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Factor Rank and Its Preservers of Integer Matrices

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ABSTRACT. We characterize the linear operators which preserve the factor rank of integer matrices. That is, if $\mathcal{M}$ is the set of all $m \times n$ matrices with entries in the integers and $\min(m, n) > 1$, then a linear operator T on $\mathcal{M}$ preserves the factor rank of all matrices in $\mathcal{M}$ if and only if T has the form either $T(X) = UXV$ for all $X \in \mathcal{M}$, or $m = n$ and $T(X) = UX^tV$ for all $X \in \mathcal{M}$, where U and V are suitable nonsingular integer matrices. Other characterizations of factor rank-preservers of integer matrices are also given.

1. Introduction

The research of Linear Preserver Problems is an active area of matrix theory (see [1]-[7]). Many researchers have studied on the ranks and their preservers of matrices over fields ([1]-[5]). Also (nonnegative) integer matrices are combinatorially interesting matrices and hence it has been a subject of many research works ([6], [7]).

If $\mathbb{F}$ is an algebraically closed field, which linear operators T on the space of $m\times n$ matrices over $\mathbb{F}$ preserve the rank of each matrix? Evidently if U and V are $m \times m$ and $n \times n$ nonsingular matrices, respectively, then $X \to UXV$ is a rank-preserving linear operator. When $m = n$, $X \to UX^tV$ is also. Already in 1957 Marcus and Moyls [4] found that such (U, V)-operators were the only rank preservers. Later they [5] obtained that T preserves all ranks if and only if T preserves rank 1. In 1981, Lautemann [3] extended these results to an arbitrary field, and found that T preserves all ranks if and only if T is bijective and preserves rank 1 if and only if T is a (U, V)-operator.

In this paper, we characterize linear operators which preserve the factor ranks of all matrices over the ring of integers. That is, if $\mathcal{M}$ is the set of all $m \times n$ matrices with entries in the integers and $\min(m, n) > 1$, then a linear operator T on $\mathcal{M}$ preserves the factor rank of all matrices in $\mathcal{M}$ if and only if T has the form either $T(X) = UXV$ for all $X \in \mathcal{M}$, or $m = n$ and $T(X) = UX^tV$ for all $X \in \mathcal{M}$, where U and V are suitable nonsingular integer matrices. Other characterizations of factor rank-preservers of integer matrices are also given.

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2. Preliminaries and basic results

Let $\mathcal{M}_{m\times n}(\mathbb{Z})$ denote the set of all $m \times n$ matrices with entries in the ring, $\mathbb{Z}$ of integers. Addition, multiplication by scalars, and the product of matrices are defined as if $\mathbb{Z}$ were a field. Let $\mathbb{E}_{m,n} = \{E_{ij} \mid i = 1, \cdots, m \text{ and } j = 1, \cdots, n\}$, where E_{ij} is the $m \times n$ matrix whose $(i, j)^{\text{th}}$ entry is 1 and whose other entries are 0. We call each member of $\mathbb{E}_{m,n}$ a *cell*.

Lowercase, boldface letters will represent vectors, a vector $\boldsymbol{u}$ is column vector ($\boldsymbol{u}^t$ is a row vector). A nonzero vector $\boldsymbol{p} = [p_i]$ in $\mathbb{Z}^n$ is *irreducible* if the greatest common divisor of nonzero p_i's is 1 (that is, $\gcd(p_1, \cdots, p_n) = 1$). A subset $S = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_d\}$ of $\mathbb{Z}^n$ is called *linearly dependent* if there exist $\alpha_1, \alpha_2, \cdots, \alpha_d$ in $\mathbb{Z}$, not all zeros, such that $\sum_{i=1}^{d} \alpha_i \boldsymbol{s}_i = \boldsymbol{0}$; S is called *linearly independent* if it is not linearly dependent.

An $n\times n$ integer matrix A is called *nonsingular* if for any vector $\boldsymbol{x}$ in $\mathbb{Z}^n$, $A\boldsymbol{x} = \boldsymbol{0}$ implies that $\boldsymbol{x} = \boldsymbol{0}$. We note that nonsingularity and invertibility of a square integer matrix are not equivalent. For example, consider a matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ in $\mathcal{M}_{2\times 2}(\mathbb{Z})$. Then we can easily show that A is nonsingular but not invertible in $\mathcal{M}_{2\times 2}(\mathbb{Z})$.

