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On Certain Novel Subclasses of Analytic and Univalent Functions

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Abstract. The purpose of the present paper is to introduce two novel subclasses $\mathcal{T}_\mu(n,\lambda,\alpha)$ and $\mathcal{H}_\mu(n,\lambda,\alpha;\kappa)$ of analytic and univalent functions with negative coefficients, involving Ruscheweyh derivative operator. The various results investigated in this paper include coefficient estimates, distortion inequalities, radii of close-to-convexity, starlikenes, and convexity for the functions belonging to the class $\mathcal{T}_\mu(n,\lambda,\alpha)$. These results are then appropriately applied to derive similar geometrical properties for the other class $\mathcal{H}_\mu(n,\lambda,\alpha;\kappa)$ of analytic and univalent functions. Relevant connections of these results with those in several earlier investigations are briefly indicated.

1. Introduction and preliminaries

Let $\mathcal{A}(n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbf{N} = \{1,2,3,\cdots\}), \tag{1.1}$$

which are *analytic* and *univalent* in the unit open disk $\mathcal{U} = \{z \in \mathbf{C} : |z| < 1\}$.

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If $f(z)$ is given by (1.1), and the function $g(z)$ is defined by

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \quad (n \in \mathbf{N}), \tag{1.2}$$

then the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$ is defined (in usual manner) by

$$(f * g)(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k \quad (n \in \mathbf{N}). \tag{1.3}$$

The Ruscheweyh derivative operator ([7]):

$$\mathcal{D}^{\mu} : \mathcal{A} \to \mathcal{A} \ \ (\mathcal{A} := \mathcal{A}(1)),$$

is defined by

$$\mathcal{D}^{\mu}\{f(z)\} = \frac{z}{(1-z)^{1+\mu}} * f(z) \quad (\mu > -1; f(z) \in \mathcal{A}), \tag{1.4}$$

which in view of (1.1) becomes

$$\begin{aligned} \mathcal{D}^{\mu}\{f(z)\} &= z + \sum_{k=2}^{\infty} \binom{k+\mu-1}{k-1} a_k z^k \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+\mu)}{(k-1)!\Gamma(1+\mu)} a_k z^k \quad (\mu > -1; f(z) \in \mathcal{A}). \end{aligned} \tag{1.5}$$

In particularly, when $\mu := m$ ($m \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}$), then

$$\mathcal{D}^{m}\{f(z)\} = \frac{z[z^{m-1} f(z)]^{(m)}}{m!} \quad (m \in \mathbf{N}_0). \tag{1.6}$$

Definition 1.1. A function $f(z) \in \mathcal{A}(n)$ is said to belong to to the class $\mathcal{K}_{\mu}(n, \lambda, \alpha)$, if and only if,

$$\left| \frac{\lambda z[\mathcal{D}^{\mu}\{f(z)\}]' + (1-\lambda) z[\mathcal{D}^{1+\mu}\{f(z)\}]'}{\lambda \mathcal{D}^{\mu}\{f(z)\} + (1-\lambda)\mathcal{D}^{1+\mu}\{f(z)\}} - 1 \right| < \alpha, \tag{1.7}$$

where $\mu > -1$, $0 \leq \lambda \leq 1$, $0 < \alpha \leq 1$, and $z \in \mathcal{U}$.

Let $\mathcal{T}(n)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0; n \in \mathbf{N}). \tag{1.8}$$

Further, we define the class $\mathcal{T}_\mu(n,\lambda,\alpha)$ by

$$\mathcal{T}_\mu(n,\lambda,\alpha) := \mathcal{K}_\mu(n,\lambda,\alpha) \cap \mathcal{T}(n). \tag{1.9}$$

In view of the above definition of the class $\mathcal{T}_\mu(n,\lambda,\alpha)$, we deem it worthwhile to point out the relevance of this class of functions with some known classes . Indeed, we have

