K'$  by making the real crossing corresponding to the chord  $\overrightarrow{PQ}$  to a virtual crossing respectively. Thus  $v_{2,1}(K') - v_{2,1}(K)$  is a  $\mathbb{Z}$ -linear combination of  $d_1(\mathbf{a})$ ,  $d_2(\mathbf{a})$ ,  $d_3(\mathbf{a})$  and  $d_4(\mathbf{a})$  by the induction hypothesis.

Similarly we can prove the statement for the invariant  $v_{2,2}$ . □

In particular if  $a_1, \dots, a_m$  are nonzero even integers with the same sign, then

$$d_1(\mathbf{a}) = \sum_{k=1}^m b_k = d_2(\mathbf{a}), \quad d_3(\mathbf{a}) - d_4(\mathbf{a}) = \sum_{k=1}^m (-1)^k |b_k|,$$

$$d_4(\mathbf{a}) = \sum_{k=1}^m \sum_{j=1}^{|b_k| - \delta_k} j + \sum_{1 \leq i < j \leq m} b_i b_j,$$

where  $\delta_k = 1$  if  $k$  is even and  $\delta_k = 0$  if  $k$  is odd.

Thus we get the following

**Corollary 3.5.** *Let  $a_1, \dots, a_m$  be nonzero even integers with the same sign. Assume that two virtual knots  $K$  and  $K'$  are  $t(a_1, \dots, a_m)$ -equivalent. For an integer  $k$ , we set*

$$\delta_k = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Let  $b_i = \frac{a_i}{2}$  for  $i = 1, \dots, m$  and  $d$  be the positive greatest common divisor of the following

$$\sum_{k=1}^m b_k, \quad \sum_{k=1}^m (-1)^k |b_k|, \quad \sum_{k=1}^m \sum_{j=1}^{|b_k| - \delta_k} j + \sum_{1 \leq i < j \leq m} b_i b_j.$$

Then we have

$$\begin{cases} v_{2,1}(K) \equiv v_{2,1}(K') \pmod{d}, \\ v_{2,2}(K) \equiv v_{2,2}(K') \pmod{d}. \end{cases}$$

For a classical knot  $K$ , the second coefficient  $a_2(K)$  of the Conway polynomial  $\nabla_K(z)$  can be given by  $v_{2,1}(L)$ , where  $L$  is a long knot obtained from  $K$  by cutting it in a point ([1], [3], [6], [7], [12], [14]). If two classical knots  $K$  and  $K'$  are  $t(2i)$  equivalent for some positive integer  $i$ , then the two knots are  $t_{2i}$ -equivalent. Therefore we get the following

**Corollary 3.6 ([8]).** *Let  $K$  and  $K'$  be two  $t_{2i}$ -equivalent knots for a positive integer  $i$ . Then*

$$\begin{cases} a_2(K) \equiv a_2(K') \pmod{\frac{i}{2}} & \text{if } i \text{ is even,} \\ a_2(K) \equiv a_2(K') \pmod{i} & \text{if } i \text{ is odd.} \end{cases}$$

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