Lemma 2.1. *Let $\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_n$ be linearly independent vectors in $\mathbb{Z}^n$. Then for any nonzero vector $\boldsymbol{b}$ in $\mathbb{Z}^n$, there exist nonzero integer β and integers α_i, not all zero, such that $\beta\boldsymbol{b} = \alpha_1\boldsymbol{p}_1 + \alpha_2\boldsymbol{p}_2 + \cdots + \alpha_n\boldsymbol{p}_n$.*

Proof. Let A be the $n \times n$ matrix whose columns are $\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_n$. Then A is nonsingular, and hence $\det(A)$ is a nonzero integer. Consider a system $A\boldsymbol{x} = \boldsymbol{b}$ of n linear equations in n unknowns. By Cramer's rule, this system has a unique solution $x_i = \frac{\det(A_i)}{\det(A)}$ in the rational numbers for all $i = 1, 2, \cdots, n$, where A_i is the matrix obtained by replacing the entries in the i^{th} column of A by the entries in $\boldsymbol{b}$. Then we have

$$\boldsymbol{b} = \frac{\det(A_1)}{\det(A)}\boldsymbol{p}_1 + \frac{\det(A_2)}{\det(A)}\boldsymbol{p}_2 + \cdots + \frac{\det(A_n)}{\det(A)}\boldsymbol{p}_n.$$

If we take $\beta = \det(A)$ and $\alpha_i = \det(A_i)$, then the result follows. □

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are nonzero vectors in $\mathbb{Z}^n$, we denote $\boldsymbol{a} \simeq \boldsymbol{b}$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ have an irreducible common factor. That is, $\boldsymbol{a} \simeq \boldsymbol{b}$ if and only if there exists an irreducible vector $\boldsymbol{p}$ in $\mathbb{Z}^n$ such that $\boldsymbol{a} = \alpha\boldsymbol{p}$ and $\boldsymbol{b} = \beta\boldsymbol{p}$ for some nonzero integers α and β. Then we can easily show that $\simeq$ is an equivalence relation in $\mathbb{Z}^n$.

Proposition 2.2. *If $\boldsymbol{a}$ and $\boldsymbol{b}$ are nonzero vectors in $\mathbb{Z}^n$ with $\alpha\boldsymbol{a} = \beta\boldsymbol{b}$ for some nonzero integers α and β, then we have $\boldsymbol{a} \simeq \boldsymbol{b}$.*

Proof. Let $\boldsymbol{a} = [a_1, \cdots, a_n]$, $\boldsymbol{b} = [b_1, \cdots, b_n]$ and $\alpha' = \gcd(a_1, \cdots, a_n)$. Then there exists an irreducible vector $\boldsymbol{p}$ in $\mathbb{Z}^n$ such that $\boldsymbol{a} = \alpha'\boldsymbol{p}$. Thus $\alpha\boldsymbol{a} = \beta\boldsymbol{b}$ becomes

$$\alpha\alpha'\boldsymbol{p} = \beta\boldsymbol{b}. \tag{2.1}$$

Let $\gamma = \gcd(\alpha\alpha', \beta)$, $\gamma_1 = \frac{\alpha\alpha'}{\gamma}$ and $\gamma_2 = \frac{\beta}{\gamma}$. Then γ_1 and γ_2 are nonzero in $\mathbb{Z}$, and (2.1) becomes

$$\gamma_1 \boldsymbol{p} = \gamma_2 \boldsymbol{b}. \tag{2.2}$$

Therefore we have that γ_1 divides every $\gamma_2 b_i$ for all $i = 1, \cdots, n$. Since $\gcd(\gamma_1, \gamma_2) = 1$ and $\boldsymbol{p}$ is an irreducible vector, $\gamma_2 = \pm 1$ so that $\boldsymbol{b} = \pm\gamma_1 \boldsymbol{p}$. Therefore $\boldsymbol{a}$ and $\boldsymbol{b}$ have an irreducible common factor $\boldsymbol{p}$, and thus $\boldsymbol{a} \simeq \boldsymbol{b}$. □

The *factor rank*, $fr(A)$, of a nonzero matrix $A \in \mathcal{M}_{m\times n}(\mathbb{Z})$ is defined as the least integer k for which there exist $m \times k$ and $k \times n$ matrices B and C, respectively, with $A = BC$. If the matrices were considered as matrices in the real field, then the factor ranks of them are the same as their ranks. The factor rank of a zero matrix is zero.

