KYUNGPOOK Math. J. 46(2006), 315-328

Boundedness for Multilinear Littlewood-Paley Operators on Certain Hardy Spaces

Lanzhe Liu
College of Mathematics, Changsha University of Science and Technology, Changsha 410077, P. R. of China
e-mail : lanzheliu@263.net

Abstract. In this paper, the Boundedness for the multilinear Littlewood-Paley operator on certain Hardy and Herz-Hardy spaces are obtained.

1. Introduction and definitions

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [1]-[8]). From [6] and [3], we know that the commutators and multilinear operators are bounded on $L^p(R^n)$ for $1 < p < \infty$. However, it was observed that the commutators and multilinear operators are not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$ for $0 < p \leq 1$. But, if $H^p(R^n)$ is replaced by a suitable atomic space(see [1], [18]), then the commutators and multilinear operators are bounded from the suitable atomic space to $L^p(R^n)$ for $p \in (n/(n+1), 1]$. In recent years, the theory of Herz space and Herz type Hardy space, as a local version of Lebesgue space and Hardy space, have been developed (see [9]-[11]). The main purpose of this paper is to establish the boundedness properties of some multilinear operator related to Littlewood-Paley operator on Hardy and Herz type Hardy spaces.

Let ψ be a fixed function on R^n which satisfies the following properties:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \leq C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y) - \psi(x)| \leq C|y|(1+|x|)^{-(n+2)}$ when $2|y| < |x|$.

Let m be a positive integer and A be a function on R^n. The multilinear Littlewood-Paley operator is defined by

$$g_\psi^A(f)(x) = \left[\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t}\right]^{1/2},$$

Received January 6, 2005, and, in revised form, July 18, 2005.
2000 Mathematics Subject Classification: 42B25, 42B20.
Key words and phrases: Littlewood-Paley operator, Multilinear operator, $BMO(R^n)$, Hardy space, Herz-Hardy space.

where

$$F_t^A(f)(x) = \int_{R^n} \frac{f(y)\psi_t(x-y)}{|x-y|^m} R_{m+1}(A;x,y)dy,$$

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\beta|\le m} \frac{1}{\beta!} D^\beta A(y)(x-y)^\beta$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(x) = f * \psi_t(x)$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t}\right)^{1/2},$$

which is the Littlewood-Paley operator(see [20]).

Note that when $m = 0$, g_ψ^A is just the commutator of Littlewood-Paley operator (see [2], [12]-[14]). It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [3]-[5], [7]-[8]). The main purpose of this paper is to consider the continuity of the multilinear Littlewood-Paley operators on certain Hardy and Herz-Hardy spaces. Let us first introduce some definitions(see [1], [9]-[11], [15]-[17]).

Definition 1. Let A be a function on R^n and m be a positive integer, $0 < p \le 1$. A bounded measurable function a on R^n is said to be a $(p, D^m A)$ atom if

i) supp$a \subset B = B(x_0, r)$,

ii) $\|a\|_{L^\infty} \le |B|^{-1/p}$,

iii) $\int_{R^n} a(y)dy = \int_{R^n} a(y)D^\beta A(y)dy = 0, |\beta| = m$.

A temperate distribution f is said to belong to $H^p_{D^m A}(R^n)$, if, in the Schwartz distributional sense, it can be written as

$$f(x) = \sum_{j=0}^\infty \lambda_j a_j(x),$$

where a_j's are $(p, D^m A)$ atoms, $\lambda_j \in C$ and $\sum_{j=0}^\infty |\lambda_j|^p < \infty$. Moreover, $\|f\|_{H^p_{D^m A}} \approx \left(\sum_{j=0}^\infty |\lambda_j|^p\right)^{1/p}$, where, and in what follows, $K \approx L$ means that there are positive constants C_1, C_2, independent of K and L, such that $K \le C_1 L \le C_2 K$.

Let $B_k = \{x \in R^n : |x| \le 2^k\}$, $C_k = B_k \setminus B_{k-1}$, $\tilde m_0(\lambda, f) = |\{x \in B_0 : |f(x)| > \lambda\}|$, $m_k(\lambda, f) = |\{x \in C_k : |f(x)| > \lambda\}|$ for $k \in Z$ and $\tilde m_k(\lambda, f) = m_k(\lambda, f)$ for $k \in N$. Denote $\chi_k = \chi_{C_k}$ for $k \in Z$ and $\chi_0 = \chi_{B_0}$, where χ_E is the characteristic function of the set E.

