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Nonlinear Functional Boundary Value Problems in Banach Algebras Involving Carathéodories

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Abstract. In this paper, an existence theorem for a nonlinear two point boundary value problem of second order differential equations in Banach algebras is proved using a nonlinear alternative based on Leray-Schauder alternative.

1. Introduction

Given a closed and bounded interval $J = [a, b]$, $a < b$, of the real line $\mathbb{R}$, consider the nonlinear two point functional boundary value problem (in short FBVP) of second order ordinary differential equations

$$
(1.1) \qquad \begin{cases} -\Big(\dfrac{x(t)}{f(t, x(\mu(t)))}\Big)'' = g(t, x(\sigma(t)), x'(\eta(t))) \quad \text{a.e. } t \in J, \\ x(a) = 0 = x(b), \end{cases}
$$

where $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $\mu, \sigma, \eta : J \to J$.

By a *solution* of the above FBPV (1.1) we mean a function $x \in AC^1(J, \mathbb{R})$ that satisfies the differential equation and the boundary conditions of (1.1), where $AC^1(J, \mathbb{R})$ is the space of all continuous real-valued functions on $J = [a, b]$, whose first derivative exists and is absolutely continuous on J. Note that the second derivative of the solution $x(t)$ exists for almost all $t \in J$.

The main idea is to write the FBPV (1.1) into an equivalent operator equation $x = Ax\,Bx$ and to prove that it has a solution in $AC^1(J, \mathbb{R})$.

The FBVP (1.1) has not been studied in the literature before, so the results of this paper are new to the theory of differential equations in Banach algebras. The special cases of the FBVP (1.1) have already been discussed in the literature by several authors for various aspects of the solutions. For example, if $f(t, x) = 1$

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whenever $(t,x) \in J \times \mathbb{R}$, then the FBPV (1.1) reduces to

$$\left\{ \begin{array}{l} -x''(t) = g(t,x(t),x'(t)) \quad \text{for a.e. } t \in J, \\ x(a) = 0 = x(b). \end{array} \right. \tag{1.2}$$

There is an abundance literature on the BVP (1.2), see for example, Baily et. al. [1], Bernfield and Lakshmikantham [2] and the reference therein. The importance of the FBVP (1.1) concerning the applications is yet to be investigated. However, it is new to the literature on the theory of nonlinear two point boundary value problems of ordinary differential equations. This is the main motivation to study the FBVP (1.1) in the present paper. The rest of this paper is organized as follows. Section 2 deals with the preliminaries and the fixed point results for the operator equations involving the product of two operators in Banach algebras. Section 3 deals with the existence theorems for the FBVP (1.1) under certain generalized Lipschitz and Carathéodory conditions. Finally, an illustrative example is given at the end of this paper.

2. Preliminaries

Let X be a Banach space with norm $\|\cdot\|$. A mapping $A : X \to X$ is called **$\mathcal{D}$-Lipschitz** if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \le \psi(\|x-y\|) \tag{2.1}$$

for all $x, y \in X$ with $\psi(0) = 0$. Sometimes we call the function ϕ a **$\mathcal{D}$-function** of A on X. In the special case when $\psi(r) = \alpha\, r,\ \ \alpha > 0,\ \ A$ is called a Lipschitz with the Lipschitz constant α. In particular if $\alpha < 1$, A is called a **contraction** with the contraction constant α. Further if $\psi(r) < r$ for $r > 0$, then A is called a **nonlinear $\mathcal{D}$-contraction** on X.

An operator $B : X \to X$ is called **totally compact** if $\overline{B(S)}$ is a compact subset of X for any $S \subset X$. Again $B : X \to X$ is called **compact** if B maps a bounded subset of X into a relatively compact subset of X. Similarly, $B : X \to X$ is called **totally bounded** if B maps a bounded subset of X into a totally bounded subset of X. Finally, $B : X \to X$ is called a **completely continuous** operator if it is a continuous and compact operator on X. It is clear that every compact operator is totally bounded, but the converse may not be true. However, these two notions are equivalent on bounded subsets of a Banach space X.

The following nonlinear alternative is fundamental and has been used extensively in the theory of differential and integral equations for proving the existence results under certain compactness conditions.

Theorem 2.1 [Zeidler [12]]**.** *Let K be a convex subset of a normed linear space E,*

U an open subset of K with $0 \in U$, and $N : \overline{U} \to K$ a continuous and compact map. Then either

(a) *N has a fixed point in $\overline{U}$; or,*

(b) *there is an element $u \in \partial U$ such that $u = \lambda N u$ for some real number $\lambda \in (0,1)$, where ∂U is a boundary of U.*

Before presenting the main results of this section, we give some preliminaries needed in the sequel.