It is obvious that for a matrix A in $\mathcal{M}_{m\times n}(\mathbb{Z})$, $fr(A) = 1$ if and only if there exist two nonzero vectors $\boldsymbol{a} \in \mathbb{Z}^m$ and $\boldsymbol{x} \in \mathbb{Z}^n$ such that $A = \boldsymbol{a}\boldsymbol{x}^t$. We call $\boldsymbol{a}$ the *left factor*, and $\boldsymbol{x}$ the *right factor* of A.

For any index $i \in \{1, \cdots, n\}$, we denote $\boldsymbol{e}_i^{(n)}$ as the irreducible vector in $\mathbb{Z}^n$ with "1" in i^{th} position and zero elsewhere.

Lemma 2.3. *Let A and B be factor rank-1 matrices in $\mathcal{M}_{m\times n}(\mathbb{Z})$ with factorizations $A = \boldsymbol{a}\boldsymbol{x}^t$ and $B = \boldsymbol{b}\boldsymbol{y}^t$, where $A + B \neq 0$. Then $fr(A+B) = 1$ if and only if $\boldsymbol{a} \simeq \boldsymbol{b}$ or $\boldsymbol{x} \simeq \boldsymbol{y}$.*

Proof. Suppose that $fr(A + B) = 1$. Let

$$A = \boldsymbol{a}\,\boldsymbol{x}^t = [x_1\boldsymbol{a}, \cdots, x_n\boldsymbol{a}] = [a_1\boldsymbol{x}^t, \cdots, a_m\boldsymbol{x}^t]^t$$

and

$$B = \boldsymbol{b}\,\boldsymbol{y}^t = [y_1\boldsymbol{b}, \cdots, y_n\boldsymbol{b}] = [b_1\boldsymbol{y}^t, \cdots, b_m\boldsymbol{y}^t]^t.$$

If $A + B$ has exactly one nonzero i^{th} row or exactly one nonzero j^{th} column, so do A and B. In this case, A and B have an irreducible common left factor $\boldsymbol{e}_i^{(m)}$ or an irreducible common right factor $\boldsymbol{e}_j^{(n)}$. Thus we can assume that $A + B$ has at least two nonzero rows and at least two nonzero columns. Furthermore, without loss of generality, we may assume that columns of $A + B$ are all nonzero.

Case 1) $x_i y_i = 0$ for some $i \in \{1, \cdots, n\}$. If $x_i = 0$, then $y_i \neq 0$ because $A + B$ has no zero column. Since A is not a zero matrix, there exists an index j different from i such that $x_j \neq 0$. Therefore, the i^{th} and j^{th} columns of $A + B$ are $y_i\boldsymbol{b}$ and $x_j\boldsymbol{a} + y_j\boldsymbol{b}$, respectively. Since $fr(A + B) = 1$, there exist nonzero scalars α, β in $\mathbb{Z}$ such that $\alpha y_i \boldsymbol{b} = \beta(x_j\boldsymbol{a} + y_j\boldsymbol{b})$, equivalently $\beta x_j \boldsymbol{a} = (\alpha y_i - \beta y_j)\boldsymbol{b}$. Since $\beta x_j \neq 0$, we have $\alpha y_i - \beta y_j \neq 0$. It follows from Proposition 2.2 that $\boldsymbol{a} \simeq \boldsymbol{b}$. Similarly, a parallel argument holds if $y_i = 0$.

Case 2) $x_i y_i \neq 0$ for all $i = 1, \cdots, n$. Consider any distinct i^{th} and j^{th} columns of $A + B$. Since $fr(A + B) = 1$, there exist two nonzero scalars α and β in $\mathbb{Z}$ such that $\alpha(x_i\boldsymbol{a} + y_i\boldsymbol{b}) = \beta(x_j\boldsymbol{a} + y_j\boldsymbol{b})$, equivalently $(\alpha x_i - \beta x_j)\boldsymbol{a} = (\beta y_j - \alpha y_i)\boldsymbol{b}$. If $\alpha x_i - \beta x_j \neq 0$, then we have $\beta y_j - \alpha y_i \neq 0$. By Proposition 2.2, we have $\boldsymbol{a} \simeq \boldsymbol{b}$. Now, if $\alpha x_i - \beta x_j = 0$, then $\alpha x_i - \beta x_j = \beta y_j - \alpha y_i = 0$. Thus,

$$\alpha x_i = \beta x_j \quad \text{and} \quad \beta y_j = \alpha y_i.$$

This shows that $x_i y_j = x_j y_i$ for all $i, j = 1, \cdots, n$. Thus there exist nonzero integers s and t such that $sx_i = ty_i$ for all $i = 1, \cdots, n$. Therefore we have $s\boldsymbol{x} = t\boldsymbol{y}$. It follows from Proposition 2.2 that $\boldsymbol{x} \simeq \boldsymbol{y}$. Thus we have shown the sufficiency.