Definition 2. Let $0 < p$, $q < \infty$, $\alpha \in R$.

(1) The homogeneous Herz space is defined by

$$\dot K_q^{\alpha,p}(R^n) = \{f \in L^q_{loc}(R^n \setminus \{0\}) : \|f\|_{\dot K_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|_{L^q}^p\right]^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f \in L_{loc}^q(R^n) : \|f\|_{K_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p}\|f\chi_k\|_{L^q}^p + \|f\chi_0\|_{L^q}^p\right]^{1/p}.$$

Definition 3. Let m be a positive integer and A be a function on R^n, $\alpha \in R$, $0 < p < \infty$, and $1 < q \leq \infty$. A function $a(x)$ on R^n is called a central $(\alpha, q, D^m A)$-atom (or a central $(a, q, D^m A)$-atom of restrict type), if

1) Supp $a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),
2) $\|a\|_{L^q} \leq |B(0, r)|^{-\alpha/n}$,
3) $\int_{R^n} a(x)dx = \int_{R^n} a(x)D^\beta A(x)dx = 0$, $|\beta| = m$.

A temperate distribution f is said to belong to $H\dot{K}_{q,D^m A}^{\alpha,p}(R^n)$(or $HK_{q,D^m A}^{\alpha,p}(R^n)$), if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$(or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, where a_j is a central $(\alpha, q, D^m A)$-atom (or a central $(\alpha, q, D^m A)$-atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$ (or $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$), moreover, $\|f\|_{H\dot{K}_{q,D^m A}^{\alpha,p}}$ (or $\|f\|_{HK_{q,D^m A}^{\alpha,p}}$)$\approx \left(\sum_j |\lambda_j|^p\right)^{1/p}$.

2. Theorems and proofs

We begin with some preliminary lemmas.

Lemma 1 ([3]). *Let A be a function on R^n and $D^\beta A \in L^q(R^n)$ for $|\beta| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\beta|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\beta A(z)|^q dz\right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 2. *Let $1 < p < \infty$, $1 < r \leq \infty$, $1/q = 1/p + 1/r$ and $D^\beta A \in L^r(R^n)$ for $|\beta| = m$. Then g_ψ^A is bounded from $L^p(R^n)$ to $L^q(R^n)$, that is*

$$\|g_\psi^A(f)\|_{L^q} \leq C \sum_{|\beta|=m} \|D^\beta A\|_{L^r}\|f\|_{L^p}.$$

Proof. By Minkowski inequality and the condition of ψ, we have

$$\begin{aligned} g_\psi^A(f)(x) &\le \int_{R^n} \frac{|f(y)|\,|R_{m+1}(A;x,y)|}{|x-y|^m}\left(\int_0^\infty |\psi_t(x-y)|^2\frac{dt}{t}\right)^{1/2} dy \\ &\le C\int_{R^n} \frac{|f(y)||R_{m+1}(A;x,y)|}{|x-y|^m}\left(\int_0^\infty \frac{t^{-2n}}{(1+|x-y|/t)^{2(n+1)}}\frac{dt}{t}\right)^{1/2} dy \\ &\le C\int_{R^n} \frac{|R_{m+1}(A;x,y)|}{|x-y|^{m+n}}|f(y)|dy, \end{aligned}$$

thus, the lemma follows from [7], [8]. □

Theorem 1. *Let $1 \ge p > n/(n+1)$ and $D^\beta A \in BMO(R^n)$ for $|\beta| = m$. Then g_ψ^A is bounded from $H^p_{D^m A}(R^n)$ to $L^p(R^n)$.*

Proof. It suffices to show that there exists a constant $C > 0$ such that for every $(p, D^m A)$ atom a,

$$\|g_\psi^A(a)\|_{L^p} \le C.$$

Let a be a $(p, D^m A)$ atom supported on a ball $B = B(x_0, r)$. We write

$$\int_{R^n}[g_\psi^A(a)(x)]^p dx = \int_{|x-x_0|\le 2r}[g_\psi^A(a)(x)]^p dx + \int_{|x-x_0|>2r}[g_\psi^A(a)(x)]^p dx = I + II.$$

