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On Idempotent Reflexive Rings

Jin Yong Kim
Department of Mathematics and Institute of Natural Sciences, Kyung Hee University, Suwon 449-701, Korea
e-mail : jykim@khu.ac.kr

Jong Uk Baik
Department of Mathematics, Kyung Hee University, Seoul 130-701, Korea
e-mail : jubag@khu.ac.kr

Abstract. We introduce in this paper the concept of idempotent reflexive right ideals and concern with rings containing an injective maximal right ideal. Some known results for reflexive rings and right HI-rings can be extended to idempotent reflexive rings. As applications, we are able to give a new characterization of regular right self-injective rings with nonzero socle and extend a known result for right weakly regular rings.

Throughout this paper, R denotes an associative ring not necessarily with unity unless otherwise stated. A right ideal I is said to be *reflexive* [2] if $aRb \subseteq I$ implies $bRa \subseteq I$ for $a, b \in R$. A ring R is called *reflexive* if 0 is a reflexive ideal. In this paper we define an idempotent reflexive right ideal which is a nontrivial generalization of a reflexive right ideal. Some known results of Mason [2] are extended. For an idempotent reflexive ring R with unity, we prove that if R contains an injective maximal right ideal, then R is right self-injective. As a byproduct of this result, we obtain a new characterization of regular right self-injective rings with nonzero socle. This characterization is then used to prove that an idempotent reflexive right HI-ring is semisimple Artinian. Consequently we extend nontrivially a result in [7]. Moreover we show that if R is an idempotent reflexive ring with unity and every simple singular right R-module is p-injective then R is a right weakly regular ring.

Definition 1. A right ideal I is called *idempotent reflexive* if $aRe \subseteq I$ if and only if $eRa \subseteq I$ for $a, e = e^2 \in R$. We say that R is an *idempotent reflexive ring* when 0 is an idempotent reflexive ideal.

Note that any prime ideal is reflexive. Since an intersection of reflexive right ideals is reflexive, all semiprime ideals are reflexive. Recall that a ring R is called *abelian* if every idempotents in R is central. Obviously an abelian ring with unity is

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an idempotent reflexive ring. We say a ring R with unity satisfies condition (SI) [5] if the left annihilator $\ell(x)$, is a two-sided ideal for each $x \in R$, equivalently if $ab = 0$ then $aRb = 0$ for $a, b \in R$. Note that any ring satisfying condition (SI) is abelian. Also observe that if R has unity, then R is a reflexive ring satisfying condition (SI) if and only if $\ell(x) = r(x)$ for each $x \in R$, where $r(x)$ is the right annihilator of x. The following example is essentially due to Birkenmeier, Kim and Park [1, Example 2.8]

Example 2. There is an idempotent reflexive ring which is not reflexive.

Assume that $F\{X, Y\}$ is the free algebra over a field F generated by X and Y, and $\langle YX\rangle$ is the two-sided ideal of $F\{X, Y\}$ generated by the element YX. Let $R = F\{X, Y\}/\langle YX\rangle$. Put $x = X + \langle YX\rangle$ and $y = Y + \langle YX\rangle$ in R. Then $R = \{f_0(x) + f_1(x)y + \cdots + f_n(x)y^n | n = 0, 1, 2, \cdots ,$ and $f_i(x) \in F[x]\}$, the polynomial ring such that $yx = 0$. Now let α, β be nonzero elements in R satisfying $\alpha\beta = 0$. Say $\alpha = f_0(x) + f_1(x)y + \cdots + f_n(x)y^n$ and $\beta = g_0(x) + g_1(x)y + \cdots + g_m(x)y^m$ with $f_n(x) \neq 0$ and $g_m(x) \neq 0$.

Case 1: $f_0(x) = 0$. Then $\alpha x\beta = f_0(x)x\beta = 0$. From the fact that $yg(x) = g(0)y$ for $g(x) \in F[x]$, it can be checked that $g_0(0) = g_1(0) = \cdots = g_m(0) = 0$. Thus $\alpha y\beta = \alpha(g_0(0) + g_1(0)y + \cdots + g_m(0)y^m)y = 0$. Thus $\alpha R\beta = 0$.

Case 2: $g_0(x) = 0$. Of course we may assume that $f_0(x) \neq 0$. In this case, it also can be checked that $g_1(x) = g_2(x) = \cdots = g_m(x) = 0$, a contradiction to $g_m(x) \neq 0$.

From these we have $\alpha\beta = 0$ implies $\alpha R\beta = 0$ for $\alpha, \beta \in R$. So it is easily checked that R is an abelian ring. Hence R is an idempotent reflexive ring.

Proposition 3. *If R is an idempotent reflexive ring and e is an idempotent of R, then the following are equivalent.*

(1) *eR is an idempotent reflexive right ideal.*

(2) *eR is a two-sided ideal.*

(3) *e is central.*

Proof. (1) $\Rightarrow$ (2): Let $i \in eR = I$. Then for some $x \in R$, we have $i = ex \in R^2I$ and hence $I \subseteq R^2I$. Since eR is an idempotent reflexive right ideal and $eRy \subseteq eR$ for any $y \in R$, we have $yRe \subseteq eR$. Thus $R^2e \subseteq eR$. Hence $I = R^2I$. For any $a \in R$ and $i \in I$, we have that $ai = aex = aeex \in R^2I$. Therefore $I = eR$ is a two-sided ideal.