The necessity is an immediate consequence. □

3. Factor rank-1 preserver

Suppose that T is a linear operator on $\mathcal{M}_{m\times n}(\mathbb{Z})$. Then T is a

(i) (U,V)-*operator* if there exist nonsingular matrices U in $\mathcal{M}_{m\times m}(\mathbb{Z})$ and V in $\mathcal{M}_{n\times n}(\mathbb{Z})$ such that $T(X) = UXV$ for all X in $\mathcal{M}_{m\times n}(\mathbb{Z})$, or $m = n$ and $T(X) = UX^tV$ for all X in $\mathcal{M}_{m\times n}(\mathbb{Z})$, where X^t denotes the transpose of X;

(ii) *factor rank preserver* if $fr(T(X)) = fr(X)$ for all X in $\mathcal{M}_{m\times n}(\mathbb{Z})$;

(iii) *factor rank-k preserver* if $fr(T(X)) = k$ whenever $fr(X) = k$ for all X in $\mathcal{M}_{m\times n}(\mathbb{Z})$.

Lemma 3.1. *If T is a (U,V)-operator on $\mathcal{M}_{m\times n}(\mathbb{Z})$, then T is an injective factor rank preserver.*

Proof. It follows directly from the definition of a (U,V)-operator. □

Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and a linear operator T on $\mathcal{M}_{2\times 2}(\mathbb{Z})$ defined by $T(X) = AX$ for all X in $\mathcal{M}_{2\times 2}(\mathbb{Z})$. Then T is a (U,V)-operator because A is nonsingular. Clearly, T is injective. But T is not surjective: for any cell E_{ij} in $\mathbb{E}_{2,2}$, there is not a matrix X in $\mathcal{M}_{2\times 2}(\mathbb{Z})$ such that $T(X) = E_{ij}$. Therefore a (U,V)-operator on $\mathcal{M}_{m\times n}(\mathbb{Z})$ may not be invertible.

For any matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $\mathcal{M}_{m\times n}(\mathbb{Z})$, let $A \circ B$ denote the *Hadamard* (or *Schur*) product, the $(i,j)^{\text{th}}$ entry of $A \circ B$ is $a_{ij}b_{ij}$.

Lemma 3.2. *Let $B = [b_{ij}]$ be a factor rank-1 matrix in $\mathcal{M}_{m\times n}(\mathbb{Z})$. Then there exist diagonal matrices D in $\mathcal{M}_{m\times m}(\mathbb{Z})$ and E in $\mathcal{M}_{n\times n}(\mathbb{Z})$ such that $X \circ B = DXE$ for all X in $\mathcal{M}_{m\times n}(\mathbb{Z})$.*

Proof. If $fr(B) = 1$, then there exist vectors $\boldsymbol{d} = [d_1, d_2, \cdots, d_m]^t$ and $\boldsymbol{e} = [e_1, e_2, \cdots, e_n]^t$ such that $B = \boldsymbol{d}\boldsymbol{e}^t$, equivalently $b_{ij} = d_i e_j$ for all $i = 1, \cdots, m$ and $j = 1, \cdots, n$. Let $D = diag(d_1, \cdots, d_m)$ and $E = diag(e_1, \cdots, e_n)$. Now, the $(i,j)^{\text{th}}$ entry of $X \circ B$ is $x_{ij}b_{ij}$ and the $(i,j)^{\text{th}}$ entry of DXE is $d_i x_{ij} e_j = x_{ij}b_{ij}$. Therefore we have the results. □

Theorem 3.3. *Let T be a linear operator on $\mathcal{M}_{m\times n}(\mathbb{Z})$. Then T is an injective factor rank-*1 *preserver if and only if T is a (U,V)-operator.*

