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Two More Radicals for Right Near-Rings: The Right Jacobson Radicals of Type-1 and 2

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Abstract. Near-rings considered are right near-rings and R is a near-ring. $J_0^r(R)$, the right Jacobson radical of R of type-0, was introduced and studied by the present authors. In this paper $J_1^r(R)$ and $J_2^r(R)$, the right Jacobson radicals of R of type-1 and type-2 are introduced. It is proved that both J_1^r and J_2^r are radicals for near-rings and $J_0^r(R) \subseteq J_1^r(R) \subseteq J_2^r(R)$. Unlike the left Jacobson radical classes, the right Jacobson radical class of type-2 contains $M_0(G)$ for many of the finite groups G. Depending on the structure of G, $M_0(G)$ belongs to different right Jacobson radical classes of near-rings. Also unlike left Jacobson-type radicals, the constant part of R is contained in every right 1-modular (2-modular) right ideal of R. For any family of near-rings R_i, $i \in I$, $J_\nu^r\ (\oplus_{i\in I} R_i) = \oplus_{i\in I} J_\nu^r(R_i)$, $\nu \in \{1,2\}$. Moreover, under certain conditions, for an invariant subnear-ring S of a d.g. near-ring R it is shown that $J_2^r(S) = S \cap J_2^r(R)$.

1. Introduction

Throughout this paper we consider only right near-rings. Many radicals of near-rings and the corresponding structure theories have been developed. But almost all of them give structures of near-rings in term of ideals and left ideals but not in terms of right ideals (which are not ideals). In [2] and [3] the first author has established that only right ideals are relevant for the extension of the Wedderburn-Artin theorem to near-rings. This motivated the authors to develop and study the right Jacobson radicals for near-rings. In [4] the right Jacobson radical of type-0 was introduced and studied. In subsequent papers the authors will present the structure theorems of near-rings given by the right Jacobson radicals.
In this paper, the right Jacobson radicals of type-1 and 2 for near-rings are intro-

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duced and studied. It is proved that both of them are radicals for near-rings. Let $(G,+)$ be a finite group. We know that the left ideals of $M_0(G)$ do not depend on the structure of the group G and that left Jacobson radicals also do not depend on the structure of G. We know that $J_2(M_0(G)) = \{0\}$. But, we see in this paper that the right Jacobson radicals of $M_0(G)$ depend on the nature of G and many of them are right Jacobson radical near-rings of type-2, in contrast to the left Jacobson radicals. Unlike left Jacobson-type radicals, the constant part of R is contained in every right 1-modular (2-modular) right ideal of R. For any family of near-rings R_i, $i \in I$, $J^r_\nu(\oplus_{i\in I} R_i) = \oplus_{i\in I} J^r_\nu(R_i)$, where J^r_ν is the right Jacobson radical of type-ν, $\nu \in \{1,2\}$. Moreover, under certain conditions, for an invariant subnear-ring S of a d.g. near-ring R, it is shown that $J^r_2(S) = S \cap J^r_2(R)$.

2. Preliminaries

Throughout this paper R is a right near-ring and all notations and definitions will be as in [1].

Definition 2.1. A group $(G,+)$ is called a *right R-group* if there is a mapping $((g,r) \to gr)$ of $G \times R$ into G such that

(1) $(g+h)r = gr + hr$, and

(2) $g(rs) = (gr)s$, for all $g,\ h \in G$ and $r,\ s \in R$.

Definition 2.2. Let G be a right R-group. An element $g \in G$ is called a *generator* of G, if $gR = G$ and $g(r+s) = gr + gs$ for all $r,\ s \in R$. G is said to be *monogenic*, if G has a generator.

Definition 2.3. Let G be a right R-group. A normal subgroup B of G is called an *ideal* of G if $BR \subseteq B$. A subgroup B of G is called an *R-subgroup* of G if $BR \subseteq B$. G is said to be *simple* if $GR \neq \{0\}$ and $\{0\}$ and G are the only ideals of G.

Definition 2.4. A monogenic right R-group G is said to be of *type-0* if G is simple.

Definition 2.5. Let G, H be right R-groups. A mapping $f : G \to H$ is called an *R-homomorphism* if

(1) $f(x+y) = f(x) + f(y)$ and

(2) $f(xr) = f(x)r$, for all $x,\ y \in G$ and for all $r \in R$.

We say that G is *R-isomorphic* to H if there is a one-to-one R-homomorphism of G onto H.

Definition 2.6. A right ideal K of R is called *right modular* if there is an element $e \in R$ such that $x - ex \in K$ for all $x \in R$. In this case we say that K is *right modular by e*.

