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Conformally Flat Quasi-Einstein Spaces

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Abstract. The object of the present paper is to study a conformally flat quasi-Einstein space and its hypersurface.

1. Introduction

In 2000, M. C. Chaki and R. K. Maity [1] introduced the notion of a quasi-Einstein space. A non-flat Riemannian space M of dimension $n\,(>2)$ is said to be a quasi-Einstein space if its Ricci tensor R_{ij} of type $(0,2)$ is not identically zero and satisfies the condition

$$R_{ij} = a g_{ij} + b A_i A_j, \tag{1.1}$$

where a, b are scalars with $b \neq 0$. The scalars a and b are called associated scalars. A_i is a unit covariant vector, called generator of the space. Such a space is usually denoted by the symbol $(QE)_n$. In a recent paper [2], the first author and Gopal Chandra Ghosh studied generalized quasi-Einstein spaces.

The conformal curvature tensor ([3], p.90) C^h_{ijk} of type $(1,3)$ of a Riemannian space of dimension n is defined by

$$\begin{aligned} C^h_{ijk} &= R^h_{ijk} - \frac{1}{(n-2)}\left\{\delta^h_k R_{ij} - \delta^h_j R_{ik} + R^h_k g_{ij} - R^h_j g_{ik}\right\} \\ &\quad + \frac{R}{(n-1)(n-2)}\left\{\delta^h_k g_{ij} - \delta^h_j g_{ik}\right\}, \end{aligned} \tag{1.2}$$

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where R denotes the scalar curvature of the space. A Riemannian space of dimension $n\,(>3)$ is said to be conformally flat if its conformal curvature tensor vanishes identically. If $n=3$, then the conformal curvature tensor vanishes identically. The purpose of the present paper is to study a conformally flat quasi-Einstein space. This paper is organized as follows:

In section 2, we first prove that a conformally flat quasi-Einstein space is a space of quasi-constant curvature [4]. After that we find necessary and sufficient conditions for a conformally flat quasi-Einstein space to be semi-symmetric [5].

Section 3 deals with necessary and sufficient conditions for a confomally flat quasi-Einstein space to be recurrent [6] or locally symmetric.

Finally, in section 4 we study totally umbilical hypersurface ([7], p.43) of a conformally flat quasi-Einstein space.

2. Necessary and sufficient conditions for a conformally flat quasi-Einstein space to be semi-symmetric

Since the space under consideration is conformally flat, from (1.2) it follows that

$$R^h_{ijk} = \frac{1}{(n-2)}\left\{\delta^h_k R_{ij} - \delta^h_j R_{ik} + R^h_k g_{ij} - R^h_j g_{ik}\right\} - \frac{R}{(n-1)\,(n-2)}\left\{\delta^h_k g_{ij} - \delta^h_j g_{ik}\right\}. \tag{2.1}$$

Since the space is quasi-Einstein, its Ricci tensor R_{ij} of type $(0,2)$ can be expressed in the form

$$R_{ij} = a g_{ij} + b A_i A_j \tag{2.2}$$

where a, b are scalars, $b \neq 0$ and A_i is a unit covariant vector. Transecting with g^{ij} from (2.2) we get

$$R = an + b. \tag{2.3}$$

Using (2.2) and (2.3) in (2.1) we get

$$R^h_{ijk} = p\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) + q\left(\delta^h_k A_i A_j - \delta^h_j A_i A_k + g_{ij} A^h A_k - g_{ik} A^h A_j\right), \tag{2.4}$$

where

$$p = \frac{a(n-2)-b}{(n-1)(n-2)} \quad \text{and} \quad q = \frac{b}{n-2}$$

are scalars. From (2.4) it follows that a conformally flat quasi-Einstein space is a space of quasi-constant curvature [4].

A Riemannian space M of dimension n is said to be semi-symmetric [5] if its curvature tensor R^h_{ijk} of type $(1,3)$ satisfies the condition

$$R^h_{ijk,\,lm} - R^h_{ijk,\,ml} = 0. \tag{2.5}$$

Contracting h and k we obtain from the above equation

$$R_{ij,\,lm} = R_{ij,\,ml}. \tag{2.6}$$

If the space under consideration is a semi-symmetric quasi-Einstein space, then it satisfies both the conditions (1.1) and (2.6) . From (1.1) we get

$$R_{ij,\,lm} = a_{,\,lm}g_{ij} + b_{,\,lm}A_iA_j + bA_{i,\,lm}A_j + bA_iA_{j,\,lm}. \tag{2.7}$$

Therefore,

$$R_{ij,\,lm} - R_{ij,\,ml} = b\left\{\left(A_{i,\,lm} - A_{i,\,ml}\right)A_j + A_i\left(A_{j,\,lm} - A_{j\,ml}\right)\right\}. \tag{2.8}$$

Combining (2.6) and (2.8) we get

$$\left(A_{i,\,lm} - A_{i,\,ml}\right)A_j + A_i\left(A_{j,\,lm} - A_{j,\,ml}\right) = 0, \tag{2.9}$$

since b is non-zero. Using Ricci identity ([3], p. 30) we get from (2.9)

$$\left(A_hR^h_{ilm}\right)A_j + A_i\left(A_hR^h_{jlm}\right) = 0. \tag{2.10}$$

Transecting with A^j we get

$$A_hR^h_{ilm} + A_iA_hA^jR^h_{jlm} = 0. \tag{2.11}$$

Since $A_iA_hA^jR^h_{jlm} = 0$, we have $A_hR^h_{ilm} = 0$ if and only if

$$A_{i,\,ml} = A_{i,\,lm}. \tag{2.12}$$

Hence we find that if a conformally flat quasi-Einstein space $(QE)_n$ is semi-symmetric, then the generator of the space satisfies the condition (2.12).