Proof. The sufficiency follows from Lemma 3.1. So, we shall show the necessity. For any cell E_{ij} in $\mathbb{E}_{m,n}$, we can write $T(E_{ij}) = \boldsymbol{u}^{ij}{\boldsymbol{v}_{ij}}^t$ for all $i = 1, \cdots, m$ and $j = 1, \cdots, n$, where $\boldsymbol{u}^{ij} \in \mathbb{Z}^m$ and $\boldsymbol{v}_{ij} \in \mathbb{Z}^n$ are nonzero vectors. Let j and k be arbitrary integers in $\{1, \cdots, n\}$. Since $E_{ij} + E_{ik}$ is of factor rank-1, the factor rank of $T(E_{ij} + E_{ik}) = \boldsymbol{u}^{ij}{\boldsymbol{v}_{ij}}^t + \boldsymbol{u}^{ik}{\boldsymbol{v}_{ik}}^t$ must be 1. It follows from Lemma 2.3 that $\boldsymbol{u}^{ij} \simeq \boldsymbol{u}^{ik}$ or $\boldsymbol{v}_{ij} \simeq \boldsymbol{v}_{ik}$. Now, we will show that for a fixed i in $\{1, \cdots, m\}$, either

$$\boldsymbol{u}^{i1} \simeq \boldsymbol{u}^{i2} \simeq \cdots \simeq \boldsymbol{u}^{in} \quad \text{or} \quad \boldsymbol{v}_{i1} \simeq \boldsymbol{v}_{i2} \simeq \cdots \simeq \boldsymbol{v}_{in}. \tag{3.1}$$

Suppose that $\boldsymbol{v}_{i1} \not\simeq \boldsymbol{v}_{ij}$ for some index j. By Lemma 2.3, we have $\boldsymbol{u}^{i1} \simeq \boldsymbol{u}^{ij}$ because $fr(T(E_{i1} + E_{ij})) = 1$. If $\boldsymbol{u}^{i1} \not\simeq \boldsymbol{u}^{ik}$ for some index k, then we have $\boldsymbol{v}_{i1} \simeq \boldsymbol{v}_{ik}$ by Lemma 2.3. Therefore $\boldsymbol{v}_{ij} \not\simeq \boldsymbol{v}_{ik}$ because $\simeq$ is an equivalence relation. But then $\boldsymbol{u}^{ij} \simeq \boldsymbol{u}^{ik}$ and this would imply $\boldsymbol{u}^{i1} \simeq \boldsymbol{u}^{ik}$ because $\boldsymbol{u}^{i1} \simeq \boldsymbol{u}^{ij}$. This contradicts to $\boldsymbol{u}^{i1} \not\simeq \boldsymbol{u}^{ik}$, and thus (3.1) is established.

Similarly, we can show that for a fixed j in $\{1, \cdots, n\}$, either

$$\boldsymbol{u}^{1j} \simeq \boldsymbol{u}^{2j} \simeq \cdots \simeq \boldsymbol{u}^{mj} \tag{3.2}$$

or

$$\boldsymbol{v}_{1j} \simeq \boldsymbol{v}_{2j} \simeq \cdots \simeq \boldsymbol{v}_{mj}. \tag{3.3}$$

If $\boldsymbol{u}^{i1} \simeq \boldsymbol{u}^{i2} \simeq \cdots \simeq \boldsymbol{u}^{in}$, there exist an irreducible vector $\boldsymbol{p}_i$ in $\mathbb{Z}^m$ and nonzero integers c_j such that $\boldsymbol{u}^{ij} = c_j\boldsymbol{p}_i$ for all $j = 1, \cdots, n$. Thus we have $T(E_{ij}) = \boldsymbol{p}_i(c_j\boldsymbol{v}_{ij})^t$ for all $j = 1, \cdots, n$. We can therefore restate (3.1) as follows. For a fixed i in $\{1, \cdots, m\}$, either

$$\boldsymbol{u}^{i1} = \boldsymbol{u}^{i2} = \cdots = \boldsymbol{u}^{in} = \boldsymbol{p}_i \tag{3.4}$$

or

$$\boldsymbol{v}_{i1} = \boldsymbol{v}_{i2} = \cdots = \boldsymbol{v}_{in} = \boldsymbol{q}_i, \tag{3.5}$$

where $\boldsymbol{p}_i$ and $\boldsymbol{q}_i$ are irreducible vectors.