A Kuratowskii measure of noncompactness α of a bounded set A in X is a nonnegative real number $\alpha(A)$ defined by

$$\alpha(A) = \inf\left\{ r > 0 : A = \bigcup_{i=1}^{n} A_i,\ \operatorname{diam}(A_i) \le r,\ \forall i \right\}. \tag{2.2}$$

The function α enjoys the following properties:

(α_1) $\alpha(A) = 0 \Longleftrightarrow A$ is precompact.

(α_2) $\alpha(A) = \alpha(\overline{A}) = \alpha(\overline{co}\, A)$, where $\overline{A}$ and $\overline{co}\, A$ denote respectively the closure and the closed convex hull of A.

(α_3) $A \subset B \Rightarrow \alpha(A) \le \alpha(B)$

(α_4) $a(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

(α_5) $\alpha(\lambda A) = |\lambda| \alpha(A), \forall \lambda \in \mathbb{R}$.

(α_6) $\alpha(A + B) \le \alpha(A) + \alpha(B)$.

The details of measures of noncompactness and their properties appear in Banas and Goebel [3], Deimling [5] and Zeidler [12].

Definition 2.1. A mapping $T : X \to X$ is called **condensing** if for any bounded subset A of X, $T(A)$ is bounded and $\alpha(T(A)) < \alpha(A),\ \ \alpha(A) > 0$.

Note that contraction and completely continuous mappings are condensing, but the converse may not be true. The following generalization of Theorem 2.1 for condensing mappings in Banach spaces is well-known and will be used in the sequel.

Theorem 2.2. *Let U and $\overline{U}$ be respectively open and closed subsets of a Banach space X such that $0 \in U$. If $N(\overline{U})$ is bounded and $N : \overline{U} \to X$ a continuous and condensing map, then either*

(a) *N has a fixed point in $\overline{U}$; or,*

(b) *there is an element $u \in \partial U$ such that $u = \lambda N u$ for some $\lambda \in (0,1)$, where ∂U is a boundary of U.*

Our main result of this section is

Theorem 2.3. *Let U and $\overline{U}$ be open-bounded and closed-bounded subsets of a Banach algebra X such that $0 \in U$ and let $A, B : \overline{U} \to X$ be two operators satisfying*

(a) *A is $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function ϕ,*

(b) *B is continuous and compact, and*

(c) *$M\phi(r) < r$, where $M = \|B(\overline{U})\| = sup\{\|B(x)\| : x \in \overline{U}\}$.*

Then either

(i) *the equation $AxBx = x$ has a solution in $\overline{U}$, or*

(ii) *there is an element $u \in \partial U$ such that $u = \lambda Au\, Bu$ for some $\lambda \in (0, 1)$, where ∂U is a boundary of U.*

Proof. Define the mapping $T : \overline{U} \to X$ by

$$Tx = Ax\, Bx,\ x \in \overline{U}. \tag{2.3}$$

Obviously, the mapping T is continuous on $\overline{U}$. The result follows immediately from Theorem 2.1 if the operator T is condensing on $\overline{U}$. Let S be a set in $\overline{U}$. Then we have the following estimates concerning the operators A and B on $\overline{U}$. Let x^* be a fixed element of S. Then by the hypothesis (a),

$$\begin{aligned} \|Ax\| &\leq \|Ax^*\| + \|Ax^* - Ax\| \\ &\leq \|Ax^*\| + \phi(\|x^* - x\|) \\ &\leq \beta \end{aligned}$$

for all $x \in S$, where

$$\beta = \|Ax^*\| + \phi(\text{diam S}) < \infty, \tag{2.4}$$

because S is bounded. Similarly, since B is compact, $B(S)$ is a precompact subset of X. Hence for $\eta > 0$, there exist subsets $G_1, G_2, \cdots, G_m$ of X such that

$$B(S) = \bigcup_{j=1}^{m} G_j \text{ and } \operatorname{diam}(G_j) < \frac{\eta}{\beta}.$$

This further gives that

$$S = \bigcup_{j=1}^{m} B^{-1}(G_j).$$

Let $\epsilon > 0$ be given and suppose that

$$S \subseteq \bigcup_{i=1}^{n} S_i$$

with

$$\operatorname{diam}(S_i) < \alpha(S) + \epsilon$$

for all $i = 1, 2, \cdots, n$. We put $F_{ij} = S_i \bigcap B^{-1}(G_j)$, then $S \subset \bigcup F_{ij}$.