(i) $\mathcal{P}(n,\lambda,\alpha) := \mathcal{T}_0(n,\lambda,1-\alpha)$ $(0 \le \alpha < 1)$, where $\mathcal{P}(n,\lambda,\alpha)$ was studied by Altıntaş [1],

(ii) $\mathcal{S}_n(\gamma,\lambda,\alpha) := \mathcal{T}_0(n,\lambda,\alpha|\gamma|)$ $(\gamma \in \mathbf{C}-\{0\})$, where $\mathcal{S}_n(\gamma,\lambda,\alpha)$ was investigated by Altıntaş et al. [2],

(iii) $\mathcal{S}_n(\gamma,\mu,\alpha) := \mathcal{T}_\mu(n,1,\alpha|\gamma|)$ $(\gamma \in \mathbf{C}-\{0\})$, where $\mathcal{S}_n(\gamma,\mu,\alpha)$ was very recently studied by Murugusundaramoorthy and Srivastava [5].

The other subclass $\mathcal{H}_\mu(n,\lambda,\alpha;\kappa)$ consisting of functions of the form (1.8) is now defined here as follows.

Definition 1.2. A function $f(z) \in \mathcal{T}(n)$ is said to belong to the class $\mathcal{H}_\mu(n,\lambda,\alpha;\kappa)$, if $w = f(z)$ satisfies the non-homogenous Cauchy-Euler type differential equation:

$$z^2\frac{d^2w}{dz^2} + 2(1+\kappa)z\frac{dw}{dz} + \kappa(1+\kappa)w = (1+\kappa)(2+\kappa)g(z), \tag{1.10}$$

where $g(z) \in \mathcal{T}_\mu(n,\lambda,\alpha)$ and $\kappa > -1$.

Several other interesting subclasses of the classes $\mathcal{A}(n)$ and/or $\mathcal{T}(n)$ were investigated recently, for example, by Chen et al. [3], Irmak and Raina [4], Raina and Srivastava [6], and in certain papers of [8]. One may refer to [2] for the investigation of the class of functions generated by (1.10).

This paper first investigates the geometric characteristics of the class of functions $\mathcal{T}_\mu(n,\lambda,\alpha)$, which give the coefficient estimates, distortion inequalities, radii of close-to-convexity, starlikeness and convexity properties of this class. These results are then applied to obtain similar properties of the class of functions $\mathcal{H}_\mu(n,\lambda,\alpha;\kappa)$.

2. Basic properties of the class $\mathcal{T}_\mu(n,\lambda,\alpha)$

We begin by proving a necessary and sufficient condition for a function belonging to the class $\mathcal{T}(n)$ to be in the class $\mathcal{T}_\mu(n,\lambda,\alpha)$. The result is contained in the following theorem.

Theorem 2.1. *Let the function $f(z)$ be defined by (1.8). Then, $f(z)$ is in the class $\mathcal{T}_\mu(n,\lambda,\alpha)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{(k+\alpha-1)[k+\mu-\lambda(k-1)]\Gamma(k+\mu)}{(k-1)!}a_k \le \alpha\Gamma(2+\mu), \tag{2.1}$$

where $\mu > -1$, $0 \leq \lambda \leq 1$, *and* $0 < \alpha \leq 1$. *The result is sharp for the function* $f(z)$ *given by*

$$f(z) = z - \frac{\alpha n!\Gamma(2+\mu)}{(n+\alpha)\Gamma(n+\mu+1)[1+\mu+n(1-\lambda)]} z^{n+1}. \tag{2.2}$$

