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New Lacunary Strong Convergence Difference Sequence Spaces Defined by Sequence of Moduli

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Abstract. In this paper, we define $\triangle^m$-Lacunary strongly convergent sequences defined by sequence of moduli and give various properties and inclusion relations on these sequence spaces.

1. Introduction

Let ω be the set of all sequences of real or complex numbers and $l_\infty,\ c$ and c_0 be the sets of all bounded, convergent sequences and sequences convergent to zero respectively, normed by

$$||x||_\infty = \sup_k |x_k|,$$

where $k \in \mathbb{N} = \{1, 2, \cdots\}$, the set of positive integers.

The difference sequence space $X(\triangle)$ was introduced by Kizmaz [3] as follows

$$X(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in X\} \text{ for } X = \ l_\infty,\ c \text{ and } c_0,$$

where $\triangle x_k = (x_k - x_{k+1})$ for all $k \in \mathbb{N}$.

The difference sequence spaces were generalized by Et and Colak [1] as follows

$$X(\triangle^m) = \{x = (x_k) \in \omega : \triangle^m x = (\triangle^m x_k) \in X\} \text{ for } X = l_\infty,\ c \text{ and } c_0,$$

where $\triangle^m x_k = (\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1})$.

A sequence of positive integers $\theta = (k_r)$ is called "lacunary" if $k_0 = 0,\ 0 < k_r < K_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r)$ and $q_r = \frac{k_r}{k_{r-1}}$. The space of lacunary strongly convergent sequence L_θ was defined by Freedman et al [2] as:

$$L_\theta = \{x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - l| = 0 \text{ for some } l\}.$$

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The space L_θ is a BK-space with the norm

$$||x||_\theta = \sup_r \frac{1}{h_r} \sum_{k \in I_r} |x_k|.$$

L_θ^0 denotes the subset of L_θ those sequences for which $l = 0$ in the definition of L_θ. $(L_\theta^0, ||.||_\theta)$ is also a BK-space. There is a relation (see [2]) between L_θ and the space $|\sigma_1|$ of strongly Cesaro summable sequences defined by

$$|\sigma_1| = \{x \in \omega : \frac{1}{n} \sum_{k=1}^{n} |x_k - l| = 0, \text{ for some } l\}.$$

For $\theta = (2^r)$ we have $L_\theta = |\sigma_1|$.

Definition 1.1. A function $f : [0, \infty) \to [0, \infty)$ is called a modular if

(1) $f(t) = 0$ if and only if $t = 0$,

(2) $f(t + u) \leq f(t) + f(u)$ for all $t,\ u \geq 0$,

(3) f is increasing, and

(4) f is continuous from the right of 0.

Let X be a sequence space. Then the sequence space $X(f)$ is defined as

$$X(f) = \{x = (x_k) : (f(|x_k|)) \in X\}$$

for a modulus f([6], [8]).

Kolk [4], [5] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$ i.e.,

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.$$

2. Main results

For a sequence $F = (f_k)$ of moduli, we define following sequence spaces

$$\begin{aligned}
L_\theta(\triangle^m, F) &= \{x \in \omega : \lim_r \frac{1}{h_r} \sum_{k \in I_r} f_k(|\triangle^m x_k - l|) = 0 \text{ for some } l\},\\
L_\theta^0(\triangle^m, F) &= \{x \in \omega : \lim_r \frac{1}{h_r} \sum_{k \in I_r} f_k(|\triangle^m x_k|) = 0 \},\text{ and}\\
L_\theta^\infty(\triangle^m, F) &= \{x \in \omega : \lim_r \frac{1}{h_r} \sum_{k \in I_r} f_k(|\triangle^m x_k|) < \infty\}.
\end{aligned}$$

Theorem 2.1. *The sets* $L_\theta^0(\triangle^m, F)$, $L_\theta(\triangle^m, F)$ *and* $L_\theta^\infty(\triangle^m, F)$ *are linear spaces.*