Now

$$\begin{aligned} T(S) &\subseteq \bigcup_{i,j} T(F_{ij}) \\ &\subset \bigcup_{i,j} T\left(S_i \bigcap B^{-1}(G_j)\right) \\ &= \bigcup_{i,j} Y_{ij}. \end{aligned}$$

If $w_0, w_1 \in Y_{ij}$, for some $i = 1, \cdots, n$ and $j = 1, \cdots, m$, then there exist $x_0, x_1 \in F_{ij} = S_i \bigcap B^{-1}(G_j)$ such that $Tx_0 = w_0$ and $Tx_1 = w_1$.

Since ϕ is nondecreasing, one has

$$\begin{aligned} \|Tx_0 - Tx_1\| &= \|Ax_0Bx_0 - Ax_1Bx_1\| \\ &\leq \|Ax_0Bx_0 - Ax_1Bx_0\| + \|Ax_1Bx_0 - Ax_1Bx_1\| \\ &\leq \|Ax_0 - Ax_1\|\|Bx_0\| + \|Ax_1\|\|Bx_0 - Bx_1\| \\ &\leq \phi(\|x_0 - x_1\|)\|Bx_0\| + \|Ax_1\|\|Bx_0 - Bx_1\| \\ &< \phi(\operatorname{diam}(F_{ij}))\|B(\overline{U})\| + \|A(S)\|\|Bx_0 - Bx_1\| \\ &\leq M\phi(\operatorname{diam}(F_{ij})) + \eta. \end{aligned}$$

Since η is arbitrary, one has

$$\|Tx_0 - Tx_1\| \leq M\phi(\operatorname{diam}(F_{ij})).$$

This further implies that

$$\|Tx_0 - Tx_1\| \leq M\phi(\operatorname{diam}(S_i)) < M\phi(\alpha(S) + \epsilon).$$

This is true for every $w_0, w_1 \in Y_{ij}$, and so

$$\operatorname{diam}(Y_{ij}) < M\phi(\alpha(S) + \epsilon),$$

for all $i = 1, 2, \cdots, n$. Thus we have

$$\alpha(T(S)) = \max_{i,j} \operatorname{diam}(Y_{ij}) < M\phi(\alpha(S) + \epsilon).$$

Since ϵ is arbitrary, we have

$$\alpha(T(S)) \leq M\phi(\alpha(S)) < \alpha(S),$$

whenever $\alpha(S) > 0$.

This shows that T is a condensing on $\overline{U}$. Now the desired conclusion follows by an application of Theorem 2.1. This completes the proof. □

As a consequence of Theorem 2.3 we obtain the following corollary in its applicable form to nonlinear differential and integral equations.

Corollary 2.1. *Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be open and closed balls in a Banach algebra X centered at origin 0 of radius r, for some real number $r > 0$ and let $A, B : \overline{\mathcal{B}_r(0)} \to X$ be two operators satisfying*

(a) *A is Lipschitz with Lipschitz constant α,*

(b) *B is continuous and compact, and*

(c) *$\alpha M < 1$, where $M = \|B(\overline{\mathcal{B}_r(0)})\| = sup\left\{\|B(x)\| : x \in \overline{\mathcal{B}_r(0)}\|\right\}$.*

Then either

(i) *the equation $\lambda Ax\, Bx = x$ has a solution in $\overline{\mathcal{B}_r(0)}$, or*

(ii) *there is an element $u \in X$ such that $\|u\| = r$ satisfying $\lambda AuBu = u$, for some $0 < \lambda < 1$.*

Remark 2.1. Theorem 2.1 is an improvement of nonlinear alternatives of Leray-Schauder type due to Dhage [6] and Dhage and O'Regan [9] under weaker conditions.

3. Existence results

Let $C^1(J, \mathbb{R})$ be the space of all continuous real-valued functions on interval $J = [a, b]$ whose first derivative exists and is absolutely continuous equipped with the norm

$$\|x\|_{C^1} = \max\left\{\sup_{t\in J}|x(t)|\,,\ \sup_{t\in J}|x'(t)|\right\}. \tag{3.1}$$

Clearly, $C^1(J, \mathbb{R})$ is a complete normed linear space with respect to this norm. Define an equivalent norm $\|\cdot\|$ in $C^1(J, \mathbb{R})$ by $\|x\| = 2\|x\|_{C^1}$. Then it is easy to prove the following lemma.

Lemma 3.1. $\left(C^1(J, \mathbb{R}), \|\cdot\|\right)$ *is a Banach algebra with respect to the multiplication composition "$\cdot$" defined by $(x \cdot y)(t) = x(t)y(t)$, $t \in J$.*

Remark 3.1. Note that if the operator $A : C^1(J, \mathbb{R}) \to C^1(J, \mathbb{R})$ is Lipschitz with respect to the norm $\|\cdot\|_{C^1}$, then it is also Lipschitz with respect to the equivalent norm $\|\cdot\|$.