Proof. Let $f(z)$ defined by (1.8) satisfy the inequality (2.1). If we let $z \in \partial\mathcal{U}$, then on using (1.5) and (1.7), we find that

$$\begin{aligned}
&\left|\lambda z[\mathcal{D}^{\mu}\{f\}]' + (1-\lambda)z[\mathcal{D}^{1+\mu}\{f\}]' - \lambda\mathcal{D}^{\mu}\{f\} + (\lambda-1)\mathcal{D}^{1+\mu}\{f\}\right| \\
&\quad - \alpha\left|\lambda\mathcal{D}^{\mu}\{f\} + (1-\lambda)\mathcal{D}^{1+\mu}\{f\}\right| \\
= &\left|\sum_{k=n+1}^{\infty} \frac{(k-1)\Gamma(k+\mu)[k+\mu-\lambda(k-1)]}{\Gamma(2+\mu)(k-1)!} a_k z^{k-1}\right| \\
&\quad - \alpha\left|1 - \sum_{k=n+1}^{\infty} \frac{\Gamma(k+\mu)[k+\mu-\lambda(k-1)]}{\Gamma(2+\mu)(k-1)!} a_k z^{k-1}\right| \\
\leq &\sum_{k=n+1}^{\infty} \frac{(k+\alpha-1)\Gamma(k+\mu)[k+\mu-\lambda(k-1)]}{\Gamma(2+\mu)(k-1)!} a_k - \alpha \\
\leq &\ 0, \quad (n \in \mathbf{N}; \mu > -1; 0 \leq \lambda \leq 1; 0 < \alpha \leq 1; z \in \partial\mathcal{U}).
\end{aligned}$$

Hence, by maximum modulus principal, the function $f(z)$ given by (1.8) belongs to the class $\mathcal{T}_{\mu}(n, \lambda, \alpha)$.

Conversely, suppose the function $f(z)$ given by (1.8) belongs to the class $\mathcal{T}_{\mu}(n, \lambda, \alpha)$. Then, in view of (1.5) and (1.7), we readily obtain

$$\begin{aligned}
&\left|\frac{\lambda z[\mathcal{D}^{\mu}\{f\}]' + (1-\lambda)z[\mathcal{D}^{1+\mu}\{f\}]'}{\lambda\mathcal{D}^{\mu}\{f\} + (1-\lambda)\mathcal{D}^{1+\mu}\{f\}} - 1\right| \\
= &\left|\frac{-\sum_{k=n+1}^{\infty} \frac{(k-1)\Gamma(k+\mu)[k+\mu-\lambda(k-1)]}{\Gamma(2+\mu)(k-1)!} a_k z^{k-1}}{1 - \sum_{k=n+1}^{\infty} \frac{\Gamma(k+\mu)[k+\mu-\lambda(k-1)]}{\Gamma(2+\mu)(k-1)!} a_k z^{k-1}}\right| < \alpha.
\end{aligned} \tag{2.3}$$

Putting $z = r$ $(0 \leq r < 1)$ on the right-hand side of (2.3), and noting the fact that for $r = 0$, the resulting expression in the denominator is positive, and remains so for all $r \in (0, 1)$, the desired inequality (2.1) follows upon letting $r \to 1-$.

Finally, by observing that the function $f(z)$ given by (2.2) is indeed an extremal function for the assertion (2.1), we complete the proof of Theorem 2.1. □

We next prove the following growth and distortion property for the functions of the form (1.8) belonging to the class $\mathcal{T}_{\mu}(n, \lambda, \alpha)$.

Theorem 2.2. *Let the function* $f(z)$ *given by* (1.8) *be in the class* $\mathcal{T}_{\mu}(n, \lambda, \alpha)$. *Then*

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\alpha n!\Gamma(2+\mu)}{(n+\alpha)\Gamma(n+\mu+1)[1+\mu+n(1-\lambda)]}, \tag{2.4}$$

and

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{\alpha(n+1)!\Gamma(2+\mu)}{(n+\alpha)\Gamma(n+\mu+1)[1+\mu+n(1-\lambda)]}. \tag{2.5}$$

Proof. Making using of Theorem 2.1, we find from (2.1) that

$$\begin{aligned} &\frac{(n+\alpha)\Gamma(n+\mu+1)[1+\mu+n(1-\lambda)]}{n!} \sum_{k=n+1}^{\infty} a_k \\ \leq\ & \sum_{k=n+1}^{\infty} \frac{(k+\alpha-1)\Gamma(k+\mu)[k+\mu-\lambda(k-1)]}{(k-1)!} a_k \\ \leq\ & \alpha\Gamma(2+\mu), \end{aligned}$$

which immediately yields the assertion (2.4) of Theorem 2.2.