Definition 2.7. An element $a \in R$ is called *right quasi-regular* if and only if the right ideal of R generated by the set $\{x - ax \mid x \in R\}$ is R.

We need the following results of [4].

Proposition 2.8. *Let R_c be the constant part of R. Then each element of R_c is right quasi-regular.*

Proposition 2.9. *Let G be a right R-group. Then G is monogenic if and only if there is a right modular right ideal K of R such that G is R-isomorphic to R/K.*

Proposition 2.10. *Let G be a right R-group. G is a right R-group of type-0 if and only if there is a maximal right modular right ideal K of R such that G is R-isomorphic to R/K.*

Proposition 2.11. *Let R be a zero-symmetric near-ring and K be a right ideal of R right modular by e. Then $(K : R) = (K : eR)$ and the largest ideal of R contained in K is the largest ideal of R contained in $(K : R)$.*

Theorem 2.12. *A right 0-primitive ideal of R is a prime ideal of R.*

Proposition 2.13. *Let G be a monogenic right R-group. If R is a distributively generated (d.g.) near-ring then there is a subset T of G such that $h(a+b) = ha+hb$ for all $h \in T$ and $a,\ b \in R$ and T generates $(G, +)$.*

3. Right Jacobson radicals of type-1 and 2

Definition 3.1. A right R-group G of type-0 is said to be of *type-1* if G has exactly two R-subgroups, namely $\{0\}$ and G.

Remark 3.2. Let G be a right R-group. Then $\{g \in G \mid gR = \{0\}\}$ is an ideal of G.

Definition 3.3. A right R-group G of type-0 is said to be of *type-2*, if $gR = G$ for all $0 \neq g \in G$.

Remark 3.4. Clearly a right R-group of type-2 is of type-1.

Now we give some examples of right R-groups of type-0, 1 and 2.

Example 3.5. Let $(G, +)$ be a finite non-abelian simple group. Since $\{0\}$ is the maximal normal subgroup of $(G, +)$, $\{0\}$ is the maximal right ideal of $M_0(G)$ and hence $M_0(G)$ is a right $M_0(G)$-group of type-0. But $M_0(G)$ is not a $M_0(G)$-group of type-1. Let $0 \neq a \in G$ and let H be the cyclic subgroup of G generated a. Now $H \neq \{0\}$ and $H \neq G$. So $(H : G) = \{f \in M_0(G) \mid f(x) \in H, \text{ for all } x \in G\}$ is a right $M_0(G)$-subgroup of $M_0(G)$ and $(H : G) \neq \{0\}$, $(H : G) \neq M_0(G)$. Therefore, $M_0(G)$ is not a right $M_0(G)$-group of type-1.

Example 3.6. Let $(G, +)$ be a finite cyclic group of prime order greater than

2. Then $M_0(G)$ is a right $M_0(G)$-group of type-1. Since $\{0\}$ is the only proper subgroup of G, $\{0\}$ is the only proper right $M_0(G)$-subgroup of $M_0(G)$ and is right modular by the identity element of $M_0(G)$. Therefore, $M_0(G)$ is a right $M_0(G)$-group of type-1. Clearly $M_0(G)$ is not a right $M_0(G)$-group of type-2, as $M_0(G)$ is not a near-field.

Example 3.7. A near-field R is a right R-group of type-2.

Definition 3.8. Let $\nu \in \{0, 1, 2\}$. A right modular right ideal K of R is called *right ν-modular* if R/K is a right R-group of type-ν.

Remark 3.9. Let K be a right ideal of R. K is right 0-modular if and only if K is a maximal right modular right ideal of R and K is a right 1-modular if and only if K is a maximal right modular right R-subgroup of R.

Definition 3.10. Let $\nu \in \{0, 1, 2\}$. An ideal P of R is called *right ν-primitive* if P is the largest ideal of R contained in a right ν-modular right ideal of R.

Definition 3.11. Let $\nu \in \{0, 1, 2\}$. $D^r_\nu(R)$ denotes the intersection of all right ν-modular right ideals of R. If R has no right ν-modular right ideals then $D^r_\nu(R)$ is defined as R.

Definition 3.12. Let $\nu \in \{0, 1, 2\}$. $J^r_\nu(R)$ denotes the intersection of all right ν-primitive ideals of R. If R has no right ν-primitive ideals then $J^r_\nu(R)$ is defined as R. J^r_ν is called the *right Jacobson radical of type-ν*.

Let G be a finite group and $R = M_0(G)$. Then, we have that $\{0\} = P(R) = N(R) = J_0(R) = J_s(R) = J_1(R) = J_2(R) = J_3(R)$. But the following remarks show that for many finite groups G, $M_0(G)$ is a right Jacobson radical near-ring of type-ν, for some $\nu \in \{0, 1, 2\}$.