Conversely, let us assume that the generator of a conformally flat quasi-Einstein space satisfies the condition (2.12). Now, from (1.2) we get,

$$\begin{aligned} (2.13)\quad & R^h_{ijk,\,lm} \\ = \;& p_{,\,lm}\left(\delta^h_kg_{ij} - \delta^k_jg_{ik}\right) + q_{,\,lm}\left\{\delta^h_kA_iA_j - \delta^h_jA_iA_k + g_{ij}A^hA_k - g_{ik}A^hA_j\right\} \\ & +q\left\{\delta^h_k\left(A_{i,\,lm}A_j + A_iA_{j,\,lm}\right) - \delta^h_j\left(A_{i,\,lm}A_k + A_iA_{k,\,lm}\right)\right\} \\ & +q\left\{g_{ij}\left(A^h_{,\,lm}A_k + A^hA_{k,\,lm}\right) - g_{ik}\left(A^h_{,\,lm}A_j + A^hA_{j,\,lm}\right)\right\}. \end{aligned}$$

This gives,

$$R^h_{ijk,\,lm} - R^h_{ijk,\,ml} = 0 \tag{2.14}$$

i.e., the space is semi-symmetric. Thus we can state the following:

Theorem 1. *A conformally flat quasi-Einstein space $(QE)_n$ $(n > 3)$ is semi-symmetric if and only if the generator A_i of the space $(QE)_n$ satisfies*

$$A_{i,\,lm} = A_{i,\,ml}.$$

This theorem also shows that a conformally flat quasi-Einstein space $(QE)_n$ is not always a semi-symmetric space. Hence it is not always symmetric or recurrent [6]. As sufficient conditions the following may be easily obtained.

Corollary. *A conformally flat quasi-Einstein space* $(QE)_n$ $(n > 3)$ *is semi-symmetric if the generator* A^i *satisfies one of the following conditions*:

(i) A^i *is a parallel vector field, i.e.,* $A^i_{,j} = 0$;

(ii) A^i *is concurrent, i.e.,* $A^i_{,j} = c\delta^i_j$, *where c is a constant.*

Now, the condition (2.12) is equivalent to

$$A_h R^h_{ilm} = 0.$$

Expressing this with respect to p and q we get from (1.2)

$$A_h R^h_{ijk} = 0$$

$$\text{i.e., } (p+q)\,(A_k g_{ij} - A_j g_{ik}) = 0. \tag{2.15}$$

Transecting with $g^{ij} A^k$ we get from (2.15)

$$(p+q)\,(n-1) = 0.$$

For $n > 3$ we have $p + q = 0$. This gives

$$(a+b)\,(n-2) = 0.$$

Therefore $a + b = 0$. Obviously, this condition is equivalent to (2.12). Hence we can state:

Theorem 2. *A conformally flat quasi-Einstein space* $(QE)_n$ $(n > 3)$ *is semi-symmetric if and only if the sum of associated scalars is zero.*

3. Necessary and sufficient condition for a conformally flat Quasi-Einstein space to be Recurrent

Now we seek a necessary and sufficient condition for a conformally flat quasi-Einstein space $(QE)_n$, $(n > 3)$ to be recurrent [6].

First we assume that the space under consideration is recurrent. Then the space is semi-symmetric [5]. Since the space is semi-symmetric, using Theorem 2, we get $a + b = 0$. Hence the equation (1.1) can be written as

$$R_{ij} = a\,(g_{ij} - A_i A_j)\,. \tag{3.1}$$

On contraction this yields

$$R = a\,(n-1)\,. \tag{3.2}$$

Since the space is recurrent, we can write

$$R^h_{ijk,\,l} = \lambda_l R^h_{ijk}, \tag{3.3}$$

where λ_l is a non-zero covariant vector. From (3.3) we get

$$R_{ij,\,l} = \lambda_l R_{ij} \tag{3.4}$$

and

$$R_{,\,l} = \lambda_l R. \tag{3.5}$$

Combining (3.2) and (3.5) we get

$$(n-1)\,a_{,\,l} = \lambda_l R$$

$$\text{i.e., } \lambda_l = \frac{1}{a}\,a_{,\,l}. \tag{3.6}$$

Now from (3.1), (3.4) and (3.6) it follows that

$$R_{ij,\,l} = \frac{1}{a} a_{,\,l} R_{ij}$$

and hence we get

$$A_{i,\,l}A_j + A_i A_{j,\,l} = 0. \tag{3.7}$$

since $a = -b \neq 0$. Transecting with A^j we get

$$A_{i,\,l} = 0,$$

i.e., A_i is parallel.

