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Oscillations of Difference Equations with Several Terms

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Abstract. In this paper, we obtain sufficient conditions for the oscillation of every solution of the difference equation

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} + q x_{n-z} = 0, \ n = 0, 1, 2, \cdots ,$$

where $p_i \in \mathbb{R}$, $k_i \in \mathbb{Z}$ for $i = 1, 2, \cdots, m$ and $z \in \{-1, 0\}$. Furthermore, we obtain sufficient conditions for the oscillation of all solutions of the equation

$$\Delta^r x_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0\,, \ n = 0, 1, 2, \cdots ,$$

where $p_i \in \mathbb{R}$, $k_i \in \mathbb{Z}$ for $i = 1, 2, \cdots, m$. The results are given terms of the p_i and the k_i for each $i = 1, 2, \cdots, m$.

1. Introduction

The concept of the oscillatory behavior of solutions of difference equations has been extensively investigated, see [1]-[8] and the reference cited therein. In [7], Ladas established a theorem for the oscillatory behavior of all solutions for the following difference equation

$$x_{n+1} - x_n + p x_{n-k} + q x_{n-z} = 0\,, \quad n = 0, 1, 2, \cdots , \tag{1.1}$$

where $p, q \in \mathbb{R}$, $k \in \mathbb{Z}$ and $z \in \{-1, 0\}$. In equation (1.1), the case $q = 0$ was examined in [3], [6], [7].

In [5] is investigated for the oscillatory behavior of the higher order difference equation

$$\Delta^r x_n + p x_{n-k} = 0,$$

where $p \in \mathbb{R}$, $k \in \mathbb{Z}$ and $r \geq 1$. (See also [1]).

In this paper, we obtain sufficient conditions for the oscillation of all solutions of the difference equation

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} + q x_{n-z} = 0\ , \ n = 0, 1, 2, \cdots , \tag{1.2}$$

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where $p_i \in \mathbb{R}$, $k_i \in \mathbb{Z}$ and $z \in \{-1, 0\}$.

In section 3, we obtain sufficient conditions for the oscillation of all solutions of the difference equation

$$\Delta^r x_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0\ ,\ n = 0, 1, 2, \cdots ,\tag{1.3}$$

where $p_i \in \mathbb{R}$, $k_i \in \mathbb{Z}$ for $i = 1, 2, \cdots, m$.

Let $l = \max\{k, z\}$. Then by a solution of the equation (1.1) we mean a sequence $\{x_n\}$ which is defined $n \geq -l$ and which satisfies equation (1.1) for $n \geq 0$. A solution $\{x_n\}$ of equation (1.1) is said to oscillate if the terms x_n are not eventually positive or eventually negative. Otherwise, the solution is called nonoscillatory.

We need the following lemma, which proved in [4].

Lemma 1.1. *We consider the following difference equation*

$$x_{n+k} + P_1 x_{n+k-1} + \cdots + P_k = 0.\tag{1.4}$$

Assume that $P_1, P_2, \cdots, P_k \in R^{rxr}$ and that I is the $r \times r$ identity matrix. Then every solution of equation (1.3) *oscillates* (*componentwise*) *if and only if the characteristic equation*

$$\det\left[\lambda^k I + \lambda^{k-1} P_1 + \cdots + \lambda P_{k-1} + P_k\right] = 0$$

has no positive roots.

We should remark that, in this paper, we will use the case $r = 1$ in Lemma 1.1.

2. Sufficient conditions for oscillation of (1.2)

In this section, we obtain sufficient conditions for the oscillation of equation (1.2). The conditions will be given in terms of the $p_i,\ q \in \mathbb{R}$ and $k_i \in \mathbb{Z}$ for each $i = 1, 2, \cdots, m$. Throughout this section we will use the convention that $0^0 = 1$.