Remark 3.13. Let R be the near-ring given in Example 3.5. We get that $J^r_0(R) = \{0\}$ and $J^r_1(R) = R$.

Remark 3.14. Let R be the near-ring given in Example 3.6. It is clear that $J^r_1(R) = \{0\}$ and $J^r_2(R) = R$.

The following result shows that the abelian property of $(R, +)$ depends on the abelian property of a monogenic faithful right R-group, if R is d.g..

Proposition 3.15. *Let G be a monogenic right R-group and R be a d.g. near-ring. If $(0 : G) = \{0\}$ and G is abelian then $(R, +)$ is also an abelian group.*

Proof. Suppose that $(0 : G) = \{0\}$ and G is abelian. Since R is d.g., by Proposition 2.13 we get a subset H of G such that H generates $(G, +)$ and $h(r+s) = hr+hs$, for all $h \in H$ and $r,\ s \in R$. Let $r,\ s \in R$. Now $h(r + s - r - s) = hr + hs - hr - hs = 0$ as G is abelian. So, $r + s - r - s \in (0 : H) = (0 : G) = \{0\}$. Therefore, $r + s = s + r$ and hence $(R, +)$ is an abelian group. □

The following propositions follow easily from the definitions.

Proposition 3.16. *Let $\nu \in \{1, 2\}$. Let K be a right ν-modular right ideal of R. Let I be an ideal of R contained in K. Then K/I is also a right ν-modular right ideal of R/I.*

Proposition 3.17. *Let $\nu \in \{1, 2\}$. Let I be an ideal of R. If K/I is a right ν-modular right ideal of R/I then K is also a right ν-modular right ideal of R.*

We see now that for $\nu \in \{1, 2\}$, J^r_ν is a radical map.

Theorem 3.18. *$R \to J^r_\nu(R)$ is a radical map for $\nu \in \{1, 2\}$.*

Proof. Let $\nu \in \{1, 2\}$. First suppose that R has no right ν-modular right ideal. Now $R = J^r_\nu(R)$ and hence $R/J^r_\nu(R) = \{0\}$. So $J^r_\nu(R/J^r_\nu(R)) = \{0\}$. Suppose now that R has a right ν-modular right ideal. Let $\{K_\alpha \mid \alpha \in \Delta\}$ be the collection of all right ν-modular right ideals of R. Since K_α is a right ν-modular right ideal of R and $J^r_\nu(R) \subseteq K_\alpha$, $K_\alpha/J^r_\nu(R)$ is a right ν-modular right ideal of $R/J^r_\nu(R)$ for all $\alpha \in \Delta$. So, $J^r_\nu(R/J^r_\nu(R)) \subseteq \cap_{\alpha\in\Delta} (K_\alpha/J^r_\nu(R)) = (\cap_{\alpha\in\Delta} K_\alpha)/J^r_\nu(R)$. Since $J^r_\nu(R)$ is the largest ideal of R contained in $\cap_{\alpha\in\Delta} K_\alpha$, we get that the largest ideal of $R/J^r_\nu(R)$ contained in $\cap_{\alpha\in\Delta}(K_\alpha/J^r_\nu(R))$ is the zero ideal. Therefore $J^r_\nu(R/J^r_\nu(R)) = \{0\}$.

Let h be a homomorphism of the the near-ring R onto a near-ring S. If S has no right ν-modular right ideal then $J^r_\nu(S) = S$. Then clearly $h(J^r_\nu(R)) \subseteq S = J^r_\nu(S)$. Suppose that S has a right ν-modular right ideal. Let $\{L_\alpha \mid \alpha \in \Delta\}$ be the collection of all right ν-modular right ideals of S.

Now $h^{-1}(L_\alpha)$ is a right ν-modular right ideal of R for each $\alpha \in \Delta$. Let $K_\alpha = h^{-1}(L_\alpha)$, $\alpha \in \Delta$. We have that $h(h^{-1}(L_\alpha)) = L_\alpha$, for all $\alpha \in \Delta$ and also $J^r_\nu(R) \subseteq \cap_{\alpha\in\Delta} K_\alpha$. So $h(J^r_\nu(R)) \subseteq h(\cap_{\alpha\in\Delta} K_\alpha) \subseteq \cap_{\alpha\in\Delta} h(K_\alpha) = \cap_{\alpha\in\Delta} L_\alpha$. Since $h(J^r_\nu(R))$ is an ideal of S and $J^r_\nu(S)$ is the largest ideal of S contained in $\cap_{\alpha\in\Delta} L_\alpha$, $h(J^r_\nu(R)) \subseteq J^r_\nu(S)$. Therefore $R \to J^r_\nu(R)$ is a radical map. □