(2) $\Rightarrow$ (3): For any $x \in R$, $xe = xee \in x(eR) \subseteq eR$. Thus $xe = er$ for some $r \in R$. Hence $exe = er = xe$. Also for any $s \in R$, we have that $sexe = sxe$. Thus $(se - s)xe = 0$, so $(se - s)Re = 0$. Since R is idempotent reflexive, $eR(se - s) = 0$. Thus $ese = es$ for any $s \in R$. Therefore we have $ex = xe$ for any $x \in R$.

(3) $\Rightarrow$ (1): Assume that $xRf \subseteq eR$ where f is an idempotent of R. For any $r \in R$, it follows that $xrf = ey$ for some $y \in R$. Thus $exrf = ey = xrf$, so

$(ex-x)rf = 0$. Since R is idempotent reflexive and $(ex-x)Rf = 0$, we have $fR(ex-x) = 0$. Hence $frex = frx$. Now e is central, so $frx \in eR$. Thus $fRx \subseteq eR$ and hence eR is idempotent reflexive. □

Corollary 4. *If every principal right ideal of R is idempotent reflexive, then R is abelian.*

In general the existence of an injective maximal right ideal in a ring R can not guarantee the right self-injectivity of R. But we have the following result.

Proposition 5. *Let R be an idempotent reflexive ring with unity. If R contains an injective maximal right ideal, then R is right self-injective.*

Proof. Let M be an injective maximal right ideal of R. Then $R = M \oplus N$, where N is a minimal right ideal. Hence we have $M = eR$ and $N = (1-e)R$ for some $0 \neq e = e^2 \in R$. If $NM = 0$, then we have $(1-e)Re = 0$. Since R is idempotent reflexive, $eR(1-e) = 0$. So e is central. Hence we can write $R = M \oplus N = Re \oplus R(1-e)$. Thus the left module ${}_R(R/M)$ is projective (hence flat). By a result of Ramanurthi [4], the right module $(R/M)_R$ is injective. Hence N_R is injective. If $NM \neq 0$, then $NM = N$. So there exists $b \in N$ such that $bM \neq 0$, whence $N = bM$. Let $f : M \longrightarrow N$ be the map defined by $f(m) = bm$ for each $m \in M$. Then f is an epimorphism. Since the right module N_R is projective and $M/\ker f \simeq N$, we have $M \simeq \ker f \oplus M/\ker f \simeq \ker f \oplus N$ as right R-modules. Thus N_R is injective. At any rate, N is injective. Hence $R = M \oplus N$ is right self-injective. □

Corollary 6. *Let R be a semiprime* (*or an abelian*) *ring with unity. If R contains an injective maximal right ideal, then R is right self-injective.*

Recall that a ring R is called a right *pp* if every principal right ideal of R is projective. As an application of Proposition 5, we have the following result.

Corollary 7. *For a ring R with unity, the following are equivalent.*

(1) *R is a regular right self-injective ring with $Soc(R_R) \neq 0$, where $Soc(R_R)$ is the right socle of R.*

(2) *R is an idempotent reflexive right pp-ring containing an injective maximal right ideal.*

Proof. (1) ⇒ (2): Since R is a regular ring, R is an idempotent reflexive right *pp*-ring. If every maximal right ideal of R is essential, then $Soc(R_R)$ is contained in the Jacobson radical of R, which is absurd. So there is a maximal right ideal M of R which is not essential. Therefore M is a direct summand of R. Since R is right self-injective, M is an injective right ideal.

(2) ⇒ (1): By Proposition 5, R is right self-injective. Hence R is regular because R is right *pp*. Also we have $Soc(R_R) \neq 0$ since there is an injective maximal right ideal. □

By [7], a ring R is called a *right HI-ring* if R is a right hereditary ring containing an injective maximal right ideal. Osofsky [3] proves that a right self-injective right hereditary ring is semisimple Artinian.

The next corollary extends Theorem 8 in [7].

Corollary 8. *For a ring R with unity, the following statements are equivalent.*

(1) *R is semisimple Artinian.*

(2) *R is an idempotent reflexive right HI-ring.*

Proof. (1) $\Rightarrow$ (2): Obvious.
(2) $\Rightarrow$ (1): Proposition 5 and Osofsky's theorem in [3]. □

Recall that a right R-module M is called to be *right p-injective* if every right R-homomorphism from a principal right ideal aR to M extends to one from R to M. Ming [6] proved that if R is a semiprime ring whose simple singular right R-module is p-injective then R is a right weakly regular ring. We extend this result as follows.

Lemma 9. *If every simple singular right R-module is p-injective, then for every element $a \in R$, there exists a right ideal K such that $R = (RaR + r(a)) \oplus K$.*

Proof. See [6, Lemma 1]. □

Proposition 10. *Let R be an idempotent reflexive ring with unity. If every simple singular right R-module is p-injective, then R is a right weakly regular ring.*

Proof. For every $a \in R$, by lemma 9, we have $R = (RaR + r(a)) \oplus K$ for some right ideal K. Let $K = eR$ where $e = e^2 \in R$. Then $eRaR = KRaR \subseteq RaR \cap K = 0$, hence $eRa = 0$. Thus $aRe = 0$ since R is idempotent reflexive. Hence $K \subseteq ReR \subseteq r(a)$. Thus $K = 0$. Therefore $R = RaR + r(a)$ for every $a \in R$. Hence R is a right weakly regular ring. □

Corollary 11. *Let R be a semiprime (or an abelian) ring with unity. If every simple singular right R-module is p-injective, then R is right weakly regular.*

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