Theorem 2.1. *Let $p_i,\ q \in \mathbb{R}$, $k_i \in \mathbb{Z}$ and $z = -1$ for $i = 1, 2, \cdots, m$. Suppose that for $i = 1, 2, \cdots, m$, either*

$$p_i \geq 0,\ q > 0 \quad and \quad k_i \in \{0, 1, 2, \cdots\}\tag{2.1}$$

or

$$p_i \leq 0,\ q \in (-1, 0) \quad and \quad k_i \in \{\cdots, -3, -2, -1\}.\tag{2.2}$$

If

$$\sum_{i=1}^{m} p_i (1+q)^{k_i} \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1,\tag{2.3}$$

then every solution of equation (1.2) *oscillates.*

Proof. Assume that (2.1) holds. In view of Lemma 1.1, it is enough to prove that the characteristic equation of equation (1.2)

$$f(\lambda) = \lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i} + q\lambda = 0 \tag{2.4}$$

has no positive roots. It is clear that equation (2.4) has no roots in $[1, \infty)$. Then observe that for $i = 1, 2, \cdots, m$, we have

$$\min_{0<\lambda<1} \frac{\lambda^{-k_i}}{1 - \lambda(1+q)} = (1+q)^{k_i} \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}},$$

therefore for $0 < \lambda < 1$, we get

$$\begin{aligned} f(\lambda) &= [1 - \lambda(1+q)] \left(-1 + \sum_{i=1}^{m} p_i \frac{\lambda^{-k_i}}{1 - \lambda(1+q)} \right) \\ &\geq [1 - \lambda(1+q)] \left(-1 + \sum_{i=1}^{m} p_i (1+q)^{k_i} \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} \right) \\ &> 0 \end{aligned}$$

which completes the proof.

Now, assume that (2.2) holds. Then it is clear that equation (2.4) has no roots in $(0, 1]$. So, we have for $i = 1, 2, \cdots, m$,

$$\min_{\lambda>1} \frac{\lambda^{-k_i}}{\lambda(1+q) - 1} = -(1+q)^{k_i} \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}}.$$

Hence for $\lambda > 1$, we get

$$\begin{aligned} f(\lambda) &= [\lambda(1+q) - 1] \left(1 + \sum_{i=1}^{m} p_i \frac{\lambda^{-k_i}}{\lambda(1+q) - 1} \right) \\ &\leq [\lambda(1+q) - 1] \left(1 - \sum_{i=1}^{m} p_i (1+q)^{k_i} \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} \right) \\ &< 0 \end{aligned}$$

which completes the proof. □

Corollary 2.2. *Let* $p_i, q \in \mathbb{R}$, $k_i \in \mathbb{Z}$ *and* $z = -1$ *for* $i = 1, 2, \cdots, m$. *Assume that either* (2.1) *or* (2.2) *holds and suppose that*

$$m \left(\prod_{i=1}^{m} |p_i| \right)^{\frac{1}{m}} (1+q)^k \left| \frac{(k+1)^{k+1}}{k^k} \right| > 1, \tag{2.5}$$

where $k = \frac{1}{m}\sum_{i=1}^{m} k_i$. *Then every solution of equation* (1.2) *oscillates.*

Proof. Assume that (2.1) holds. Then by using Theorem 2.1 and applying the arithmetic-geometric mean inequality, we conclude that

$$\begin{aligned} f(\lambda) &= [1-\lambda(1+q)]\left(-1+\sum_{i=1}^{m} p_i \frac{\lambda^{-k_i}}{1-\lambda(1+q)}\right) \\ &\geq [1-\lambda(1+q)]\left[-1+m\left(\prod_{i=1}^{m} p_i\right)^{\frac{1}{m}} \frac{\lambda^{-k}}{1-\lambda(1+q)}\right] \\ &\geq [1-\lambda(1+q)]\left[-1+m\left(\prod_{i=1}^{m} p_i\right)^{\frac{1}{m}} (1+q)^k \frac{(k+1)^{k+1}}{k^k}\right] \\ &> 0 \end{aligned}$$

and so the proof is complete. The proof when (2.2) holds is similar and will be omitted. □

Now, by using similar idea in the proofs of the Theorem 2.1 and Corollary 2.2 the following results follows immediately. So we only state them without their proofs.

Theorem 2.3. *Let* $p_i, q \in \mathbb{R}$, $k_i \in \mathbb{Z}$ *and* $z = 0$ *for* $i = 1, 2, \cdots, m$. *Suppose that for* $i = 1, 2, \cdots, m$, *either*

$$p_i \geq 0,\ q \in (0,1) \ \text{ and } \ k_i \in \{0, 1, 2, \cdots\} \tag{2.6}$$

or

$$p_i \leq 0,\ q < 0 \ \text{ and } \ k_i \in \{\cdots, -3, -2, -1\}. \tag{2.7}$$