For I, taking $q > 1$, by Hölder's inequality and the L^q-boundedness of g_ψ^A(see Lemma 2), we see that

$$I \le C\|g_\psi^A(a)\|^p_{L^q}\cdot|B(x_0,2r)|^{1-p/q} \le C\|a\|^p_{L^q}|B|^{1-p/q} \le C.$$

To obtain the estimate of II, we need to estimate $g_\psi^A(a)(x)$ for $x \in (2B)^c$. Let $\tilde{B} = 5\sqrt{n}B$ and $\tilde{A}(x) = A(x) - \sum_{|\beta|=m}\frac{1}{\beta!}(D^\beta A)_{\tilde{B}}x^\beta$. Then $R_m(A;x,y) = R_m(\tilde{A};x,y)$. We write, by the vanishing moment of a,

$$\begin{aligned} F_t^A(a)(x) &= \int_B\left[\frac{\psi_t(x-y)}{|x-y|^m} - \frac{\psi_t(x-x_0)}{|x-x_0|^m}\right]R_m(\tilde{A};x,y)a(y)dy \\ &\quad + \int_B \frac{\psi_t(x-x_0)}{|x-x_0|^m}[R_m(\tilde{A};x,y) - R_m(\tilde{A};x,x_0)]a(y)dy \\ &\quad - \sum_{|\beta|=m}\frac{1}{\beta!}\int_B\frac{\psi_t(x-y)(x-y)^\beta}{|x-y|^m}(D^\beta A(y) - (D^\beta A)_B)a(y)dy. \end{aligned}$$

Thus

$$\begin{aligned} g_\psi^A(a)(x) &\leq \left[\int_0^\infty \left(\int_B \left|\frac{\psi_t(x-y)}{|x-y|^m} - \frac{\psi_t(x-x_0)}{|x-x_0|^m}\right| |R_m(\tilde{A};x,y)||a(y)|dy\right)^2 \frac{dt}{t}\right]^{1/2} \\ &+ \left[\int_0^\infty \left(\int_B \frac{|\psi_t(x-x_0)|}{|x-x_0|^m} |R_m(\tilde{A};x,y) - R_m(\tilde{A};x,x_0)||a(y)|dy\right)^2 \frac{dt}{t}\right]^{1/2} \\ &+ \left[\int_0^\infty \left|\sum_{|\beta|=m} \frac{1}{\beta!}\int_B \frac{\psi_t(x-y)(x-y)^\beta}{|x-y|^m}(D^\beta A(y) - (D^\beta A)_B)a(y)dy\right|^2 \frac{dt}{t}\right]^{1/2} \\ &= II_1 + II_2 + II_3. \end{aligned}$$

By Lemma 1, for $y \in B$ and $x \in 2^{k+1}B \setminus 2^k B$, we know

$$|R_m(\tilde{A};x,y)| \leq C|x-y|^m \sum_{|\beta|=m} |D^\beta A(x) - (D^\beta A)_{2^k B}|.$$

By the condition on ψ and Minkowski's inequality, noting that $|x-y| \sim |x-x_0|$ for $y \in B$ and $x \in R^n \setminus B$, we obtain

$$\begin{aligned} II_1 &\leq \int_B |R_m(\tilde{A};x,y)||a(y)| \left(\int_0^\infty \left|\frac{\psi_t(x-y)}{|x-y|^m} - \frac{\psi_t(x-x_0)}{|x-x_0|^m}\right|^2 \frac{dt}{t}\right)^{1/2} dy \\ &\leq C|x-x_0|^{-(m+n+1)}|B|^{1/n-1/p}\left(\int_B |R_m(\tilde{A};x,y)|dy\right) \\ &\leq Ck|x-x_0|^{-n-1}|B|^{1/n-1/p+1} \sum_{|\beta|=m} |D^\beta A(x) - (D^\beta A)_{2^{k+1}B}|. \end{aligned}$$