Theorem 3.19. *$D^r_1(R)$ contains all right quasi-regular right R-subgroups of R.*

Proof. If $D^r_1(R) = R$, then we get the result. Suppose that $D^r_1(R) \neq R$. Let K be a right quasi-regular right R-subgroup of R. Assume that $K \not\subseteq D^r_1(R)$. We get a right 1-modular right ideal M of R such that $K \not\subseteq M$. So, $M + K = R$. Let M be right modular by e. Now $m + k = e$, $m \in M$, $k \in K$. It is clear that $x - ex \in M$, for all $x \in R$. Since $e - k = m \in M$, $ex - kx \in M$, for all $x \in R$. Now $x - kx = (x - ex) + (ex - kx) \in M$, for all $x \in R$. This is a contradiction to the fact that k is right quasi-regular. Therefore, $K \subseteq D^r_1(R)$. Hence, $D^r_1(R)$ contains all right quasi-regular right R-subgroups of R. □

Corollary 3.20. *R_c is a subset of $D^r_1(R)$, where R_c is the constant part of R.*

Proof. By Proposition 2.8, R_c is right quasi-regular. Since R_c is right R-subgroup of R, by Theorem 3.19, R_c is contained in $D^r_1(R)$. □

Corollary 3.21. *If $D^r_1(R)$ is an ideal of R, then $R/D^r_1(R)$ is zero-symmetric.*

Corollary 3.22. *If $D^r_1(R) = \{0\}$, then R is zero-symmetric.*

Theorem 3.23. *$D_2^r(R)$ contains all right quasi-regular subsets of R of the form $aR = \{ax \mid x \in R\}$, $a \in R$.*

Proof. If $D_2^r(R) = R$, then we get the result. Suppose that $D_2^r(R) \neq R$. Let aR be a right quasi-regular subset of R, $a \in R$. Assume that $aR \nsubseteq D_2^r(R)$. We get a right 2-modular right ideal M of R such that $aR \nsubseteq M$. Therefore, $M + aR = R$. Let M be right modular by e. Now $m + ab = e$, $m \in M$, $b \in R$. We have that $x - ex \in M$, for all $x \in R$. Since $e - ab = m \in M$, $ex - abx \in M$, for all $x \in R$. Now $x - abx = (x - ex) + (ex - abx) \in M$, for all $x \in R$. This is a contradiction to the fact that ab is right quasi-regular. Therefore, $aR \subseteq D_2^r(R)$. Hence, $D_2^r(R)$ contains all right quasi-regular subsets of R of the form aR, $a \in R$. □

Corollary 3.24. *If $D_2^r(R)$ is an ideal of R, then $R/D_2^r(R)$ is zero-symmetric.*

Proof. By Corollary 3.20, $R_c \subseteq D_1^r(R) \subseteq D_2^r(R)$. Therefore, $R/D_2^r(R)$ is zero symmetric. □

Corollary 3.25. *If $D_2^r(R) = \{0\}$, then R is zero-symmetric.*

Definition 3.26. R is called a *right ν-primitive near-ring* if $\{0\}$ is a right ν-primitive ideal of R, $\nu \in \{1, 2\}$.

Theorem 3.27. *An ideal P of R is right ν-primitive if and only if R/P is a right ν-primitive near-ring, $\nu \in \{1, 2\}$.*

Proof. Let $\nu \in \{1, 2\}$. Let P be a right ν-primitive ideal of R. we get a right ν-modular right ideal M of R such that P is the largest ideal of R contained in M. M/P is a right ν-modular right ideal of R/P. Since P is the largest ideal of R contained in M, the zero ideal of R/P is the largest ideal of R/P contained in M/P. Therefore R/P is a right ν-primitive near-ring. On the other hand, suppose that R/P is a right ν-primitive near-ring. We get a right ν-modular right ideal M/P of R/P such that the largest ideal of R/P contained in M/P is the zero ideal of R/P. Clearly M is a right ν-modular right ideal of R and P is the largest ideal of R contained in M. Therefore, P is a right ν-primitive ideal of R. □

Theorem 3.28. *A right ν-primitive ideal of R is prime, $\nu \in \{1, 2\}$.*

Proof. Let $\nu \in \{1, 2\}$. Let P be a right ν-primitive ideal of R. We have that a right 0-primitive ideal of R is prime. Since a right ν-primitive ideal of R is right 0-primitive, from Theorem 2.12, we get that P is prime.

4. Properties of the right Jacobson radicals of type-1 and 2

We now see the relation between right ν-primitive ideals of a zero-symmetric near-ring R and the annihilators of the right R-groups of type-ν.