We denote by $L^1(J, \mathbb{R})$ the space of all Lebesgue integrable functions on J with the norm

$$\|x\|_{L^1} = \int_a^b |x(t)|\, dt. \tag{3.2}$$

Definition 3.1. A function $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be Carathéodory if

(i) $t \mapsto g(t, x, y)$ is measurable for all x, $y \in \mathbb{R}$, and

(ii) $(x, y) \mapsto g(t, x, y)$ is continuous for almost all $t \in J$.

We consider the following set of hypotheses imposed on the functions $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\mu, \sigma, \eta : J \to J$:

(A_1) The functions μ, σ and η are continuous.

(A_2) The function $\mu : J \to J$ has continuous first derivative and $m_1 = \sup_{t \in J} |\mu'(t)|$.

(A_3) The function f and the partial derivatives f_t and f_2 are continuous and there exist bounded functions p, $p_1, p_2 : J \to \mathbb{R}^+$ such that

$$\left\{ \begin{array}{l} |f(t,x) - f(t,y)| \le p(t)|x-y| \\ |f_t(t,x) - f_t(t,y)| \le p_1(t)|x-y| \\ |f_2(t,x) - f_2(t,y)| \le p_2(t)|x-y| \end{array} \right. \tag{3.3}$$

for all $t \in J$ and $x, y \in \mathbb{R}$, where $f_2(t,x) = \frac{\partial}{\partial w} f(t,w)\big|_{w=x}$.

(A_4) The function g is a Carathéodory.

(A_5) There exists a function $\psi \in L^1(J, \mathbb{R}_+)$ and an increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|g(t,x,y)| \le \psi(t)\phi(\max\{|x|,|y|\}) \quad \text{for a.e. } t \in J \tag{3.4}$$

whenever x, $y \in \mathbb{R}$.

Assume that conditions (A_4)–(A_5) hold. Then the FBVP (1.1) is equivalent to a functional integral equation (in short FIE)

$$x(t) = \big[f(t, x(\mu(t)))\big]\Big(\int_a^b G(t,s) g(s, x(\sigma(s)), x'(\eta(s)))\, ds\Big), \tag{3.5}$$

for all $t \in J$, where $G : J \times J \to \mathbb{R}$ is a Green's function associated with the linear homogeneous BVP

$$\left. \begin{array}{c} -y''(t) = 0 \ \text{ a.e. } t \in J \\ y(a) = 0 = y(b) \end{array} \right\} \tag{3.6}$$

and is given by

$$G(t,s) = \begin{cases} \dfrac{(t-a)(b-s)}{b-a} & \text{if } a \le t \le s \le b, \\[2ex] \dfrac{(s-a)(b-t)}{b-a} & \text{if } a \le s \le t \le b. \end{cases} \tag{3.7}$$

It is known that the Green's function $G(t,s)$ is continuous and nonnegative real-valued function on $J \times J$ satisfying

$$0 \le G(t,s) \le \frac{b-a}{4}, \quad \forall\, t,s \in J, \tag{3.8}$$

and

$$\int_a^b G(t,s)ds \le \frac{(b-a)^2}{8}. \tag{3.9}$$

Again, $G_t(t,s)$ is continuous in $(a,b)\times(a,b)\setminus\{(t,t)\,|\,t\in J\}$ and they satisfy

$$|G_t(t,s)| = G_t(t,s) = \frac{1}{b-a}\begin{cases} b-s, & a<s<t<b \\ s-a, & a<t<s<b \end{cases} \le 1. \tag{3.10}$$

Theorem 3.1. *Assume that the hypotheses* (A_1)-(A_5) *hold. Further if there exists a real number* $r>0$ *such that*

$$M(r)\,\max\{M_o,\, \max\{M_1, m_1c_2\} + m_1M_2r\} < 1, \tag{3.11}$$

where

$$\begin{cases} M(r) = 2\max\left\{1, \frac{b-a}{4}\right\}\, \|\psi\|_{L^1}\phi(r), \\ M_o = \sup_{t\in J} p(t), \quad M_1 = \sup_{t\in J} p_1(t), \quad M_2 = \sup_{t\in J} p_2(t), \\ c_2 = \sup\{|f_2(t,0)|\,|\,t\in J\}, \end{cases}$$

and

$$r > \frac{(b-a)}{2}(M_o\,r + c_o)\max\left\{1\,,\ \frac{4}{b-a} + rm_1(M_2r + c_2) + M_1r + c_1\right\}\|\psi\|_{L^1}\phi(r) \tag{3.12}$$

where $c_0 = \sup\{|f(t,0)|\,|\,t\in J\}$ *and* $c_1 = \sup\{|f_t(t,0)|\,|\,t\in J\}$. *Then the FBVP* (1.1) *has a solution* $x \in C^1(J,\mathbb{R})$ *with* $\|x\| \le r$.