Assume that (3.4) holds for some i. If $\boldsymbol{v}_{i1}, \boldsymbol{v}_{i2}, \cdots, \boldsymbol{v}_{in}$ are linearly dependent, then there exist $\alpha_1, \alpha_2, \cdots, \alpha_n$ in $\mathbb{Z}$, not all zeros, such that $\sum_{j=1}^{n} \alpha_j \boldsymbol{v}_{ij} = \boldsymbol{0}$. Consider a factor rank-1 matrix $X = \sum_{j=1}^{n} \alpha_j E_{ij}$. Then we have

$$T(X) = T\left(\sum_{j=1}^{n} \alpha_j E_{ij}\right) = \boldsymbol{p}_i \left(\sum_{j=1}^{n} \alpha_j \boldsymbol{v}_{ij}\right)^t = \boldsymbol{0},$$

a contradiction to the fact that T is a factor rank-1 preserver. Thus $\boldsymbol{v}_{i1}, \boldsymbol{v}_{i2}, \cdots, \boldsymbol{v}_{in}$ are linearly independent. Analogous statements are satisfied in case (3.2), (3.3) or (3.5).

Next, we will show that if (3.4) holds for a fixed i, then (3.3) must hold for all $j = 1, \cdots, n$, and consequently (3.4) must hold for all i. Suppose that (3.2) holds for some $j = 1, \cdots, n$. Then $\boldsymbol{u}^{ij} (= \boldsymbol{p}_i)$ appears both in (3.4) and (3.2). It follows from (3.2) that there exist nonzero integers α_s such that $\boldsymbol{u}^{sj} = \alpha_s \boldsymbol{p}_i$ for all $s = 1, \cdots, m$. Note that $\boldsymbol{v}_{i1}, \boldsymbol{v}_{i2}, \cdots, \boldsymbol{v}_{in}$ are linearly independent since (3.4) is satisfied. By Lemma 2.1, there exist nonzero integer β_s and integers β_{sk}, not all zero, such that $\beta_s \boldsymbol{v}_{sj} = \sum_{k=1}^{n} \beta_{sk} \boldsymbol{v}_{ik}$ for all $s = 1, \cdots, m$. Then we have

$$\beta_s \boldsymbol{u}^{sj} {\boldsymbol{v}_{sj}}^t = \sum_{k=1}^{n} \beta_{sk} \boldsymbol{u}^{sj} {\boldsymbol{v}_{ik}}^t = \sum_{k=1}^{n} \beta_{sk} \alpha_s \boldsymbol{p}_i {\boldsymbol{v}_{ik}}^t = \sum_{k=1}^{n} \beta_{sk} \alpha_s \boldsymbol{u}^{ik} {\boldsymbol{v}_{ik}}^t,$$

equivalently $T(\beta_s E_{sj}) = T\Big(\sum_{k=1}^{n} \beta_{sk} \alpha_s E_{ik} \Big)$ for all $s \in \{1, \cdots, m\} \setminus \{i\}$. This contradicts to the fact that T is injective. Thus we have established that either

$$\boldsymbol{u}^{ij} = \boldsymbol{p}_i \quad \text{and} \quad \boldsymbol{v}_{ij} = b_{ij} \boldsymbol{q}_j \tag{3.6}$$

for all $i = 1, \cdots, m$ and $j = 1, \cdots, n$, where $\boldsymbol{p}_1, \cdots, \boldsymbol{p}_m$ and $\boldsymbol{q}_1 \cdots, \boldsymbol{q}_n$ are linearly independent irreducible vectors and b_{ij} are nonzero integers, or

$$\boldsymbol{v}_{ij} = \boldsymbol{q}_i \quad \text{and} \quad \boldsymbol{u}^{ij} = b_{ij} \boldsymbol{p}_j \tag{3.7}$$

for all $i = 1, \cdots, m$ and $j = 1, \cdots, n$, where $\boldsymbol{q}_1, \cdots, \boldsymbol{q}_m$ and $\boldsymbol{p}_1 \cdots, \boldsymbol{p}_n$ are linearly independent irreducible vectors and b_{ij} are nonzero integers.

If $m \neq n$, (3.7) is not possible. For, if $m < n$, then the set $\{\boldsymbol{p}_1, \cdots, \boldsymbol{p}_n\}$ would be linearly dependent by Lemma 2.1. Similar conclusion follows if $m > n$. Hence, if $m \neq n$, only (3.6) is possible.