Also, (2.1) yields

$$\frac{\Gamma(n+\mu+1)[1+\mu+n(1-\lambda)]}{n!} \sum_{k=n+1}^{\infty} (k+\alpha-1) a_k \leq \alpha\Gamma(2+\mu),$$

which implies

$$\begin{aligned} &\frac{\Gamma(n+\mu+1)[1+\mu+n(1-\lambda)]}{n!} \sum_{k=n+1}^{\infty} k a_k \\ \leq\ & \alpha\Gamma(2+\mu) + \frac{\Gamma(n+\mu+1)[1+\mu+n(1-\lambda)]}{n!} \cdot (1-\alpha) \sum_{k=n+1}^{\infty} a_k. \end{aligned} \tag{2.6}$$

In view of the coefficient inequality (2.4), we arrive at once at the desired assertion (2.5) of Theorem 2.2.

If we apply the coefficients assertions (2.4) and (2.5) to the modulus of the functions $f(z)$ and $f'(z)$, respectively, then the following distortion inequalities can be easily established. The proof can well be omitted. □

Theorem 2.3. *If $f(z) \in \mathcal{T}(n)$ is in the class $\mathcal{T}_\mu(n, \lambda, \alpha)$, then*

$$||f(z)| - |z|| \leq \frac{\alpha n!\Gamma(2+\mu)}{(n+\alpha)\Gamma(n+\mu+1)[1+\mu+n(1-\lambda)]} |z|^{n+1}, \tag{2.7}$$

and

$$||f'(z)| - 1| \leq \frac{\alpha(n+1)!\Gamma(2+\mu)}{(n+\alpha)\Gamma(n+\mu+1)[1+\mu+n(1-\lambda)]} |z|^{n}, \tag{2.8}$$

where $z \in \mathcal{U}$. The above results are sharp for the function $f(z)$ given by (2.2).

The properties of close-to-convexity, starlikeness and convexity are given by the assertions contained in the following theorem.

Theorem 2.4. *If $f(z) \in \mathcal{T}_\mu(n, \lambda, \alpha)$, then $f(z)$ is, respectively, close-to-convex of order β in $|z| < r_1 := r_1(n, \lambda, \alpha, \mu, \beta)$, starlike of order β in $|z| < r_2 := r_2(n, \lambda, \alpha, \mu, \beta)$, and convex of order β in $|z| < r_3 := r_3(n, \lambda, \alpha, \mu, \beta)$, where*

$$r_1(n, \lambda, \alpha, \mu, \beta) := \inf_{k \geq n+1} \left(\frac{1-\beta}{k} \cdot \Theta(\lambda, \alpha, \mu; k) \right)^{1/(k-1)}, \tag{2.9}$$

$$r_2(n, \lambda, \alpha, \mu, \beta) := \inf_{k \geq n+1} \left(\frac{1-\beta}{k-\beta} \cdot \Theta(\lambda, \alpha, \mu; k) \right)^{1/(k-1)}, \tag{2.10}$$

and

$$r_3(n, \lambda, \alpha, \mu, \beta) := \inf_{k \geq n+1} \left(\frac{1-\beta}{k(k-\beta)} \cdot \Theta(\lambda, \alpha, \mu; k) \right)^{1/(k-1)}, \tag{2.11}$$

where

$$\Theta(\lambda, \alpha, \mu; k) := \frac{(k+\alpha-1)[k+\mu-\lambda(k-1)]\Gamma(k+\mu)}{\alpha(k-1)!\Gamma(2+\mu)}. \tag{2.12}$$

$(0 < \alpha \leq 1; \mu > -1; 0 \leq \beta < 1; 0 \leq \lambda \leq 1; n \in \mathcal{N})$. *Each of these results is sharp for the function $f(z)$ given by* (2.2).