Proof. Let $X = \big(C^1(J,\mathbb{R}), \|\cdot\|\big)$ and consider the closed ball $\overline{\mathcal{B}_r(0)}$ in X, where the real number $r>0$ satisfies the inequalities (3.11) and (3.12). Define two operators $A, B : \overline{\mathcal{B}_r(0)} \to X$ by

$$Ax(t) = f(t, x(\mu(t))), \quad t\in J \tag{3.13}$$

and

$$Bx(t) = \int_a^b G(t,s)g(t, x(\sigma(s)), x'(\eta(s)))\,ds, \quad t \in J. \tag{3.14}$$

Now the FBVP (1.1) and FIE (3.11) have the same solutions which are also the solutions of the operator equation

$$Ax(t)Bx(t) = x(t), \quad t\in J. \tag{3.15}$$

We shall show that the operators A and B satisfy all the conditions of Theorem 2.2. First we show that the operator B is continuous and compact in $\overline{\mathcal{B}_r(0)}$. Let $(x_n)_{n=0}^{\infty}$ be a converging sequence in $\overline{\mathcal{B}_r(0)}$ such that $\lim_{n\to\infty} x_n = x$. Then we have

$$\max\{\sup\{|x(t)-x_n(t)|\,|\,t\in J\},\sup\{|x'(t)-x_n'(t)|\,|\,t\in J\}\}\to 0 \text{ as } n\to\infty.$$

Then by assumptions (A_4)–(A_5) and by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty}\int_a^b G(t,s)g(s,x_n(\sigma(s)),x_n'(\eta(s)))\,ds\\ &= \int_a^b G(t,s)g(s,x(\sigma(s)),x'(\eta(s)))\,ds\\ &= Bx(t)\end{aligned}$$

for all $t\in J$, and

$$\begin{aligned}\lim_{n\to\infty}(Bx_n)'(t) &= \lim_{n\to\infty}\int_a^b G_t(t,s)g(s,x_n(\sigma(s)),x_n'(\eta(s)))\,ds\\ &= \int_a^b G_t(t,s)g(s,x(\sigma(s)),x'(\eta(s)))\,ds\\ &= (Bx)'(t).\end{aligned}$$

for all $t\in J$.Thus B is continuous on $\overline{\mathcal{B}_r(0)}$.

Assume that y in an element of $\overline{\mathcal{B}_r(0)}$. Then $\|y\|\le r$, and from the condition (A_5) and inequality (3.8), it follows that

$$\begin{aligned}|By(t)| &\le \int_a^b G(t,s)|g(s,y(\sigma(s)),y'(\eta(s)))|\,ds\\ &\le \int_a^b\Big(\frac{b-a}{4}\Big)\psi(s)\phi(\max\{|y(\sigma(s))|,|y'(\eta(s)))|\})\,ds\\ &\le \Big(\frac{b-a}{4}\Big)\|\psi\|_{L^1}\phi(r)),\end{aligned}\tag{3.16}$$

for all $t\in J$. Furthermore,

$$\begin{aligned}|(By)'(t)| &\le \int_a^b G_t(t,s)|g(s,y(\sigma(s)),y'(\eta(s)))|\,ds\\ &\le \int_a^b \psi(s)\phi(\max\{|y(\sigma(s))|,|y'(\eta(s)))|\})\,ds\\ &\le \|\psi\|_{L^1}\phi(r),\end{aligned}\tag{3.17}$$

for all $t\in J$. Hence

$$\|By\| = 2\|Ty\|_{C^1}\le M(r) = 2\max\Big\{1,\frac{b-a}{4}\Big\}\|\psi\|_{L^1}\phi(r),$$

whenever $y \in \overline{\mathcal{B}_r(0)}$.

As a result, $B[\overline{\mathcal{B}_r(0)}]$ is a uniformly bounded subset of X. We show next that the image $B[\overline{\mathcal{B}_r(0)}]$ of the closed ball $\overline{\mathcal{B}_r(0)}$ under the operator B is equi-continuous. Let $x \in \overline{\mathcal{B}_r(0)}$ and $a \le \tau \le t \le b$. Then by (3.4),

$$\begin{aligned} |Bx(t) - Bx(\tau)| &\le \int_a^b |G(t,s) - G(\tau,s)||g(s, x(\sigma(s)), x'(\eta(s)))|\,ds \\ &\le \int_a^b |G(t,s) - G(\tau,s)|\psi(s)\phi(\max\{|y(\sigma(s))|, |y'(\eta(s)))|\})\,ds \\ &\le \int_a^b |G(t,s) - G(\tau,s)|\psi(s)\phi(r)\,ds \end{aligned}$$

and

$$\begin{aligned} |(Bx)'(t) - (Bx)'(\tau)| &\le \int_a^b |G_t(t,s) - G_\tau(\tau,s)||g(s, x(\sigma(s)), x'(\eta(s)))|\,ds \\ &= 0. \end{aligned}$$

Hence

$$\begin{aligned} \max\Big\{|Bx(t) - Bx(\tau)|\,,\, |(Bx)'(t) - (Bx)'(\tau)|\Big\} \\ \le \int_a^b |G(t,s) - G(\tau,s)|\psi(s)\phi(r)\,ds \end{aligned}$$

whenever $x \in \overline{\mathcal{B}_r(0)}$ and $t,\, \tau \in J$.