Assume that (3.6) holds. Let U' be the $m \times m$ matrix whose columns are $\boldsymbol{p}_1, \cdots, \boldsymbol{p}_m$ and let V' be the $n \times n$ matrix whose rows are $\boldsymbol{q}_1, \cdots, \boldsymbol{q}_n$. Then U' and V' are nonsingular, and

$$T(E_{ij}) = \boldsymbol{u}^{ij} {\boldsymbol{v}_{ij}}^t = \boldsymbol{p}_i b_{ij} {\boldsymbol{q}_j}^t = U'(b_{ij} E_{ij}) V'$$

for all $i = 1, \cdots, m$ and $j = 1, \cdots, n$. It follows that for any matrix X in $\mathcal{M}_{m \times n}(\mathbb{Z})$, we have $T(X) = U'(X \circ B)V'$, where $B = [b_{ij}]$ as above. Now, we claim $fr(B) = 1$. If not, there exists a 2×2 submatrix $B' = \begin{bmatrix} b_{ij} & b_{ik} \\ b_{lj} & b_{lk} \end{bmatrix}$ of B such that $fr(B') = 2$. Consider a factor rank-1 matrix $Y = E_{ij} + E_{ik} + E_{lj} + E_{lk}$. Then the factor rank of

$$T(Y) = \boldsymbol{p}_i (b_{ij} \boldsymbol{q}_j + b_{ik} \boldsymbol{q}_k)^t + \boldsymbol{p}_l (b_{lj} \boldsymbol{q}_j + b_{lk} \boldsymbol{q}_k)^t$$

must be 1. Since $\boldsymbol{p}_i \not\simeq \boldsymbol{p}_l$, it follows that $b_{ij} \boldsymbol{q}_j + b_{ik} \boldsymbol{q}_k \simeq b_{lj} \boldsymbol{q}_j + b_{lk} \boldsymbol{q}_k$. Therefore there exist an irreducible vectors $\boldsymbol{q}$ and nonzero integers α and β such that $b_{ij} \boldsymbol{q}_j + b_{ik} \boldsymbol{q}_k =$

$\alpha\boldsymbol{q}$ and $b_{lj}\boldsymbol{q}_j + b_{lk}\boldsymbol{q}_k = \beta\boldsymbol{q}$, equivalently $(b_{ij}\beta - b_{lj}\alpha)\boldsymbol{q}_j = (b_{lk}\alpha - b_{ik}\beta)\boldsymbol{q}_k$. It follows from $\boldsymbol{q}_j \not\approx \boldsymbol{q}_k$ that $b_{ij}\beta - b_{lj} = b_{lk}\alpha - b_{ik}\beta = 0$ so that $b_{ij}b_{lk} = b_{ik}b_{lj}$. This implies that the factor rank of B' is 1, a contradiction. Therefore we have $fr(B) = 1$. By Lemma 3.2, there exist diagonal matrices D in $\mathcal{M}_{m\times m}(\mathbb{Z})$ and E in $\mathcal{M}_{n\times n}(\mathbb{Z})$ such that $X \circ B = DXE$ for all X in $\mathcal{M}_{m\times n}(\mathbb{Z})$. Since B has no zero entries, it follows that D and E are nonsingular. Let $U = U'D$ and $V = EV'$. Then U and V are nonsingular. Furthermore, we have $T(X) = UXV$ for all matrix X in $\mathcal{M}_{m\times n}(\mathbb{Z})$. Therefore T is a (U, V)-operator.

If (3.7) holds, then $m = n$ and we can easily establish that for any matrix X in $\mathcal{M}_{m\times n}(\mathbb{Z})$, $T(X) = UX^tV$ for some $n\times n$ nonsingular matrices U and V. Therefore T is a (U, V)-operator. □

4. Factor rank preserver

In this section, we characterize the linear operators which preserve the factor rank of all matrices over the ring of integers.