Proposition 4.1. *Let P be an ideal of a zero-symmetric near-ring R. Then P is right ν-primitive if and only if P is the largest ideal of R contained in $(0 : G)$ for some right R-group G of type-ν, $\nu \in \{1, 2\}$.*

Proof. Let P be an ideal of a zero-symmetric near-ring R. Suppose that P is a right ν-primitive ideal of R. So, we get a right ν-modular right ideal K of R such that P is the largest ideal of R contained in K. Now R/K is a right R-group of type-ν. By Proposition 2.11, P is the largest ideal of R contained in $(K : R) = (0 : R/K)$. Conversely, suppose that P is the largest ideal of R contained in $(0 : G)$, where G is a right R-group of type-ν. Now G is R-isomorphic to R/K for some right ν-modular right ideal K of R. So, $(0 : G) = (0 : R/K) = (K : R)$. Since P is the largest ideal of R contained in $(0 : G) = (K : R)$, by Proposition 2.11, P is the largest ideal of R contained in K. Hence P is a right ν-primitive ideal of R. □

Proposition 4.2. *Let R be a d.g. near-ring. Then $J^r_\nu(R) = D^r_\nu(R)$, $\nu \in \{1, 2\}$.*

Proof. If $J^r_\nu(R) = R$, then the result is obvious. Suppose that $J^r_\nu(R) \neq R$. Let P be a right ν-primitive ideal of R, $\nu \in \{1, 2\}$. Since R is d.g., we get a right R-group G of type-ν, such that $P = (0 : G)$. By Proposition 2.13, we get a subset T of G such that T generates $(G, +)$ and $t(r+s) = tr+ts$ for all $t \in T$ and $r,\ s \in R$. If $Tr = \{0\}$ then $Gr = \{0\}$. Therefore, $(0 : T) = (0 : G)$. Let $0 \neq t \in T$. Since tR is a right R-subgroup of G and $tR \neq \{0\}$, $tR = G$. Define $h : R \to G$ by $h(r) = tr$, for all $r \in R$. Clearly h is a R-homomorphism of R onto G with kernel $(0 : t)$. Therefore, $R/(0 : t)$ is R-isomorphic to G. So, $(0 : t)$ is a right ν-modular right ideal of R. Therefore, $(0 : G) = (0 : T) = \cap_{0 \neq s \in T}(0 : s)$ and hence $D^r_\nu(R) \subseteq J^r_\nu(R)$. It is clear that $J^r_\nu(R) \subseteq L$ for each right ν-modular right ideal L of R. So, $J^r_\nu(R) \subseteq D^r_\nu(R)$. Hence, $J^r_\nu(R) = D^r_\nu(R)$. □

Corollary 4.3. *If R is a d.g. near-ring, then $J^r_1(R)$ contains all right quasi-regular right R-subgroups of R.*

Corollary 4.4. *If R is a d.g. near-ring, then $J^r_2(R)$ contains all right quasi-regular subsets of R of the form $aR = \{ax \mid x \in R\}$, $a \in R$.*

We see some of the properties of the J^r_2-radical.

Theorem 4.5. *Let G be a right R-group of type-2. If I is a left invariant ideal of R and $GI \neq \{0\}$, then G is also a right I-group of type-2.*

Proof. We have that G is a right R-group of type-2, I is a left invariant ideal of R and $GI \neq \{0\}$. So for all $0 \neq g \in G$, $gR = G$. Also the right R-group G has a generator h. Now clearly G is a right I-group. Let $0 \neq g \in G$. We claim that $gI = G$. We first show that $gI \neq 0$. We have that $gI \supseteq g(RI) = (gR)I = GI \neq \{0\}$. So, $gI \neq \{0\}$ and hence $(gI)R = G$. Therefore, $G = (gI)R = g(IR) \subseteq gI$. Hence, $gI = G$. This also shows that h is a generator of the right I-group G. Therefore, G is right I-group of type-2. □

Now we want to study the hereditary property of the semisimple class of the radical J^r_2.

Theorem 4.6. *Let R be a d.g. near-ring and I be a non-zero ideal of R. If R is right 2-primitive, then I is also right 2-primitive.*

Proof. Suppose that R is right 2-primitive. Since R is zero-symmetric, we get a right R-group G of type-2 such that $\{0\}$ is the largest ideal of R contained in $(0:G)$. Since R is d.g., $(0:G)$ is an ideal of R and hence $(0:G)=\{0\}$. Now, since $GI \neq \{0\}$, by Theorem 4.5, G is a right I-group of type-2. The largest ideal of I contained in $(0:G)_I$ is a right 2-primitive ideal of I as G is right I-group of type-2. Since $(0:G)_I=\{0\}$, $\{0\}$ is a right 2-primitive ideal of I, that is, I is right 2-primitive. □

Now we develop a few results to prove that for any family of near-rings R_i, $i \in I$, $J^r_\nu(\oplus_{i\in I}R_i)=\oplus_{i\in I}J^r_\nu(R_i)$, $\nu \in \{0,1,2\}$.