If

$$\sum_{i=1}^{m} p_i (1-q)^{k_i+1} \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1,$$

then every solution of equation (1.2) *oscillates. Furthermore, suppose that* $p_i \geq 0, q \geq 1, z = 0$ *and that* $p_i \leq 0,\ q \leq -1, z = 0$ *for* $i = 1, 2, \cdots, m$, *respectively. Then every solution of equation* (1.2) *oscillates.*

Corollary 2.4. *Assume that* (2.6) *or* (2.7) *holds and that* $z = 0$. *If*

$$m\left(\prod_{i=1}^{m} |p_i|\right)^{\frac{1}{m}} (1-q)^{k+1} \left|\frac{(k+1)^{k+1}}{k^k}\right| > 1,$$

where $k = \frac{1}{m}\sum_{i=1}^{m} k_i$, *then every solution of equation* (1.2) *oscillates.*

Now, we have the following results.

Corollary 2.5. *Let* $p_i \geq 0$, $q \in (-1, 0)$, $k_i \in \mathbb{N}$ *and* $z = -1$. *If* (2.3) *or* (2.5) *holds, then every solution of equation* (1.2) *oscillates.*

Corollary 2.6. *Let* $p_i \geq 0, q < 0$, $k_i \in \{0, 1, 2, \cdots\}$ *for* $i = 1, 2, \cdots, m$ *and let* $z = 0$. *If*

$$\sum_{i=1}^{m} p_i \frac{1}{(1+q)^{k_i+1}} \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1,$$

then every solution of equation (1.2) *oscillates.*

Corollary 2.7. *Let* $p_i \leq 0, q > 0$, $k_i \in \{\cdots, -2, -1\}$ *for* $i = 1, 2, \cdots, m$ *and let* $z = -1$. *If* (2.3) *or* (2.5) *holds, then every solution of equation* (1.2) *oscillates.*

3. Sufficient conditions for oscillation of (1.3)

In this section, we provide a sufficient conditions for the oscillation of every solution of higher-order difference equation (1.3). The conditions will be given in terms of p_i and k_i for each $i = 1, 2, \cdots, m$.

Theorem 3.1. *Let* $p_i \in \mathbb{R}$ *and* $k_i \in Z$ *for* $i = 1, 2, \cdots, m$ *and let* r *is an odd positive integer. Assume that for* $i = 1, 2, \cdots, m$, *either*

$$p_i \geq 0 \ \textit{and} \ k_i \in \{0, 1, 2, \cdots\} \tag{3.1}$$

or

$$p_i \leq 0 \ \textit{and} \ k_i \in \{\cdots, -(r+1), -r\}. \tag{3.2}$$

If

$$\sum_{i=1}^{m} p_i \frac{(k_i+r)^{(k_i+r)}}{k_i^{k_i}} > r^r, \tag{3.3}$$

then every solution of equation (1.3) *oscillates.*

Proof. We will assume that (3.1) holds. In view of Lemma 1.1, it suffices to prove that the characteristic equation of (1.3)

$$f(\lambda) = (\lambda - 1)^r + \sum_{i=1}^{m} p_i \lambda^{-k_i} = 0 \tag{3.4}$$

has no positive roots. Then equation (3.4) has no roots in $[1, \infty)$. Consider the function g defined by $g(\lambda) = \dfrac{\lambda^{-k_i}}{(\lambda-1)^r}$. Then observe that $g'(\frac{k_i}{k_i+r}) = 0$ and $g''(\frac{k_i}{k_i+r}) < 0$. Therefore we get

$$\max_{0<\lambda<1} \frac{\lambda^{-k_i}}{(\lambda-1)^r} = \left(-\frac{1}{r^r}\right) \frac{(k_i+r)^{(k_i+r)}}{k_i^{k_i}}.$$

Hence for $0 < \lambda < 1$, we get

$$\begin{aligned} f(\lambda) &= (\lambda-1)^r \left[1+\sum_{i=1}^{m} p_i \frac{\lambda^{-k_i}}{(\lambda-1)^r}\right] \\ &\geq (\lambda-1)^r \left[1-\sum_{i=1}^{m} p_i \frac{1}{r^r}\frac{(k_i+r)^{(k_i+r)}}{k_i^{k_i}}\right] \\ &> 0, \end{aligned}$$

which completes the proof when (3.1) holds.