On the other hand, by the formula(see [3]):

$$R_m(\tilde{A};x,y) - R_m(\tilde{A};x,x_0) = \sum_{|\gamma|<m} \frac{1}{\gamma!} R_{m-|\gamma|}(D^\gamma \tilde{A};y,x_0)(x-x_0)^\gamma$$

and Lemma 1, we get

$$|R_m(\tilde{A};x,y) - R_m(\tilde{A};x,x_0)| \leq C\sum_{|\gamma|<m}\sum_{|\beta|=m} |x_0-y|^{m-|\gamma|}|x-x_0|^{|\gamma|}\|D^\beta A\|_{BMO},$$

so that

$$\begin{aligned} II_2 &\leq C\int_B |x-x_0|^{-(n+m)}\sum_{|\gamma|<m}\left|R_{m-|\gamma|}(D^\gamma\tilde{A};y,x_0)\right|\left|x-x_0\right|^{|\gamma|}|a(y)|dy \\ &\leq C\int_B |x-x_0|^{-(n+m)}\sum_{|\gamma|<m}|x_0-y|^{m-|\gamma|}|x-x_0|^{|\gamma|}\sum_{|\beta|=m}\|D^\beta A\|_{BMO}|a(y)|dy \\ &\leq C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\int_B \frac{|x_0-y|}{|x-x_0|^{n+1}}|a(y)|dy \\ &\leq C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}|x-x_0|^{-n-1}|B|^{1/n-1/p+1}. \end{aligned}$$

For II_3 , we write

$$\begin{aligned} &\int_B \frac{\psi_t(x-y)(x-y)^\beta}{|x-y|^m}(D^\beta A(y)-(D^\beta A)_B)a(y)dy \\ = &\int_B \left[\frac{\psi_t(x-y)(x-y)^\beta}{|x-y|^m}-\frac{\psi_t(x-x_0)(x-x_0)^\beta}{|x-x_0|^m}\right][D^\beta A(y)-(D^\beta A)_B]a(y)dy, \end{aligned}$$

similar to the estimate of II_1, we obtain

$$\begin{aligned} II_3 &\leq C\sum_{|\beta|=m}|x-x_0|^{-(n+1)}\int_B |x_0-y\|D^\beta A(y)-(D^\beta A)_B\|a(y)|dy \\ &\leq C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}|B|^{1/n-1/p+1}|x-x_0|^{-n-1}. \end{aligned}$$

Therefore, recall that $p>n/(n+1)$,

$$\begin{aligned} II &\leq \sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}[g_\psi^A(a)(x)]^p dx \\ &\leq C\sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB} k^p|x-x_0|^{-p(n+1)}|B|^{p(1+1/n-1/p)} \\ &\quad \left(\sum_{|\beta|=m}|D^\beta A(x)-(D^\beta A)_{2^{k+1}B}|\right)^p dx \\ &\quad +C\left(\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\right)^p\sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}|x-x_0|^{-p(n+1)}|B|^{p(1+1/n-1/p)}dx \end{aligned}$$

$$\leq C\left(\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\right)^p\sum_{k=1}^{\infty}k^p 2^{k(n-p-pn)}$$
$$\leq C\left(\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\right)^p,$$

which together with the estimate for I yields the desired result. This finishes the proof of Theorem 1. □

Remark. In general, when $p = n/(n+1)$, Theorem 1 is false, that is g_ψ^A is not bounded from $H^p_{D^mA}(R^n)$ to $L^p(R^n)$(see [1], [13]-[14]).

Theorem 2. *Let $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \leq \alpha < n(1-1/q)+1$ and $D^\beta A \in BMO(R^n)$ for $|\beta| = m$. Then g_ψ^A is bounded from $H\dot{K}^{\alpha,p}_{q,D^mA}(R^n)$ to $\dot{K}^{\alpha,p}_q(R^n)$.*

Proof. Let $f \in H\dot{K}^{\alpha,p}_{q,D^mA}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 3. We write

$$\|g_\psi^A(f)\|_{\dot{K}^{\alpha,p}_q} \leq C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|\;\|g_\psi^A(a_j)\chi_k\|_{L^q}\right)^p\right]^{1/p}$$
$$+C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|\|g_\psi^A(a_j)\chi_k\|_{L^q}\right)^p\right]^{1/p}$$
$$= J + JJ.$$