Proposition 4.7. *Let I be an ideal of R and let I be a direct summand of R. If K is a right ν-modular right ideal of I, then there is a right ν-modular right ideal M of R such that $K=R\cap M$, $\nu \in \{0,1,2\}$.*

Proof. Let $R=I\oplus J$, J is an ideal of R. Suppose that K is a right ν-modular right ideal of I, $\nu \in \{0,1,2\}$. Let $M=K+J$. Now $M\cap I=K$. Let K be right modular by $e\in I$. Now M is a right ideal of R and right modular in R by e. Define $f: R \longrightarrow I/K$ by $f(r=i+j)=i+K$, for all $r\in R$, where $i\in I$, $j \in J$. Clearly f is a R-homomorphism of R onto I/K and kernel of f is M. Therefore, R/M is R-isomorphic to I/K. So, R/M is isomorphic to I/K as right I-groups. Since I/K is a right I-group of type-ν, R/M is also a right I-group of type-ν, that is, R/M is a right R/J-group of type-ν. Hence, R/M is a right R-group of type-ν, that is, M is a right ν-modular right ideal of R. □

Proposition 4.8. *Let R be the direct sum of its ideals I_1 and I_2 and M be a right ideal of R. Let $M_i=M\cap I_i$, $i=1,2$. If M is a right ν-modular right ideal of R, then for some $i\in\{1,2\}$, M_i is a right ν-modular right ideal of I_i, $\nu\in\{0,1,2\}$.*

Proof. Let M be a right ν-modular right ideal of R. Now M_i is a right ideal of I_i. If $M_i=I_i$, for $i=1,2$, then $M=R$, a contradiction. So, we may assume that $M_1\neq I_1$. Suppose that M is right modular by e_1+e_2, $e_i\in I_i$. If e_2 is in the zero symmetric part of I_2, then as $i_1-(e_1+e_2)i_1=i_1-e_1i_1\in M_1$, for all $i_1\in I_1$, we get that M_1 is right modular by e_1 in I_1. Suppose now that e_2 is in the constant part of I_2. If e_1 is also in the constant part of I_1, then e_1+e_2 is in the constant part of R and hence it must be right quasi-regular, which is a contradiction to the fact that M is right modular by e_1+e_2 and $M\neq R$. Therefore, e_1 is in the zero-symmetric part of I_1. Now $0-(e_1+e_2)0=-e_2\in M$. So, $i_1-(e_1+e_2)i_1=(i_1-e_1i_1)-e_2\in M$ and hence $i_1-e_1i_1\in M$, for all $i_1\in I_1$. Therefore, in this case also M_1 is right modular by e_1 in I_1.

(a) We see now that M_1 is a maximal right ideal of I_1. Suppose that N is a proper right ideal of I_1 properly containing M_1. Now, as N is not contained in M, we have that $N+M=R$. Let $x\in I_1-N$. $x=n+m$, $n\in N$, $m\in M$. Clearly $m\in I_1$ and hence $x\in M_1$, a contradiction. So, M_1 is a maximal right ideal of I_1.

(b) Suppose that M is right 1-modular right ideal of R. So, M is a maximal right

R-subgroup of R. Using the arguments similar to those in (a), we get that M_1 is a maximal right I_1-subgroup of I_1, and hence M_1 is right 1-modular right ideal of I_1.

(c) Suppose that M is a right 2-modular right ideal of R. Let $a \in I_1 - M_1$. Since $a \notin M$, $M + aR = R$. Let $b \in I_1$. $b = m + ar$, $m \in M$ and $r \in R$. Since b, $ar \in I_1$, $m \in I_1$ we get that $m \in M_1$. If $r = i_1 + i_2$, $i_1 \in I_1$, $i_2 \in I_2$ then as $ar \in I_1$, $ar = ai_1 \in aI_1$. Therefore $M_1 + aI_1 = I_1$ and hence M_1 is a right 2-modular right ideal of I_1.