Proof. Let $x, y \in L_\theta(\triangle^m, F)$ and $\alpha, \beta \in \mathbb{C}$. Then exists positive integers N_α and M_β such that $|\alpha| \leq N_\alpha$ and $|\beta| \leq M_\beta$. From the definition of modulus function and $\triangle^m$ we have

$$\begin{aligned} & \frac{1}{h_r}\sum_{k\in I_r} f_k(|\triangle^m(\alpha x_k + \beta y_k) - (\alpha l_1 + \beta l_2)|) \\ \leq \; & N_\alpha \frac{1}{h_r}\sum_{k\in I_r} f_k(|\triangle^m x_k - l_1|) + M_\beta \frac{1}{h_r}\sum_{k\in I_r} f_k(|\triangle^m y_k - l_2|) \to 0, \quad r\to\infty. \end{aligned}$$

Thus $L_\theta(\triangle^m, F)$ is a linear space. □

Lemma 1 ([7]). *Let* $F = (f_k)$ *be a sequence of moduli and let* $0 < \delta < 1$. *Then for each* $x > \delta$ *we have* $f_k(x) \leq 2f_k(1)\delta^{-1}x$.

Theorem 2.2. *Let* $F = (f_k)$ *be a sequence of moduli. Then* $L_\theta(\triangle^m) \subset L_\theta(\triangle^m, F)$.

Proof. Let $x \in L_\theta(\triangle^m)$. Then we have

$$\tau_r = \frac{1}{h_r}\sum_{k\in I_r} |\triangle^m x_k - l| \to 0 \;\; as \;\; r\to\infty, \quad \text{for some } l. \tag{2.1}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_k(u) < \epsilon$ for every u with $0 \leq u \leq \delta$. Then we can write

$$\begin{aligned} & \frac{1}{h_r}\sum_{k\in I_r} f_k(|\triangle^m x_k - l|) \\ = \; & \frac{1}{h_r}\sum_{k\in I_r, |\triangle^m x_k - l|\leq\delta} f_k(|\triangle^m x_k - l|) + \frac{1}{h_r}\sum_{k\in I_r, |\triangle^m x_k - l|>\delta} f_k(|\triangle^m x_k - l|) \\ \leq \; & \frac{1}{h_r}(h_r\epsilon) + \frac{1}{h_r} 2f_k(1)\delta^{-1}h_r\tau_r \text{ (from Lemma 1).} \end{aligned}$$

Therefore $x \in L_\theta(\triangle^m, F)$. □

Theorem 2.3. *Let* $F = (f_k)$ *be a sequence of moduli, if* $\lim_{t\to\infty}\frac{f_k(t)}{t} = \gamma > 0$, *then* $L_\theta(\triangle^m) = L_\theta(\triangle^m, F)$.

Proof. We need to show that $L_\theta(\triangle^m, F) \subset L_\theta(\triangle^m)$. Let $\gamma > 0$ and $x \in L_\theta(\triangle^m, F)$. Since $\gamma > 0$, we have $f_k(t) > \gamma t$ for all $t \geq 0$.
Hence we have

$$\frac{1}{h_r}\sum_{k\in I_r} f_k(|\triangle^m x_k - l|) \geq \frac{1}{h_r}\sum_{k\in I_r} \gamma|\triangle^m x_k - l| = \frac{1}{h_r}\gamma\sum_{k\in I_r} |\triangle^m x_k - l|.$$

Therefore we have $x \in L_\theta(\triangle^m)$. Hence $L_\theta(\triangle^m, F) \subset L_\theta(\triangle^m)$. On the other hand, by Theorem 2.2 we have $L_\theta(\triangle^m) \subset L_\theta(\triangle^m, F)$. Thus $L_\theta(\triangle^m) = L_\theta(\triangle^m, F)$. □

Theorem 2.4. *Let $m \geq 1$ be a fixed integer, then*

(1) $L^0_\theta(\triangle^{m-1}, F) \subset L^0_\theta(\triangle^m, F)$;

(2) $L_\theta(\triangle^{m-1}, F) \subset L_\theta(\triangle^m, F)$;

(3) $L^\infty_\theta(\triangle^{m-1}, F) \subset L^\infty_\theta(\triangle^m, F)$;

and the inclusions are strict.

