KYUNGPOOK Math. J. 46(2006), 433-436

Remarks on "Representable Difference Algebras"

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ABSTRACT. In this note we show that a representable difference algebra is equivalent to a commutative BCK-algebra.

1. Preliminaries

I. Chajda and P. Emanovskỳ ([1]) introduced the notion of representable difference algebras which was a particular case of difference algebra introduced formerly by J. Meng. Such an algebra can be represented as a suitable meet-semilattice with 0 where every interval $[0, x]$ has an antitone involution. The converse was true under certain conditions investigated in ([1]).

The concept of a difference algebra was introduced by J. Meng ([2]):

Definition 1.1 ([2]). An algebraic structure $(X; *, \leq, 0)$ with a binary operation $*$, a nullary operation 0 and a binary relation $\leq$ is called a *difference algebra* if it satisfies the axioms:

(D1) $(D, \leq)$ is a poset;

(D2) $x \leq y$ implies $x * z \leq y * z$;

(D3) $(x * y) * z \leq (x * z) * y$;

(D4) $0 \leq x * x$;

(D3) $x \leq y$ if and only if $x * y \leq 0$,

for any $x, y, z \in X$.

As it was pointed out in [2] and [4], difference algebra are important and very useful in certain algebraic considerations. A lot of examples of these algebras were exposed in [4]. Unfortunately, only rather work structural properties can be proved for difference algebras. The reason is that the structure of a poset cannot get a

Received May 16, 2005.
2000 Mathematics Subject Classification: 03G25, 06A06, 06F35.
Key words and phrases: (representable) difference algebra, commutative BCK-algebra.

reach enough structure. To improve this situation, I. Chajda and P. Emanovský ([1]) introduced the notion of a representable difference algebra. In this note we show that a representable difference algebra is equivalent to a commutative BCK-algebra.

Definition 1.2 ([1]). By a *representable difference algebra* we mean an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the following identities: for any $x, y, z \in X$,

(1) $x * 0 = x,\ x * x = 0,\ 0 * x = 0;$

(2) $x * (x * y) = y * (y * x);$

(3) $(x * y) * z = (x * z) * y.$

In a representable difference algebra, the following are true (see [1]): for any $x, y, z \in X$,

(d1) $(x * y) * x = 0;$

(d2) $y * (y * (y * x)) = y * x;$

(d3) $x \leq y$ if and only if $x * y = 0;$

(d4) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x.$

Theorem 1.3 ([1]). *Every representable difference algebra is a difference algebra.*

The converse of Theorem 1.3 need not be true.

Example 1.4 ([4]). Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

*	0	1	2	3
0	0	0	3	3
1	1	0	3	2
2	2	3	0	1
3	3	3	0	0

Then $(X; *, \leq, 0)$ is a difference algebra, but not a representable difference algebra since $1 * (1 * 3) = 1 * 2 = 3 \neq 0 = 3 * 3 = 3 * (3 * 1)$.

Example 1.5. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

*	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

Then $(X;*,\leq,0)$ is a difference algebra, but not a representable difference algebra since $2*(2*3)=2*3=3\neq 0=3*3=3*(3*2)$.

2. Main results

By a *BCI-algebra* ([3]) we mean an algebra $(X;*,0)$ of type $(2,0)$ satisfying the following axioms, for all $x,\ y,\ z\in X$,

(i) $((x*y)*(x*z))*(z*y)=0$;

(ii) $(x*(x*y))*y=0$;

(iii) $x*x=0$;

(iv) $x*y=0$ and $y*x=0$ imply $x=y$.

A *BCK-algebra* is a BCI-algebra satisfying the axiom: (v) $0*x=0$ for all $x\in X$.

We can define a partial ordering “ $\leq$ ” on X by $x\leq y$ if and only if $x*y=0$. In any BCI-algebra X, we have:

(b1) $x*0=x$,

(b2) $(x*y)*z=(x*z)*y$,

(b3) $x\leq y$ implies $z*z\leq y*z$ and $z*y\leq z*x$,

(b4) $(x*z)*(y*z)\leq x*y$

for any $x,y,z\in X$.

A BCK-algebra $(X;*,0)$ is said to be *commutative* ([3]) if for all $x,y\in X$, $x*(x*y)=y*(y*x)$. H. Yutani ([5]) obtained equivalent simple axioms for an algebra $(X;*,0)$ to be a commutative BCK-algebra.

Theorem 2.1 ([5]). *An algebra $(X;*,0)$ is a commutative BCK-algebra if and only if it satisfies the following*:

(a) $x*(x*y)=y*(y*x)$;

(b) $(x*y)*z=(x*z)*y$;

(c) $x*x=0$;

(d) $x*0=x$

for any $x,y,z\in X$.

Proposition 2.2. *The axiom $0*x=0$ in a representable difference algebra is superfluous.*

Proof. By (2) and (3) we obtain, for any $x,y,z\in X$,

$$\begin{aligned}(x*y)*(x*z) &= (x*(x*z))*y=(z*(z*x))*y\\ &= (z*y)*(z*x).\end{aligned}$$

Hence

$$(*) \qquad (x * y) * (x * z) = (z * y) * (z * x).$$

In the above equality $(*)$ if we let $x = y$ and $z = 0$, then by $x * x = 0$, we obtain

$$0 * x = (x * x) * (x * 0) = (0 * x) * (0 * x) = 0.$$

□

Using this concept and comparing the axiom system of representable difference algebra, we summarize :

Theorem 2.3. *An algebra $(X; *, 0)$ is a commutative BCK-algebra if and only if it is a representable difference algebra.*

Corollary 2.4. *If an algebra $(X; *.0)$ is a commutative BCK-algebra, then it is a difference algebra.*

Proof. It can be easily obtained by Theorem 1.3 and Theorem 2.3. □

The converse of Corollary 2.4 need not be true.

Example 2.5. Let $(X; *, 0)$ be a non-commutative BCK-algebra and let $\leq$ be a BCK-order on X. Then $(X; *, \leq, 0)$ is a difference algebra, but not a commutative BCK-algebra.

Example 2.6. Let $X = \{0, 1, 2, 3\}$ as in Example 1.5. Then $(X; *, \leq, 0)$ is a difference algebra but not a commutative BCK-algebra, since $2 * (2 * 3) = 2 * 3 = 3 \neq 0 = 3 * 3 = 3 * (3 * 2)$.

References

[1] I. Chajda and P. Emaonvský, *Representable difference algebras*, Kyungpook Math. J., **44**(2004), 335-342.

[2] J. Meng, *Difference algebras Selected Papers on BCK- and BCI-algebras*, **1**(1992), 33-39.

[3] J. Meng and Y. B. Jun, BCK-algrbras, Kyung Moon Sa Co., Seoul, 1994.

[4] E. H. Roh, S. Y. Kim, Y. B. Jun and W. H. Shim, *On difference algebras*, Kyungpook Math. J., **43**(2003), 407-414.

[5] H. Yutani, *On a system of axioms of a commutative BCK-algebras*, Math. Seminar Notes, **5**(1977), 255-256.