For JJ, by the boundedness of g_ψ^A on $L^q(R^n)$ (see Lemma 2), we have

$$JJ \leq C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|\|a_j\|_{L^q}\right)^p\right]^{1/p}$$
$$\leq C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|2^{-j\alpha}\right)^p\right]^{1/p}$$
$$\leq \begin{cases} C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{-j\alpha p}\right]^{1/p}, & 0<p\leq 1\\ C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{-j\alpha p/2}\right)\left(\sum_{j=k-2}^{\infty}2^{-j\alpha p'/2}\right)^{p/p'}\right]^{1/p}, & p>1\end{cases}$$
$$\leq \begin{cases} C\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p}\right)\right]^{1/p}, & 0<p\leq 1\\ C\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p/2}\right)\right]^{1/p}, & p>1\end{cases}$$

$$\leq C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p} \leq C\|f\|_{H\dot{K}^{\alpha,p}_{q,D^mA}}.$$

For J, similar to the proof of Theorem 1, we have, for $x \in C_k$, $j \leq k-3$,

$$\begin{aligned} g_\psi^A(a_j)(x) &\leq C|x-x_0|^{-n-m-1}|B_j|^{1/n}\left(\int_{B_j}|a_j(y)||R_m(\tilde{A};x,y)|dy\right) \\ &\quad +C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}(k-j)|x-x_0|^{-n-1}|B_j|^{1/n}\int_{B_j}|a(y)|dy \\ &\leq C2^{-k(n+1)}2^{j(1+n(1-1/q)-\alpha)}\left(\sum_{|\beta|=m}|D^\beta A(x)-(D^\beta A)_{B_k}|\right) \\ &\quad +C\sum_{|\beta|=m}\|D^\beta A\|_{BMO}(k-j)2^{-k(n+1)}2^{j(1+n(1-1/q)-\alpha)}. \end{aligned}$$

Thus

$$\begin{aligned} J &\leq C\left(\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left[\sum_{j=-\infty}^{k-3}|\lambda_j|2^{-k(n+1)+j(1+n(1-1/q)-\alpha)}\right.\right. \\ &\quad \left.\left.\sum_{|\beta|=m}\left(\int_{B_k}|D^\beta A(x)-(D^\beta A)_{B_k}|^q dx\right)^{1/q}\right]^p\right)^{1/p} \\ &\quad +C\left(\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left[\sum_{j=-\infty}^{k-3}|\lambda_j|(k-j)2^{-k(n+1)+j(1+n(1-1/q)-\alpha)}\right.\right. \\ &\quad \left.\left.2^{kn/q}\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\right]^p\right)^{1/p} \\ &= J_1+J_2. \end{aligned}$$

To estimate J_1 and J_2, we consider two cases:

Case 1. $0 < p \leq 1$.

$$\begin{aligned} J_1 &\leq C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\sum_{j=-\infty}^{k-3}|\lambda_j|^p 2^{[-k(n+1)+j(1+n(1-1/q)-\alpha)]p}\right. \\ &\quad \left. 2^{knp/q}(\sum_{|\beta|=m}\|D^\beta A\|_{BMO})^p\right]^{1/p} \end{aligned}$$

$$
\begin{aligned}
&= C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} 2^{(j-k)(1+n(1-1/q)-\alpha)p} \right]^{1/p} \\
&\leq C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \\
&\leq C\|f\|_{H\dot{K}^{\alpha,p}_{q,D^m A}},
\end{aligned}
$$

$$
\begin{aligned}
J_2 &\leq C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} (k-j)^p 2^{(j-k)(1+n(1-1/q)-\alpha)p} \right]^{1/p} \\
&\leq C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \\
&\leq C\|f\|_{H\dot{K}^{\alpha,p}_{q,D^m A}}.
\end{aligned}
$$

Case 2. $p > 1$. By Hölder's inequality, we deduce that

$$
\begin{aligned}
J_1 &\leq C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \left[\sum_{j=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} |\lambda_j|^p 2^{(j-k)p(1(1-1/q)-\alpha)/2} \right) \right. \\
&\qquad \left. \left(\sum_{j=-\infty}^{k-3} 2^{(j-k)p'(1+n(1-1/q)-\alpha)/2} \right)^{p/p'} \right]^{1/p} \\
&\leq C \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \\
&\leq C\|f\|_{H\dot{K}^{\alpha,p}_{q,D^m A}},
\end{aligned}
$$

$$
\begin{aligned}
J_2 &\leq C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \\
&\qquad \times \left[\sum_{j=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} |\lambda_j|^p (k-j)^p 2^{(j-k)p(1+n(1-1/q)-\alpha)/2} \right) \right. \\
&\qquad \left. \left(\sum_{j=-\infty}^{k-3} 2^{(j-k)p'(1+n(1-1/q)-\alpha)/2} \right)^{p/p'} \right]^{1/p}
\end{aligned}
$$

$$\leq C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} (k-j)^p 2^{(j-k)p(1+n(1-1/q)-\alpha)/2} \right]^{1/p}$$

$$\leq C \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \leq C\|f\|_{H\dot{K}^{\alpha,p}_{q,D^m A}}.$$