Proof. The proof of the inclusions follows from the following inequality

$$\frac{1}{h_r}\sum_{k\in I_r} f_k(|\triangle^m x_k|) \leq \frac{1}{h_r}\sum_{k\in I_r} f_k(|\triangle^{m-1} x_k|) + \frac{1}{h_r}\sum_{k\in I_r} f_k(|\triangle^{m-1} x_{k+1}|).$$

To show the inclusions are strict, let $\theta = (2^r)$ and $x = (k^m)$. Then $x \in (\triangle^m, F)$, but $x \notin (\triangle^{m-1}, F)$. If $x = (k^m)$, then $\triangle^m x = (-1)^m m!$ and $\triangle^{m-1} x_k = (-1)^{m+1} r!(k + \frac{(m-1)}{2})$. □

Theorem 2.5. *Let $\theta = (k_r)$ be a lacunary sequence. If $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, then $|\sigma_1|(\triangle^m, F) = L_\theta(\triangle^m, F)$, where*

$$|\sigma_1|(\triangle^m, F) = \{x \in \omega : \frac{1}{n}\sum_{k=1}^{n} f_k(|\triangle^m x_k - l| = 0, \text{ for some } l\}.$$

Proof. Let $\liminf_r q_r > 1$, then there exists $\delta > 0$ such that $q_r = \frac{k_r}{k_r+1} \geq 1+\delta$ for all $r \geq 1$. Furthermore we have $\frac{k_r}{h_r} \leq \frac{(1+\delta)}{\delta}$ and $\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}$ for all $r \geq 1$. Then we write

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} f_k(|\triangle^m x_i|) &= \frac{1}{h_r}\sum_{i=1}^{k_r} f_k(|\triangle^m x_i|) - \frac{1}{h_r}\sum_{i=1}^{k_{r-1}} f_k(|\triangle^m x_i|) \\ &= \frac{k_r}{h_r}\left(k_r^{-1}\frac{1}{h_r}\sum_{i=1}^{k_r} f_k(|\triangle^m x_i|) - \frac{k_{r-1}}{h_r}\left(k_{r-1}^{-1}\sum_{i=1}^{k_{r-1}} f_k(|\triangle^m x_i|)\right)\right).\end{aligned}$$

Now suppose that the $\limsup_r q_r < \infty$ and let $\epsilon > 0$ be given. Then there exists j_0 such that for every $j \geq j_0$

$$A_j = \frac{1}{h_j}\sum_{i\in I_j} f_k(|\triangle^m x_i|) < \epsilon.$$

Choose a number $M > 0$ such that $A_j \leq M$ for all j. If $\limsup_r q_r < \infty$, then there exists a number $\beta > 0$ such that $q_r < \beta$ for every r. Now let n be any integer with $k_{r-1} < n < k_r$. Then

$$\begin{aligned} & \frac{1}{n}\sum_{i=1}^{n} f_k(|\triangle^m x_i|) \le k_{r-1}^{-1}\sum_{i=1}^{k_r} f_k(|\triangle^m x_i|) \\ = \quad & k_{r-1}^{-1}\left\{\sum_{i\in I_1} f_k(|\triangle^m x_i|) + \cdots + \sum_{i\in I_r} f_k(|\triangle^m x_i|)\right\} \\ = \quad & k_{r-1}^{-1}\left\{\sum_{j=1}^{j_0}\sum_{i\in I_j} f_k(|\triangle^m x_i|) + \sum_{j=j_0+1}^{r}\sum_{i\in I_j} f_k(|\triangle^m x_i|)\right\} \\ \le \quad & k_{r-1}^{-1}\left\{\sum_{j=1}^{j_0}\sum_{i\in I_j} f_k(|\triangle^m x_i|) + \epsilon(k_r - k_{j_0})k_{r-1}^{-1}\right\} \\ = \quad & k_{r-1}^{-1} h_1 A_1 + h_2 A_2 \cdots + h_{j_0} A_{j_0} + \epsilon(k_r - k_{j_0})k_{r-1}^{-1} \\ \le \quad & k_{r-1}^{-1}(\sup_{1\le i\le j_0} A_j) k_{j_0} + \epsilon(k_r - k_{j_0})k_{r-1}^{-1} \\ < \quad & M k_{r-1}^{-1} k_{j_0} + \epsilon\beta. \end{aligned}$$

Thus $x \in |\sigma_1|(\triangle^m, F)$. □

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