Proposition 4.1. *Let A and B be matrices in $\mathcal{M}_{m\times n}(\mathbb{Z})$ with $\alpha A \neq \beta B$ for all nonzero scalars $\alpha, \beta \in \mathbb{Z}$. If $fr(A) = fr(B) = 1$, then there exists a factor rank-1 matrix C in $\mathcal{M}_{m\times n}(\mathbb{Z})$ such that $fr(A + C) = 1$ and $fr(B + C) = 2$.*

Proof. Since $fr(A) = fr(B) = 1$, it follows from $\alpha A \neq \beta B$ that either $fr(A+B) = 2$ or $fr(A + B) = 1$. For the case of $fr(A + B) = 2$, the conclusion is satisfied by letting $C = A$. So we may assume that $fr(A + B) = 1$. By Lemma 2.3, A and B have an irreducible common factor. If A and B have an irreducible common left factor, then we may write A and B as

$$A = \boldsymbol{a}\,\boldsymbol{x}^t = [x_1\boldsymbol{a}, \cdots, x_n\boldsymbol{a}] \quad \text{and} \quad B = \boldsymbol{a}\,\boldsymbol{y}^t = [y_1\boldsymbol{a}, \cdots, y_n\boldsymbol{a}],$$

where $\boldsymbol{a}$ is an irreducible vector. Then we have $\alpha\boldsymbol{x} \neq \beta\boldsymbol{y}$ for all nonzero integers α and β because $\alpha A \neq \beta B$. Since $\boldsymbol{a} = [a_i]$ is not zero-vector, $a_i \neq 0$ for some $i = 1, \cdots, m$. Let

$$C = \begin{cases} \boldsymbol{e}_j^{(m)}\,\boldsymbol{x}^t & \text{if } a_j = 0 \text{ for some } j \neq i, \\ \boldsymbol{e}_i^{(m)}\,\boldsymbol{x}^t & \text{otherwise.} \end{cases}$$

Then C is a matrix in $\mathcal{M}_{m\times n}(\mathbb{Z})$ with $fr(C) = 1$. Moreover $fr(A+C) = 1$ because A and C have a common right factor. But B and C have neither a common left factor nor a common right factor. It follows from Lemma 2.3 that $fr(B + C) = 2$.

Similarly, a parallel argument holds if A and B have an irreducible common right factor. □

Lemma 4.2. *Let T be a factor rank-1 preserver on $\mathcal{M}_{m\times n}(\mathbb{Z})$. If T is not injective, then T decreases the factor rank of some factor rank-2 matrix.*

Proof. By the similar proof to that of Theorem 3.3, we can see that T is a (U, V)-operator if T is a factor rank-1 preserver and is injective in the set of all factor rank-1 matrices in $\mathcal{M}_{m\times n}(\mathbb{Z})$. If T is not injective, then T is not a (U,V)-operator. From above fact we have that T is not injective in the set of all factor rank-1 matrices in $\mathcal{M}_{m\times n}(\mathbb{Z})$. Thus there exist distinct factor rank-1 matrices X and Y such that $T(X) = T(Y)$. Suppose that there exist distinct nonzero integers α and β such that $\alpha X = \beta Y$. Then we have

$$\alpha T(X) = T(\alpha X) = T(\beta Y) = \beta T(Y) = \beta T(X).$$

Since $\mathbb{Z}$ has no zero divisors and $T(X) \neq O$, we have $\alpha = \beta$, a contradiction. So, we may assume that $\alpha X \neq \beta Y$ for all nonzero scalars $\alpha, \beta \in \mathbb{Z}$. By Proposition 4.1, there exists a factor rank-1 matrix C such that $fr(X+C) = 1$ while $fr(Y+C) = 2$. But we then have $T(Y+C) = T(X+C)$ so that $fr(T(Y+C)) = fr(T(X+C)) = 1$ because T is a factor rank-1 preserver. Therefore T decreases the factor rank of some factor rank-2 matrix. □

Theorem 4.3. *Let T be a linear operator on $\mathcal{M}_{m\times n}(\mathbb{Z})$. Then the following are equivalent;*

(i) *T is an injective factor rank-1 preserver;*

(ii) *T is a (U,V)-operator;*

(iii) *T is a factor rank preserver;*

(iv) *T is a factor rank-1 and factor rank-2 preserver.*

Proof. It follows from Theorem 3.3 that (i) and (ii) are equivalent. Statement (ii) implies (iii) by Lemma 3.1. Clearly, (iii) implies (iv). Lemma 4.2 shows if T preserves the factor ranks of all factor rank-1 matrices and factor rank-2 matrices, then T is injective. Thus, (iv) implies (i). □

Thus we have characterized the linear operators that preserve the factor rank of integer matrices.

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