□

Proposition 4.9. *Let I be an ideal of R and I be a direct summand of R. A right ν-primitive ideal P of I, is of the form $P = I \cap Q$, where Q is a right ν-primitive ideal of R, $\nu \in \{0, 1, 2\}$.*

Proof. Let P be a right ν-primitive ideal of I, $\nu \in \{0, 1, 2\}$. We get a right ν-modular right ideal K of I such that P is the largest ideal of I contained in K. By Proposition 4.7 there is a right ν-modular right ideal M of R such that $K = M \cap I$. Let Q be the largest ideal of R contained in M. Now Q is a right ν-primitive ideal of R. Since an ideal T of I is also an ideal of R, we have that P is an ideal of R contained in $I \cap M$. So, $P \subseteq I \cap Q$. Since an ideal of R contained in I is also an ideal of I, $Q \cap I$ is an ideal of I contained in K. So, $Q \cap I \subseteq P$. Therefore $P = I \cap Q$. □

Proposition 4.10. *Let I be an ideal of R and I be a direct summand of R. If Q is a right ν-primitive ideal R, then $Q \cap I = I$ or a right ν-primitive ideal of I, $\nu \in \{0, 1, 2\}$.*

Proof. Let Q be a right ν-primitive ideal of R, $\nu \in \{0, 1, 2\}$. We get a right ν-modular right ideal M of R such that Q is the largest ideal of R contained in M. If $I \subseteq M$, then $Q \cap I = I$. If $I \nsubseteq M$, then as seen in Proposition 4.8, $I \cap M$ is a right ν-modular right ideal of I. Since $Q \cap I$ is the largest ideal of I contained in $I \cap M$, it is a right ν-primitive ideal of I. □

Theorem 4.11. *Let I be an ideal of R and I be a direct summand of R. Then, $J^r_\nu(I) = I \cap J^r_\nu(R)$, $\nu \in \{0, 1, 2\}$.*

Proof. Let $\nu \in \{0, 1, 2\}$. If I has no right ν-primitive ideal, then by Proposition 4.10, it follows that I is contained in every right ν-primitive ideal of R and hence $J^r_\nu(I) = I = I \cap J^r_\nu(R)$. Suppose now that I has right ν-primitive ideals. Now by Propositions 4.9 and 4.10, $J^r_\nu(I) = \cap\{P \mid P$ is a right ν-primitive ideal of $I\} = \cap\{Q \cap I \mid Q$ is a right ν-primitive ideal of $R\} = I \cap J^r_\nu(R)$. □

Theorem 4.12. *Let R_i, $i \in I$, be a family of near-rings. Then $J^r_\nu(\oplus_{i\in I} R_i) = \oplus_{i\in I} J^r_\nu(R_i)$, $\nu \in \{0, 1, 2\}$.*

Proof. Let $\nu \in \{0, 1, 2\}$. Since J^r_ν is a radical map, we have that $J^r_\nu(\oplus_{i\in I} R_i) \subseteq \oplus_{i\in I} J^r_\nu(R_i)$. We show that $J^r_\nu(\oplus_{i\in I} R_i) \supseteq \oplus_{i\in I} J^r_\nu(R_i)$. Since R_i is a direct summand of $\oplus_{i\in I}R_i$, by Theorem 4.11, we have $J^r_\nu(R_i) \subseteq R_i \cap J^r_\nu(\oplus_{i\in I}R_i)$. Therefore, $J^r_\nu(\oplus_{i\in I}R_i) \supseteq \oplus_{i\in I}J^r_\nu(R_i)$. Hence, $J^r_\nu(\oplus_{i\in I}R_i) = \oplus_{i\in I}J^r_\nu(R_i)$. □

Now we develop a more general result related to the hereditariness of the J^r_2-radical.

Theorem 4.13. *Let S be an invariant subnear-ring of R and let K be a right 2-modular right ideal of S. Then K is an ideal of the right R-group S.*

Proof. Let $s \in S$. Suppose that $sS \subseteq K$. We claim now that $sR \subseteq K$. On the contrary suppose that $sR \not\subseteq K$. We have $sR \subseteq S$. Let $t \in sR - K$. Now $tS + K = S$, as K is a right 2-modular right ideal of S, $t \notin K$ and $t \in S$. Now $t = sx$, $x \in R$. Since $tS = sxS \subseteq sS \subseteq K$, we have that $K = S$, a contradiction. So, $sR \subseteq K$. Therefore, as $K \subseteq S$ and $KS \subseteq K$, we have $KR \subseteq K$. Hence K is an ideal of the right R-group S. □

Theorem 4.14. *Let S be an ideal of a d.g. near-ring R. Suppose that $(S, +)$ is generated by $S \cap D$, where D is the set of all distributive elements in R. If T is a right 2-primitive ideal of S then T is an ideal of R.*