Now, assume that (3.2) holds. Then characteristic equation (3.4) has no positive roots in $(0,1]$. Consider the function g defined by $g(\lambda) = \dfrac{\lambda^{-k_i}}{(\lambda-1)^r}$. Then observe that $g'(\frac{k_i}{k_i+r}) = 0$ and $g''(\frac{k_i}{k_i+r}) > 0$. Therefore we get

$$\min_{\lambda>1} \frac{\lambda^{-k_i}}{(\lambda-1)^r} = \left(-\frac{1}{r^r}\right)\frac{(k_i+r)^{(k_i+r)}}{k_i^{k_i}}.$$

Hence for $\lambda > 1$, we have

$$\begin{aligned} f(\lambda) &= (\lambda-1)^r \left[1-\sum_{i=1}^{m} p_i \frac{1}{r^r}\frac{\lambda_0^{-k_i}}{(\lambda-1)^r}\right] \\ &\leq (\lambda-1)^r \left[1-\sum_{i=1}^{m} p_i \frac{1}{r^r}\frac{(k_i+r)^{(k_i+r)}}{k_i^{k_i}}\right] \\ &< 0, \end{aligned}$$

which completes the proof when (3.2) holds. □

Remark 3.2. If we take $r = 1$ equation (1.3), then condition (3.3) reduces to

$$\sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1$$

which is sufficient condition for the oscillation of all solutions of equation

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0\,,\ n = 0, 1, 2, \cdots .$$

Note that this result is obtained by Ladas in [7] (See also [4]).

Theorem 3.3. *Assume that* (3.1) *or* (3.2) *holds for* $i = 1, 2, \cdots, m$ *and that* r *is an odd positive integer. If*

$$m\left(\prod_{i=1}^{m} |p_i|\right)^{\frac{1}{m}} \left|\frac{(k+r)^{k+r}}{k^k}\right| > r^r, \tag{3.5}$$

where $k = \frac{1}{m}\sum_{i=1}^{m} k_i$, *then every solution of equation* (1.3) *oscillates.*

Proof. Assume that (3.1) holds. In view of Lemma 1.1, it suffices to prove that the characteristic equation (3.4) has no positive roots. Then by Theorem 3.1, the characteristic equation (3.4) has no positive roots in $[1,\infty)$. By taking into consideretion equation (3.4) for $0<\lambda<1$ and using arithmetic-geometric mean inequality, we get

$$\begin{aligned} f(\lambda) &= (\lambda-1)^r\left[1+\sum_{i=1}^{m} p_i\frac{\lambda^{-k_i}}{(\lambda-1)^r}\right] \\ &\le (\lambda-1)^r\left[1+m\left(\prod_{i=1}^{m} p_i\right)^{\frac{1}{m}}\frac{\lambda^{-k}}{(\lambda-1)^r}\right] \\ &\le (\lambda-1)^r\left[1-m\left(\prod_{i=1}^{m} p_i\right)^{\dfrac{1}{m}}\frac{1}{r^r}\frac{(k+r)^{k+r}}{k^k}\right] \\ &< 0. \end{aligned}$$

Then, it yields that equation (1.3) under assumption (3.5) has no positive roots and the proof is complete. The proof when (3.2) holds is similar and is omitted. □

Remark 3.4. If we take $r=1$ equation (1.3), then condition (3.5) reduces to

$$m\left(\prod_{i=1}^{m}|p_i|\right)^{\dfrac{1}{m}}\left|\frac{(k+1)^{k+1}}{k^k}\right|>1,$$

which is sufficient condition for the oscillation of all solutions of equation

$$x_{n+1}-x_n+\sum_{i=1}^{m} p_i x_{n-k_i}=0\,,\ \ n=0,1,2,\cdots.$$

Note that this result is obtained by Ladas in [4].

Corollary 3.5. *Assume that* $p_i>0$ *and* $k_i\in Z$ *for* $i=1,2,\cdots,m$ *and that* r *is an even positive integer. Then every solution equation* (1.3) *oscillates if and only if* $p_i>0$ *for* $i=1,2,\cdots,m$.

Proof. It is clear that the characteristic equation (1.3) becomes

$$f(\lambda)=(\lambda-1)^r+\sum_{i=1}^{m} p_i\lambda^{-k_i}=0,$$

so we get $f(\lambda)>0$ for $\lambda>0$. Therefore by Lemma 1.1, equation (1.3) oscillates if and only if $p_i>0$ for $i=1,2,\cdots,m$. □

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