Proof. Let $f(z) \in \mathcal{T}_\mu(n, \lambda, \alpha)$. Then, in order show that $f(z)$ is close-to-convex of order β $(0 \leq \beta < 1)$, it is sufficient to show that

$$|f'(z) - 1| \leq 1 - \beta \quad (|z| < r_1; 0 \leq \beta < 1). \tag{2.13}$$

Making use of (1.8) in (2.13), and in the process taking into account the coefficient bound inequality (2.1), we infer that $f(z)$ is close-to-convex of order β $(0 \leq \beta < 1)$ inside the disk $|z| = r_1$, where r_1 is given by (2.9).

Similarly, by applying the inequalities

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \beta \quad (|z| < r_2; 0 \leq \beta < 1) \tag{2.14}$$

and

$$\left| \left(1 + \frac{zf''(z)}{f'(z)} \right) - 1 \right| \leq 1 - \beta \quad (|z| < r_3; 0 \leq \beta < 1), \tag{2.15}$$

and proceeding in the same manner as mentioned above, we conclude that $f(z)$ is, respectively, starlike of order β $(0 \leq \beta < 1)$ inside $|z| = r_2$, and convex of order β $(0 \leq \beta < 1)$ inside $|z| = r_3$, where r_2 and r_3 are, respectively, given by (2.10) and (2.11). □

3. Properties of the class $\mathcal{H}_\mu(n,\lambda,\alpha;\kappa)$

Applying the results of Section 2, which were obtained for the function $f(z)$ of the form (1.8) belonging to the class $\mathcal{T}_\mu(n,\lambda,\alpha)$, we now derive the corresponding results for the function $f(z)$ belonging to the class $\mathcal{H}_\mu(n,\lambda,\alpha;\kappa)$.

Our first result is given by Theorem 3.1 below.

Theorem 3.1. *If $f(z)\in\mathcal{T}(n)$ is in the class $\mathcal{H}_\mu(n,\lambda,\alpha;\kappa)$, then*

$$||f(z)|-|z||\leq\frac{\alpha n!(1+\kappa)(2+\kappa)\Gamma(2+\mu)}{(n+\alpha)(n+\kappa+1)[1+\mu+n(1-\lambda)]\Gamma(n+\mu+1)}|z|^{n+1}, \tag{3.1}$$

and

$$||f'(z)|-1|\leq\frac{\alpha(n+1)!(1+\kappa)(2+\kappa)\Gamma(2+\mu)}{(n+\alpha)(n+\kappa+1)[1+\mu+n(1-\lambda)]\Gamma(n+\mu+1)}|z|^{n}, \tag{3.2}$$

where $z\in\mathcal{U}$. The results in (3.1) and (3.2) *are sharp for the function $f(z)$ given by*

$$f(z)=z-\frac{\alpha n!(n+\kappa+1)\Gamma(2+\mu)}{(n+\alpha)(1+\kappa)(2+\kappa)\Gamma(n+\mu+1)[1+\mu+n(1-\lambda)]}z^{n+1}. \tag{3.3}$$