This finishes the proof of Theorem 2. □

Remark. Theorem 2 also holds for nonhomogeneous Herz-type space.

Theorem 3. *Let $D^\beta A \in BMO(R^n)$ for $|\beta| = m$ and $0 < p \leq 1 \leq q < \infty$, $\alpha = n(1-1/q)+1$. Then, for any $\lambda > 0$ and $f \in HK^{\alpha,p}_{q,D^m A}(R^n)$, we have*

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, g_\psi^A(f))^{p/q} \right]^{1/p} \leq C\lambda^{-1}\|f\|_{HK^{\alpha,p}_{q,D^m A}} \left(1 + \log^+(\lambda^{-1}\|f\|_{HK^{\alpha,p}_{q,D^m A}})\right).$$

Proof. Let $f \in HK^{\alpha,p}_{q,D^m A}(R^n)$ and $f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 3. We write

$$\begin{aligned}
&\left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, g_\psi^A(f))^{p/q} \right]^{1/p} \\
\leq\ & C \left[\sum_{k=0}^{3} 2^{k\alpha p} \tilde{m}_k(\lambda, g_\psi^A(f))^{p/q} \right]^{1/p} \\
&+C \left[\sum_{k=4}^{\infty} 2^{k\alpha p} \tilde{m}_k \left(\lambda/2, \sum_{j=0}^{k-3} |\lambda_j| g_\psi^A(a_j) \right)^{p/q} \right]^{1/p} \\
&+C \left[\sum_{k=4}^{\infty} 2^{k\alpha p} \tilde{m}_k \left(\lambda/2, g_\psi^A \left(\sum_{j=k-2}^{\infty} \lambda_j a_j \right) \right)^{p/q} \right]^{1/p} \\
=\ & K_1 + K_2 + K_3.
\end{aligned}$$

For $K_1,\ K_3$, by the weak type of (q,q) boundedness for g_ψ^A and $0 < p \leq 1$, we have

$$
\begin{aligned}
K_1 &\le C\lambda^{-1}\left[\sum_{k=0}^{3} 2^{k\alpha p}\|f\|_{L^q}^p\right]^{1/p} \le C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\|a_j\|_{L^q}^p\right)^{1/p} \\
&\le C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\cdot 2^{-j\alpha p}\right)^{1/p} \le C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p} \\
&\le C\lambda^{-1}\|f\|_{HK_{q,D^mA}^{\alpha,p}}, \\
K_3 &\le C\lambda^{-1}\left[\sum_{k=4}^{\infty} 2^{k\alpha p}\left\|\sum_{j=k-2}^{\infty}\lambda_j a_j\right\|_{L^q}^p\right]^{1/p} \le C\lambda^{-1}\left[\sum_{k=4}^{\infty}2^{k\alpha p}\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{-j\alpha p}\right]^{1/p} \\
&\le C\lambda^{-1}\left[\sum_{j=0}^{\infty}|\lambda_j|^p\sum_{k=0}^{j+2}2^{(k-j)\alpha p}\right]^{1/p} \le C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p} \\
&\le C\lambda^{-1}\|f\|_{HK_{q,D^mA}^{\alpha,p}}.
\end{aligned}
$$