Since the function $t \mapsto G(t,s)$ is continuous on compact interval J, it is uniformly continuous there. Hence we have

$$\max\{|Bx(t) - Bx(\tau)|, |(Bx)'(t) - (Bx)'(\tau)|\} \to 0 \text{ as } t \to \tau.$$

As a result, the set $B[\overline{\mathcal{B}_r(0)}]$ is equi-continuous in $C^1(J,\mathbb{R})$, and so, B is a compact operator on $\overline{\mathcal{B}_r(0)}$ by Arzela-Ascoli theorem.

Assume that $x,\, y \in \overline{\mathcal{B}_r(0)}$. Then by conditions (A_2) and (A_3),

$$\begin{aligned} |Ax(t) - Ay(t)| &= |f(t, x(\mu(t))) - f(t, y(\mu(t)))| \\ &\le p(t)|x(\mu(t)) - y(\mu(t))|, \end{aligned}$$

for each $t \in J$, so that

$$|Ax(t) - Ay(t)| \le M_o \|x - y\|_{C^1}, \tag{3.18}$$

for each $t \in J$. Moreover, by the same conditions

$$\begin{aligned} |(Ax)'(t)-(Ay)'(t)| &\le |f_t(t,x(\mu(t)))-f_t(t,y(\mu(t)))| \\ &\quad +|\mu'(t)||x'(\mu(t))f_2(t,x(\mu(t)))-y'(\mu(t))f_2(t,y(\mu(t)))| \\ &\le p_1(t)|x(\mu(t))-y(\mu(t))| \\ &\quad +m_1(|x'(\mu(t))|\,|f_2(t,x(\mu(t)))-f_2(t,y(\mu(t)))| \\ &\quad +|f_2(t,y(\mu(t)))||x'(\mu(t))-y'(\mu(t))|) \\ &\le M_1|x(\mu(t))-y(\mu(t))|+m_1(M_2r|x(\mu(t))-y(\mu(t))| \\ &\quad +(M_2r+c_2)|x'(\mu(t))-y'(\mu(t))|), \end{aligned}$$

for each $t \in J$, so that

$$\begin{aligned} |(Ax)'(t)-(Ay)'(t)| &\le (M_1+m_1M_2r)|x(\mu(t))-y(\mu(t))| \\ &\quad +m_1(M_2r+c_2)|x'(\mu(t))-y'(\mu(t))| \\ &\le K_o\|x-y\|_{C^1}, \end{aligned} \tag{3.19}$$

for each $t \in J$, where

$$K_o=\max\{M_1+m_1M_2r,\ m_1c_2+m_1M_2r\}=\max\{M_1,m_1c_2\}+m_1M_2r.$$

Inequalities (3.18) and (3.19) together imply that

$$\|Ax-Ay\|_{C^1}\le \alpha\|x-y\|_{C^1},$$

whenever $x,\, y \in \overline{\mathcal{B}_r(0)}$, where the Lipschitz constant α is given by

$$\alpha=\max\{M_o\,,\,K_o\}=\max\,\{M_o\,,\,\max\{M_1\,,\,m_1c_2\}+m_1M_2r\}.$$

Hence by Remark 3.1, A is a Lipschitz operator on $\overline{\mathcal{B}_r(0)}$ with the Lipschitz constant α, and by inequality (3.11), $\alpha M<1$.