Proof. Let T be a right 2-primitive ideal of S. Since S is d.g., we have $T = (0 : G)$ for a right S-group G of type-2. Now G is S-isomorphic to S/K for some right 2-modular right ideal K of S. So, $T = (0 : S/K)_S = (K : S)_S = S \cap (K : S)$. By Theorem 4.13, K is an ideal of the right R-group S. Therefore, S/K is a right R-group. Since $S \cap D$, generates $(S, +)$, $(0 : S/K) = (K : S)$ is an ideal of R. Hence, $T = S \cap (K : S)$ is an ideal of R. □

Theorem 4.15. *Let S be an invariant subnear-ring of the d.g. near-ring R. Suppose that $(S, +)$ is generated by $S \cap D$, where D is the set of all distributive elements in R. Then, $J^r_2(S) = S \cap J^r_2(R)$.*

Proof. Let T be a right 2-primitive ideal of S. As seen in the Theorem 4.14, there is a right 2-modular right ideal K of S such that $T = S \cap (K : S)$, where K is an ideal of the right R-group S and S/K is a right R-group. Choose $d \in (S \cap D) - K$. Now $(d + K)S = S/K$ and hence $(d + K)R = S/K$. Since $d \in D$, $d + K$ is a generator of the right R-group, S/K. If $s \in S$ and $s + K \neq K$ then $(s + K)S = S/K$ and hence $(s + K)R = S/K$. Therefore, S/K is a right R-group of type-2. So, $(K : S)$ is a right 2-primitive ideal of R, as R is d.g.. Let $\mathcal{A}$ be the collection of all right 2-primitive ideals $\{T_\alpha \mid \alpha \in \Delta\}$ of S. If $\mathcal{A}$ is empty then clearly $J^r_2(S) = S \supseteq J^r_2(R) \cap S$. Suppose that $\mathcal{A}$ is not empty. For each $\alpha \in \Delta$ we get a right 2-primitive ideal I_α of R such that $T_\alpha = I_\alpha \cap S$. Now $J^r_2(S) = \cap_{\alpha\in\Delta}T_\alpha = \cap_{\alpha\in\Delta}(I_\alpha \cap S) = S \cap (\cap_{\alpha\in\Delta}I_\alpha) \supseteq S \cap J^r_2(R)$. Therefore, we have $J^r_2(S) \supseteq J^r_2(R) \cap S$.

We prove now that $J^r_2(S) \subseteq J^r_2(R) \cap S$. Let J be a right 2-primitive ideal of R. Since R is d.g., we get a right 2-modular right ideal L of R such that $J = (L : R)$, where R/L is a right R-group of type-2. If $S \subseteq L$ then $RS \subseteq S \subseteq L$ and hence $S \subseteq J$. So, we get that $J^r_2(S) \subseteq S \subseteq J \cap S$. Now Suppose that $S \not\subseteq L$. Since L is a right ideal of R, $S \cap L$ is a right ideal of S. So, $S/(S \cap L)$ is a right S-group. We show that $S \cap L$ is a right 2-modular right ideal of S, that is, $S/(S \cap L)$ is a right S-group of type-2. We have that $S \neq S \cap L$ and hence $S/(S \cap L)$ is a non zero right S-group. Let $s \in S - L$. We see that $(s + (S \cap L))S = S/(S \cap L)$, that is, $sS + (S \cap L) = S$. Since L is a right 2-modular right ideal of R, we get $ae \in R$ such that $r - er \in L$ for all $r \in R$. Now, as L is right 2-modular, $sR + L = R$. Now $sr + l = e$, for some $r \in R$ and $l \in L$. Let $t \in S$. Now $et = srt + lt$. Since $et - t \in L$ we have that $srt + lt - t = l_1$, for some $l_1 \in L$. Now $srt - t = l_2$, for some $l_2 \in L$. So, $t \in sS + L$. Therefore, $S \subseteq sS + L$ and hence $S = sS + (L \cap S)$. We get a distributive element d of R in $S - L$. Now $d + (L \cap S)$ is a generator of the right S-group $S/(L \cap S)$. Hence, $S/(L \cap S)$ is a right S-group of type-2. Since $R/L = S + L/L$ is R-isomorphic to $S/(L \cap S)$, we have that R/L is also S-isomorphic to $S/(L \cap S)$. Therefore, as $(S, +)$ is generated by $S \cap D$, we have that $((L \cap S) : S)_S$ is a right 2-primitive ideal of S and $((L \cap S) : S)_S = (L : R)_S = (L : R) \cap S = J \cap S$. So, $J \cap S$ is a right 2-primitive ideal of S. Hence, $J_2^r(S) \subseteq J \cap S$. So, $J_2^r(S) \subseteq J_2^r(R) \cap S$. Therefore, $J_2^r(S) = S \cap J_2^r(R)$. □

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