For K_2, by the same argument as the proof of Theorem 1 and 2, we have

$$
g_\psi^A(a_j)(x) \le C2^{-k(n+1)}\left(\sum_{|\beta|=m}|D^\beta A(x)-(D^\beta A)_{B_k}| + k\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\right),
$$

therefore

$$
\begin{aligned}
K_2 &\le C\left[\sum_{k=4}^{\infty}2^{k\alpha p}\tilde{m}_k\left(\lambda/4, C2^{-k(n+1)}\sum_{|\beta|=m}|D^\beta A(x)-(D^\beta A)_{B_k}|\sum_{j=0}^{\infty}|\lambda_j|\right)^{p/q}\right]^{1/p} \\
&\quad +C\left[\sum_{k=4}^{\infty}2^{k\alpha p}\tilde{m}_k\left(\lambda/4, C2^{-k(n+\varepsilon)}k\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{j=0}^{\infty}|\lambda_j|\right)^{p/q}\right]^{1/p} \\
&= K_2^{(1)} + K_2^{(2)}.
\end{aligned}
$$

For $K_2^{(1)}$, by using John-Nirenberg inequality (see [19], [20]), we gain

$$
\begin{aligned}
K_2^{(1)} &\le C\left[\sum_{k=4}^{\infty}2^{k\alpha p}\left(\exp\left(-\frac{C2^{k(n+1)}\lambda}{\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)2^{kn}\right)^{p/q}\right]^{1/p} \\
&\le C\left[\sum_{k=0}^{\infty}2^{k(n+1)p}\exp\left(-\frac{C\lambda 2^{k(n+1)}}{\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)\right]^{1/p}
\end{aligned}
$$

$$\begin{aligned}
&\le C\left[\int_0^\infty x^{p-1}\exp\left(-\frac{c\lambda x}{\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{j=0}^\infty|\lambda_j|}\right)dx\right]^{1/p}\\
&= C\lambda^{-1}\|D^\beta A\|_{BMO}\sum_{j=0}^\infty|\lambda_j|\left(\int_0^\infty t^{p-1}e^{-t}dt\right)^{1/p}\\
&\le C\lambda^{-1}\left(\sum_{j=0}^\infty|\lambda_j|^p\right)^{1/p}\\
&\le C\lambda^{-1}\|f\|_{HK^{\alpha,p}_{q,D^mA}}.
\end{aligned}$$

For $K_2^{(2)}$, by using the following fact: If there exists $u > 1$, such that $2^x/x \le u$ for $x \ge 3$, then $2^x \le cu\log^+ u$. We have, if

$$\left|\left\{x\in C_k : C2^{-k(n+1)}k\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{j=0}^\infty|\lambda_j| > \lambda/4\right\}\right| \ne 0,$$

then

$$1 < 2^{k(n+1)/k(n+1)} < C\lambda^{-1}\sum_{|\beta|=m}\|D^\beta A\|_{BMO}\sum_{j=0}^\infty|\lambda_j|,$$

thus

$$2^{k(n+1)} \le C\lambda^{-1}\sum_{j=0}^\infty|\lambda_j|\log^+\left(\lambda^{-1}\sum_{j=0}^\infty|\lambda_j|\right).$$

Let K_λ be the maximal integer k which satisfies this estimate, then

$$\begin{aligned}
I_2^{(2)} &\le C\left(\sum_{k=4}^{K_\lambda}2^{k\alpha p}2^{knp/q}\right)^{1/p} \le C2^{K_\lambda(n+1)}\\
&\le C\lambda^{-1}\sum_{j=0}^\infty|\lambda_j|\log^+\left(\lambda^{-1}\sum_{j=0}^\infty|\lambda_j|\right)\\
&\le C\lambda^{-1}\left(\sum_{j=0}^\infty|\lambda_j|^p\right)^{1/p}\log^+\left(\lambda^{-1}\left(\sum_{j=0}^\infty|\lambda_j|^p\right)^{1/p}\right)\\
&\le C\lambda^{-1}\|f\|_{HK^{\alpha,p}_{q,D^mA}}\log^+\left(\lambda^{-1}\|f\|_{HK^{\alpha,p}_{q,D^mA}}\right).
\end{aligned}$$

Now, summing up the above estimates, we have

$$\left[\sum_{k=0}^\infty 2^{k\alpha p}\tilde{m}_k(\lambda, g_\psi^A(f))^{p/q}\right]^{1/p} \le C\lambda^{-1}\|f\|_{H^{\alpha,p}_{q,D^mA}}\left(1+\log^+\left(\lambda^{-1}\|f\|_{HK^{\alpha,p}_{q,D^mA}}\right)\right).$$

This completes the proof of Theorem 3. □

Acknowledgement. The author would like to express his deep gratitude to the referee for his valuable comments and suggestions.

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