Thus the conditions (a) and (b) of Theorem 2.2 are satisfied and hence either its conclusion (i) or (ii) holds. We show that conclusion (ii) is impossible. Let u be a solution of the operator equation

$$u(t)=\lambda Au(t)\,Bu(t),\quad t\in J,$$

with $\|u\|_{C^1}=r$ for some λ, $0<\lambda<1$. By the definitions of A and B,

$$\begin{aligned} |u(t)| &= \lambda|Au(t)Bu(t)| \\ &\le \lambda\Big|f\Big(t,u(\mu(t))\Big)\Big|\Big[\int_a^b G(t,s)|g(s,u(\sigma(s)),u'(\eta(s)))|\,ds\Big] \\ &\le \lambda\Big[\Big|f\Big(t,u(\mu(t))\Big)-f(t,0)\Big|+|f(t,0)|\Big]\Big(\frac{b-a}{4}\Big)\|\psi\|_{L^1}\phi(r) \\ &\le \Big[M_o|u(\mu(t))|+\lambda c_o\Big]\Big(\frac{b-a}{4}\Big)\|\psi\|_{L^1}\phi(r)) \\ &\le \Big(M_or+c_o\Big)\Big(\frac{b-a}{4}\Big)\|\psi\|_{L^1}\phi(r)) \end{aligned} \tag{3.20}$$

for all $t \in J$, where $M_o = \sup\{p(t) \,|\, t \in J\}$ and $c_o = \sup_{t\in J} |f(t,0)|$, and

$$
\begin{aligned}
(3.21)\quad & |u'(t)| \\
\le\ & |Au(t)|\,|(Bu)'(t)| + |(Au)'(t)|\,|Bu(t)| \\
\le\ & \Big[\,\big|f(t,u(\mu(t)))\big|\,\Big] \Big(\int_a^b |G_t(t,s)\|g(s,u(\sigma(s)),u'(\eta(s)))|\,ds\Big) \\
& + \Big[\big|f_t(t,u(\mu(t)))\big|\Big] \Big(\int_a^b |G(t,s)\|g(s,u(\sigma(s)),u'(\eta(s)))|\,ds\Big) \\
& + \Big[|u'(\mu(t))|\,|\mu'(t)|\,|f_2\big(t,u(\mu(t))\big)|\Big] \Big(\int_a^b |G(t,s)\|g(s,u(\sigma(s)),u'(\eta(s)))|\,ds\Big) \\
\le\ & \Big[|f(t,u(\mu(t))) - f(t,0)| + |f(t,0)|\Big]\,\|\psi\|_{L^1}\phi(r) \\
& \quad + \Big[|f_t(t,u(\mu(t))) - f_t(t,0)| + |f_t(t,0)|\Big]\,\frac{(b-a)}{4}\|\psi\|_{L^1}\phi(r) \\
& \quad + rm_1\Big[|f_2(t,u(\mu(t))) - f_2(t,0)| + |f_2(t,0)|\Big]\,\frac{(b-a)}{4}\|\psi\|_{L^1}\phi(r) \\
\le\ & \Big[p(t)|u(\mu(t))| + c_0\Big]\|\psi\|_{L^1}\phi(r) + \Big[p_1(t)|u(\mu(t))| + c_1\Big]\frac{(b-a)}{4}\|\psi\|_{L^1}\phi(r) \\
& + rm_1\Big[p_2(t)|u(\mu(t))| + c_2\Big]\frac{(b-a)}{4}\|\psi\|_{L^1}\phi(r) \\
\le\ & \big[M_0 r + c_0\big]\|\psi\|_{L^1}\phi(r) + \big[M_1 r + c_1\big]\frac{(b-a)}{4}\|\psi\|_{L^1}\phi(r) \\
& + rm_1\big[M_2 r + c_2\big]\frac{(b-a)}{4}\|\psi\|_{L^1}\phi(r) \\
\le\ & (M_0 r + c_0)\|\psi\|_{L^1}\phi(r) + \Big[rm_1(M_2 r + c_2) + M_1 r + c_1\Big]\frac{(b-a)}{4}\|\psi\|_{L^1}\phi(r) \\
\le\ & \Big[M_0 r + c_0 + \frac{(b-a)}{4}\big\{rm_1(M_2 r + c_2) + M_1 r + c_1\big\}\Big]\|\psi\|_{L^1}\phi(r)
\end{aligned}
$$

for all $t \in J$, where $M_1 = \sup\{p_1(t) \,|\, t \in J\}$, $M_2 = \sup\{p_2(t) \,|\, t \in J\}$, and $c_1 = \sup_{t\in J} |f_t(t,0)|$, $c_2 = \sup_{t\in J} |f_2(t,0)|$.

Inequalities (3.20) and (3.21) imply that

$$
\begin{aligned}
\sup_{t\in J} |u(t)| \ &\le\ \frac{(b-a)}{4}[M_o r + c_o]\|\psi\|_{L^1}\phi(r) \quad \text{and} \\
\sup_{t\in J} |u'(t)| \ &\le\ \Big[M_0 r + c_0 + \frac{(b-a)}{4}\big\{rm_1(M_2 r + c_2) + M_1 r + c_1\big\}\Big]\|\psi\|_{L^1}\phi(r),
\end{aligned}
$$

and thus

$$
r \le \frac{(b-a)}{2}(M_o r + c_o)\max\left\{1\,,\ \frac{4}{b-a} + rm_1(M_2 r + c_2) + M_1 r + c_1\right\}\|\psi\|_{L^1}\phi(r)
$$

since $r = \|u\|$ and $0 < \lambda < 1$. This is a contradiction to (3.12), and hence the conclusion (ii) is not valid. Consequently, the conclusion (i) is valid, and the FBVP (1.1) has a solution in $\overline{\mathcal{B}_r(0)}$. □

Remark 3.2. The FBVP (1.1) has a nonzero solution if all the conditions of Theorem 2.2 are satisfied and there exists a subset I of the interval J such that $\text{meas}(I) > 0$ and $g(s,0,0) \neq 0$, whenever $s \in I$.