Proof. Assume that $f(z)\in\mathcal{T}(n)$ is given by (1.8). Also, let the function $g(z)\in\mathcal{T}_\mu(n,\lambda,\alpha)$, occurring in the non-homogenous differential equation (1.10) be of the form:

$$g(z)=z-\sum_{k=n+1}^{\infty}b_kz^k\quad(b_k\geq 0;n\in\mathbf{N}). \tag{3.4}$$

Then, we readily find from (1.10) that

$$a_k=\frac{(1+\kappa)(2+\kappa)}{(k+\kappa)(k+\kappa+1)}b_k\quad(k\geq n+1;n\in\mathbf{N}), \tag{3.5}$$

so that

$$f(z)=z-\sum_{k=n+1}^{\infty}a_kz^k=z-\sum_{k=n+1}^{\infty}\frac{(1+\kappa)(2+\kappa)}{(k+\kappa)(k+\kappa+1)}b_kz^k, \tag{3.6}$$

and

$$||f(z)|-|z||\leq|z|^{n+1}\sum_{k=n+1}^{\infty}\frac{(1+\kappa)(2+\kappa)}{(k+\kappa)(k+\kappa+1)}b_k\quad(z\in\mathcal{U}). \tag{3.7}$$

Next, since $g(z)\in\mathcal{T}_\mu(n,\lambda,\alpha)$, therefore, on using the assertion (2.4) of Theorem 2.2, we get the following coefficient inequality:

$$b_k\leq\frac{\alpha n!\Gamma(2+\mu)}{(n+\alpha)\Gamma(n+\mu+1)[1+\mu+n(1-\lambda)]}\quad(k\geq n+1;n\in\mathbf{N}), \tag{3.8}$$

which in conjunction with (3.6) and (3.7) yield

$$(3.9)\qquad ||f(z)|-|z|| \leq \frac{\alpha n!\Gamma(2+\mu)}{(n+\alpha)\Gamma(n+\mu+1)[1+\mu+n(1-\lambda)]} \cdot |z|^{n+1}\sum_{k=n+1}^{\infty}\frac{(1+\kappa)(2+\kappa)}{(k+\kappa)(k+\kappa+1)} \quad (z\in\mathcal{U}).$$

By Noting the following summation result:

$$(3.10)\qquad \sum_{k=n+1}^{\infty}\frac{(1+\kappa)(2+\kappa)}{(k+\kappa)(k+\kappa+1)} = \frac{(1+\kappa)(2+\kappa)}{n+\kappa+1},$$

where $\kappa \in \mathbf{R}^* := \mathbf{R}\setminus\{-n-1,-n-2,\cdots\}$. The assertion (3.1) of Theorem 3.1 follows from (3.9) and (3.10). The assertion (3.2) of Theorem 3.1 can be established by similarly applying (2.5), (3.5), and (3.10). □

Theorem 3.2. *If $f(z)\in\mathcal{H}_\mu(n,\lambda,\alpha;\kappa)$, then $f(z)$ is close-to-convex of order γ in the domain*

$$0<|z|<r_4 := r_4(n,\lambda,\mu,\gamma,\alpha,\kappa) \quad (\kappa\in\mathbf{R}^*; 0\leq\gamma<1),$$

where

$$r_4(n,\lambda,\mu,\gamma,\alpha,\kappa) := \inf_{k\geq n+1}\left(\Theta(\lambda,\alpha,\mu;k)\cdot\frac{(1-\gamma)(1+\kappa)(2+\kappa)}{\alpha k(k+\kappa)(k+\kappa+1)}\right)^{1/(k-1)},$$

and $\Theta(\lambda,\alpha,\mu;k)$ is given by (2.12).

Proof. Assume that $f(z)\in\mathcal{T}(n)$ is given by (1.8). Also, let the function $g(z)\in\mathcal{T}_\mu(n,\lambda,\alpha)$, occurring in the non-homogenous differential equation (1.10), be given as in the Definition (3.4). Then, it is sufficient to show that

$$|f'(z)-1|\leq 1-\gamma \ \text{ for } \ |z|<r_4.$$

Indeed, we have

$$|f'(z)-1|\leq\sum_{k=n+1}^{\infty}ka_k|z|^{k-1},$$

and by using the coefficient relation (3.5) between the functions $f(z)$ and $g(z)$, we get

$$(3.11)\qquad |f'(z)-1|\leq\sum_{k=n+1}^{\infty}\frac{(1+\kappa)(2+\kappa)}{(k+\kappa)(k+\kappa+1)}kb_k|z|^{k-1}\leq 1-\gamma.$$