Conversely, if $b = -a \neq 0$ and A_i is parallel, then we get,

$$R^h_{ijk} = \frac{a}{n-2}\left\{\delta^h_k g_{ij} - \delta^h_j g_{ik} - \delta^h_k A_i A_j + \delta^h_j A_i A_k - g_{ij}A^h A_k + g_{ik}A^h A_j\right\}.$$

From this it follows that

$$\begin{aligned} R^h_{ijk,\,l} &= \frac{1}{n-2} a_{,\,l} R^h_{ijk} \\ &= \mu_l R^h_{ijk}, \end{aligned} \tag{3.8}$$

where $\mu_l = \frac{1}{n-2} a_{,\,l}$ is a covariant vector, i.e., the space under consideration is recurrent. In view of the above, we state:

Theorem 3. *A conformally flat quasi-Einstein space $(QE)_n$ is recurrent if and only if the generator A_i is parallel and the sum of the associated scalars is zero.*

Next, for a recurrent space the curvature tensor R^h_{ijk} satisfies

$$R^h_{ijk,\,l} = \lambda_l R^h_{ijk},$$

where λ_l is a covariant vector. Obviously, such a space is locally symmetric if and only if $\lambda_l = 0$, i.e., if and only if $a_{,\,l} = 0$. Hence we get the following theorem.

Theorem 4. *A conformally flat quasi-Einstein space $(QE)_n$ of dimension n is recurrent if and only if the generator of the space A_i is parallel and the associated scalar $b = -a$ is a constant.*

4. Totally umbilical hypersurface of a conformally flat quasi-Einstein space

Let M^n be a conformally flat quasi-Einstein space of dimension n and M^{n-1} is a space of dimension $(n-1)$ immersed in M^n by a differentiable immersion $i : M^{n-1} \longrightarrow M^n$. We identify $i(M^{n-1})$ with M^{n-1} and call it is a hypersurface ([3], p. 8) of M^n.

The Gauss equation ([3], p. 149) relates the curvature tensors of type $(0,4)$ as

$$K_{hijk} = R_{\mu\nu\lambda\eta} B^\mu_h B^\nu_i B^\lambda_j B^\eta_k + H_{ij} H_{hk} - H_{ik} H_{jh}, \tag{4.1}$$

where H_{ij} is the second fundamental tensor and

$$B^\mu_h = \frac{\partial x^\mu}{\partial x^h}. \tag{4.2}$$

If on the hypersurface M^{n-1} there exists two functions α and β and a unit vector field v_λ such that

$$H_{ij} = \alpha g_{ij} + \beta v_i v_j, \tag{4.3}$$

then M^{n-1} is said to be quasi-umbilical [4].

In particular, if $\beta = 0$, then M^{n-1} is said to be totally umbilical. Again if $\alpha = \beta = 0$, then M^{n-1} is said to be totally geodesic.

Here we assume that M^n is a conformally flat quasi-Einstein space and M^{n-1} is a totally umbilical hypersurface of M^n. Since M^n is a conformally flat quasi-Einstein space, from (2.4) if follows that the space is of quasi-constant curvature.

From (2.4), (4.1), (4.2) and (4.3) we get,

$$\begin{aligned} K_{hijk} \;=\;& \left(p + \alpha^2\right)\left(g_{hk} g_{ij} - g_{hj} g_{ik}\right) \\ &+ q\left(g_{hk} A_i A_j - g_{jh} A_i A_k + g_{ij} A_h A_k - g_{ik} A_h A_j\right). \end{aligned} \tag{4.4}$$

In particular, if the generator vector A_i of M^n is orthogonal to M^{n-1} then from (4.4) we obtain

$$K_{hijk} = \left(p + \alpha^2\right)\left(g_{hk} g_{ij} - g_{hj} g_{ik}\right). \tag{4.5}$$

Thus we have the following theorem:

Theorem 5. *If the generator of a conformally flat quasi-Einstein space is orthogonal to a totally umbilical hypersurface, then the space is of constant curvature.*

Next we assume that a conformally flat quasi-Einstein space M of dimension n with associated scalars a and b is semi-symmetric. Then by Theorem 2, we have

$$a + b = 0. \tag{4.6}$$

Similarly, a totally umbilical hypersurface M^{n-1} of the conformally flat quasi-Einstein space under consideration is semi-symmetric if and only if

$$a + \alpha^2 + b = 0. \tag{4.7}$$

From (4.6) and (4.7) we get $\alpha = 0$ and hence the hypersurface M^{n-1} is totally geodesic. Thus we get the theorem:

Theorem 6. *Let a conformally flat quasi-Einstein space is semi-symmetric. Then a totally umbilical hypersurface of the space is semi-symmetric if and only if it is totally geodesic.*

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