Example 3.1. To illustrate Theorem 3.1, consider the following FBVP

$$\left\{ \begin{array}{l} -\left(\dfrac{x(t)}{f(t,\, x(\frac{t^2}{2}))}\right)'' = g\big(t,\, x(1-t),\, x'(t^2)\big), \quad \text{a.e. } t \in [0,1], \\ x(0) = 0 = x(1), \end{array} \right. \tag{3.22}$$

where

$$f(t,x) = \frac{11}{50} + \frac{1}{100} t^2 \sin x,$$

and

$$g(t,x,y) = \left\{ \begin{array}{l} \dfrac{1}{\sqrt{t}} \Big[\dfrac{1}{50} + \dfrac{1}{100}\big(|x| + |y|\big)\Big], \text{ if } t \in (0,1] \\ 0, \text{ if } \quad t = 0. \end{array} \right.$$

Here $\mu(t) = \frac{t^2}{2}$, so that $\mu'(t) = 2t$ and $m_1 = \sup_{t\in J} |\mu'(t)| = 1$.

Again $\sigma(t) = 1-t$ and $\eta(t) = t^2$. Note that the functions $\mu, \sigma, \eta : J \to J$ are continuous. It is easy to see that g is a Carathéodory function, f is nonzero continuous function and the partial derivatives f_t and f_2 are continuous on $J \times \mathbb{R}$. Here the constants c_i, $i = 0,1,2$; are

$$\begin{aligned} c_o &= \sup\{|f(t,0)| \,|\, t \in [0,1] \times \mathbb{R}\} = \frac{11}{50}, \\ c_1 &= \sup\{|f_t(t,0)| \,|\, t \in [0,1]\} = 0, \\ c_2 &= \sup\{|f_2(t,0)| \,|\, t \in [0,1]\} = \frac{1}{100}. \end{aligned}$$

Furthermore,

$$|f(t,x) - f(t,y)| = \frac{1}{100} t^2 |\sin x - \sin y| \leq \frac{1}{100}|x-y| = M_0|x-y|,$$

$$|f_t(t,x) - f_t(t,y)| = \frac{t}{50}|\sin x - \sin y| \leq \frac{1}{50}|x-y| = M_1|x-y|$$

$$|f_2(t,x) - f_2(t,y)| = \frac{t^2}{100}|\cos x - \cos y| \leq \frac{1}{100}|x-y| = M_2|x-y|$$

whenever $t \in [0,1]$ and $x,\, y \in \mathbb{R}$, so that $M_0 = \frac{1}{100}$, $M_1 = \frac{1}{50}$ and $M_3 = \frac{1}{100}$.

Furthermore,

$$\begin{aligned} |g(t,x,y)| &= \frac{1}{\sqrt{t}}\left[\frac{1}{50}+\frac{1}{100}(|x|+|y|)\right], \\ &\le \frac{1}{\sqrt{t}}\left[\frac{1}{50}+\frac{1}{50}\max\{|x|,|y|\}\right], \\ &= \psi(t)\phi\big(\max\{|x|,|y|\}\big) \end{aligned}$$

for all $t \in J$ and $x, y \in \mathbb{R}$, where $\psi(t) = \dfrac{1}{\sqrt{t}}$ and $\phi(r) = \dfrac{1}{50} + \dfrac{1}{50}r$. Clearly, $\psi \in L^1(J, \mathbb{R})$ and $\phi : \mathbb{R}^+ \to (0, \infty)$ is a continuous and nondecreasing function with

$$\|\psi\|_{L^1} = \int_0^1 \psi(t)\,dt = \int_0^1 t^{-\frac{1}{2}}\,dt = 1.$$

If we choose $r = 2$, then conditions (3.11) and (3.12) of Theorem 3.1 are satisfied. Hence, the FBVP (3.22) has a solution u in X with $\|u\| \le 2$.

Remark 3.3 Note that $g(t, 0, 0) \neq 0$ for all $t \in (0, 1]$ and hence the FBVP (3.22) has a nonzero solution on J in view of Remark 3.2.

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