Since $g(z)\in\mathcal{T}_\mu(n,\lambda,\alpha)$, and we know from the assertion (2.1) of Theorem 2.1 that

$$\sum_{k=n+1}^{\infty}\frac{(k+\alpha-1)\Gamma(k+\mu)[1+\mu+n(1-\lambda)]}{(k-1)!}b_k\leq\alpha\Gamma(2+\mu),$$

hence, (3.11) is true if

$$\frac{k}{1-\gamma}\cdot\frac{(1+\kappa)(2+\kappa)}{(k+\kappa)(k+\kappa+1)}\cdot|z|^{k-1}\leq\Theta(\lambda,\alpha,\mu;k), \tag{3.12}$$

$$(k=n+1,n+2,n+3,\ \cdots\ ;n\in\mathcal{N})$$

where $\Theta(\lambda,\alpha,\mu;k)$ is given by (2.12). Solving (3.12) for $|z|$, we obtain

$$|z|\leq\left(\Theta(\lambda,\alpha,\mu;k)\cdot\frac{(1-\gamma)(1+\kappa)(2+\kappa)}{\alpha k(k+\kappa)(k+\kappa+1)}\right)^{1/(k-1)},$$

$$(k=n+1,n+2,n+3,\ \cdots\ ;n\in\mathcal{N})$$

which obviously proves Theorem 3.2.

By suitably invoking the inequalities (2.14) and (2.15) governing the properties of starlikenes and convexity of the function $f(z)$ belonging to the class $\mathcal{H}_\mu(n,\lambda,\alpha;\kappa)$, and using (1.8), (3.5) and the coefficient bound inequality (2.1), appropriately in the process, the following results can be easily proved by following similar steps as elucidated in the proof of Theorem 3.2. We skip further details. □

Theorem 3.3. *If $f(z)\in\mathcal{H}_\mu(n,\lambda,\alpha;\kappa)$, then $f(z)$ is starlike of order γ in the domain*

$$0<|z|<r_5:=r_5(n,\lambda,\mu,\gamma,\alpha,\kappa)\quad(\kappa\in\mathbf{R}^*;0\leq\gamma<1),$$

where

$$r_5(n,\lambda,\mu,\gamma,\alpha,\kappa):=\inf_{k\geq n+1}\left(\Theta(\lambda,\alpha,\mu;k)\cdot\frac{(1-\gamma)(1+\kappa)(2+\kappa)}{\alpha(k-\gamma)(k+\kappa)(k+\kappa+1)}\right)^{1/(k-1)},$$

and the function $\Theta(\lambda,\alpha,\mu;k)$ is given by (2.12).

Theorem 3.4. *If $f(z)\in\mathcal{H}_\mu(n,\lambda,\alpha;\kappa)$, then $f(z)$ is convex of order γ in the domain*

$$0<|z|<r_6:=r_6(n,\lambda,\mu,\gamma,\alpha,\kappa)\quad(\kappa\in\mathbf{R}^*;0\leq\gamma<1),$$

where

$$r_6(n,\lambda,\mu,\gamma,\alpha,\kappa):=\inf_{k\geq n+1}\left(\Theta(\lambda,\alpha,\mu;k)\cdot\frac{(1-\gamma)(1+\kappa)(2+\kappa)}{\alpha k(k-\gamma)(k+\kappa)(k+\kappa+1)}\right)^{1/(k-1)},$$

and $\Theta(\lambda,\alpha,\mu;k)$ is given by (2.12).

We conclude this paper by remarking that by choosing suitable values of the parameters n, α, λ, and/or μ in Theorems 2.1-2.4 and Theorems 3.1-3.4, one can deduce various new and known results (given in [1], [2] and [5]), as worthwhile consequences of our main results. These obvious considerations are omitted here.

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